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# Asymptotic and Oscillatory Behavior of Solutions of a Class of Second Order Differential Equations with Deviating Arguments

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The asymptotic and oscillatory behavior of solutions of damped nonlinear second order differential equations with deviating arguments of the type $(a(t)\,\psi(x(t))\,\dot{x}(t))^{\cdot} + p(t)\,\dot{x}(t) + q(t)\,f(x[g(t)]) = 0$ $({}^{\cdot} = d/dt)$ is studied. Criteria for oscillation of all solutions when the damping coefficient "$p$" is of constant sign on $[t_0, \infty)$ are established. Results on the asymptotic and oscillatory behavior of solutions of the damped-forced equation $(a(t)\,\psi(x(t))\,\dot{x}(t))^{\cdot} + p(t)\,\dot{x}(t) + q(t)\,f(x[g(t)]) = e(t)$, where $q$ is allowed to change signs in $[t_0, \infty)$, are also presented. Some of the results of this paper extend, improve, and correlate a number of existing criteria.


## 1. INTRODUCTION

In this paper we are concerned with nonlinear differential equations with deviating arguments of the type

$$(a(t)\,\psi(x(t))\,\dot{x}(t))^{\cdot} + p(t)\,\dot{x}(t) + q(t)\,f(x[g(t)]) = 0 \qquad ({}^{\cdot} = d/dt), \quad (1)$$

where $q$, $g$, $p$, $q\colon [t_0, \infty) \to R$, $\psi$, $f\colon R \to R = (-\infty, \infty)$ are continuous, $a(t) > 0$, $q(t) \geqslant 0$ for $t \geqslant t_0$, and $q$ is not identically zero on any ray of the form $[t_1, \infty)$ for some $t_1 \geqslant t_0$, $g(t) \to \infty$ as $t \to \infty$, $\psi(x) > 0$ for all $x$, and $xf(x) > 0$ for $x \neq 0$.



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The functions appearing in Eq. (1) will be assumed to be sufficiently smooth for a local existence and uniqueness theorem to hold for Eq. (1) on $0 \leqslant t_0 \leqslant t < \infty$.

In what follows, we consider only those solutions of Eq. (1) which are defined for all large $t$. A solution of Eq. (1) is called oscillatory if it has no last zero, otherwise it is called nonoscillatory. Equation (1) is called oscillatory if all its solutions are oscillatory.

In recent years there has been an increasing interest in the study of the qualitative behavior of solutions of equations of type (1) and/or related equations; see, for example, the papers [1–21] and references cited therein.

In the study of the differential equation

$$(a(t)\,\dot{x}(t))^{\cdot} + q(t)\,f(x(t)) = 0, \tag{$*$}$$

many criteria for oscillation exist which involve the behavior of the integral of $q$, however, a common restriction, namely $\int^{\infty} (1/a(s))\,ds = \infty$ on the function $a$, is required. As examples to this study we cite the papers of Bhatia [1], Coles [3], Grace and Lalli [4], Graef, Rankin, and Spikes [9], and Wong [20].

Recently, Grace *et al.* [5–8] extended and improved some of the known oscillation criteria for Eq. $(*)$ to more general equations of the form (1):

In [13], Kulenovic and Grammatikopoulos obtained some results on the asymptotic and oscillatory behavior of the retarded strongly superlinear equation

$$(a(t)\,\dot{x}(t))^{\cdot} + q(t)\,f(x[g(t)]) = 0, \tag{$**$}$$

where the function $f$ is required to satisfy $\int^{\pm\infty} (du/f(u)) < \infty$.

Our main purpose in this paper is to study the asymptotic and oscillatory behavior of solutions of Eq. (1), where conditions on the functions $a$, $p$, and $\psi$ are different from those imposed in [5–8]. In Section 2, we present some criteria which guarantee that every solution $x(t)$ of Eq. (1) is either oscillatory or else $x(t) \to 0$ monotonically as $t \to \infty$. Such criteria can be applied to Eq. (1), where the damping coefficient "$p$" is either a nonnegative or a nonpositive continuous function on $[t_0, \infty)$. Oscillatory behavior of all solutions of Eq. (1) when it is strongly superlinear, i.e., when $\int_{\pm 0} (\psi(u)/f(u))\,du < \infty$ with retarded or advanced arguments, is established. In Section 3, we present some theorems for asymptotic and oscillatory behavior and/or behavior of the solutions of Eq. (1). These criteria are applicable to linear equations as well as equations of the type (1), where $f'(x)/\psi(x) \geqslant k > 0$ for $x \neq 0$. Finally, we consider the damped-forced equation

$$(a(t)\,\psi(x(t))\,\dot{x}(t))^{\cdot} + p(t)\,\dot{x}(t) + q(t)\,f(x[g(t)]) = e(t),$$

where $q$ is of arbitrary sign on $[t_0, \infty)$, and obtain results which ensure the oscillation of the derivative of any solution of this equation or else $x(t) \to 0$ monotonically as $t \to \infty$. Examples are inserted in the text to illustrate the relevance of the theorems.

Thus the present work is an attempt to make a systematic study of some general second order differential equations. The results are presented in a form which is essentially new.

## 2. Main Results

In this section we are concerned with the oscillatory and asymptotic behavior of strongly sublinear and strongly superlinear differential equations of the form of Eq. (1). The damping coefficient "$p$" is assumed to be nonnegative on $[t_0, \infty)$.

Theorem 2.1. *Assume that $\dot{g}(t) \geqslant 0$ for $t \geqslant t_0$ and*

$$f'(x) \geqslant 0 \qquad \text{for} \quad x \neq 0 \qquad ('=d/dx), \tag{2}$$

*and let there exist $\rho \in C^2[[t_0, \infty), (0, \infty)]$ such that*

$$\dot{\rho}(t) \leqslant 0, \qquad (p(t)\,\rho(t))^{\cdot} \leqslant 0, \qquad (a(t)\,\dot{\rho}(t))^{\cdot} \geqslant 0 \qquad \text{for} \quad t \geqslant t_0. \tag{3}$$

*If*

$$\int^{\infty} \rho(s)\, q(s)\, ds = \infty \tag{4}$$

*and*

$$\int^{\infty} \frac{1}{a(s)\,\rho(s)} \int_{t_0}^{s} \rho(\tau)\, q(\tau)\, d\tau\, ds = \infty, \tag{5}$$

*then every solution $x$ of Eq. (1) is either oscillatory or $x(t) \to 0$ monotonically as $t \to \infty$.*

*Proof.* Let $x(t)$ be a nonoscillatory solution of Eq. (1). Without loss of generality, we assume that $x(t) \neq 0$ for all $t \geqslant t_0$. Furthermore, we suppose that $x(t)$ and $x[g(t)]$ are positive for $t \geqslant t_0$, since the substitution $u = -x$ transforms Eq. (1) into an equation of the same form subject to the assumptions of the theorem. Now, we consider the following three cases for the behavior of $x$.

*Case* 1. $\dot{x}$ is oscillatory. If $\dot{x}(t_1)=0$ and $q(t_1)>0$ for some $t_1>t_0$, then

$$(a(t)\,\psi(x(t))\,\dot{x}(t))^{\boldsymbol{\cdot}}\,|_{t=t_1}=-q(t_1)\,f(x[g(t_1)])<0$$

from which we can prove that $\dot{x}(t)$ cannot have another zero after it vanishes once. Thus $\dot{x}(t)$ has a fixed sign for all sufficiently large $t$.

*Case* 2. $\dot{x}>0$ on $[t_1,\infty)$ from some $t_1\geqslant t_0$. We define

$$w(t)=\frac{a(t)\,\psi(x(t))\,\dot{x}(t)}{f(x[g(t)])}\,\rho(t)\qquad\text{for}\quad t\geqslant t_1.$$

Then for every $t\geqslant t_1$ we obtain

$$\begin{aligned}\dot{w}(t)=&-\rho(t)\,q(t)-p(t)\,\rho(t)\frac{\dot{x}(t)}{f(x[g(t)])}+a(t)\,\dot{\rho}(t)\frac{\psi(x(t))\,\dot{x}(t)}{f(x[g(t)])}\\&-a(t)\,\rho(t)\,\dot{g}(t)\frac{\psi(x(t))\,f'(x[g(t)])\,\dot{x}[g(t)]\,\dot{x}(t)}{f^2(x[g(t)])}.\end{aligned}\tag{6}$$

Using conditions (2) and (3) we get

$$\dot{w}(t)\leqslant-\rho(t)\,q(t)\qquad\text{for}\quad t\geqslant t_1.$$

Integrating the above inequality from $t_1$ to $t$ we have

$$\int_{t_1}^{t}\rho(s)\,q(s)\,ds\leqslant w(t_1)-w(t)\leqslant w(t_1)<0.$$

This contradicts condition (4).

*Case* 3. $\dot{x}<0$ on $[t_1,\infty)$ for $t_1\geqslant t_0$. Suppose that $\lim_{x\to\infty}x(t)=b$, $b\geqslant 0$. We claim that $b=0$. To prove it, assume that $b>0$, and define

$$u(t)=a(t)\,\psi(x(t))\,\dot{x}(t)\,\rho(t),\qquad t\geqslant t_1.$$

Then, for $t\geqslant t_1$ we obtain

$$\dot{u}(t)=-\rho(t)\,q(t)\,f(x[g(t)])-p(t)\,\rho(t)\,\dot{x}(t)+a(t)\,\dot{\rho}(t)\,\psi(x(t))\,\dot{x}(t).\tag{7}$$

Hence, for all $t\geqslant t_1$ we have

$$\begin{aligned}u(t)=u(t_1)&-f(x[g(t)])\int_{t_1}^{t}\rho(s)\,q(s)\,ds+\int_{t_1}^{t}f'(x[g(s)])\,\dot{x}[g(s)]\,\dot{g}(s)\\&\times\int_{t_1}^{s}\rho(\tau)\,q(\tau)\,d\tau\,ds-\int_{t_1}^{t}(q(s)\,\rho(s))\,\dot{x}(s)\,ds\\&+\int_{t_1}^{t}(a(s)\,\dot{\rho}(s))\,\psi(x(s))\,\dot{x}(s)\,ds.\end{aligned}$$

By the Bonnet theorem, for any $t \geqslant t_1$, there exist $\xi_1, \xi_2 \in [t_1, t]$ so that

$$-\int_{t_1}^{t} (p(s)\,\rho(s))\,\dot{x}(s)\,ds = -p(t_1)\,\rho(t_1)[x(\xi_1) - x(t_1)] \leqslant p(t_1)\,\rho(t_1)\,x(t_1),$$

and

$$\int_{t_1}^{t} (a(s)\,\dot{\rho}(s))(\psi(x(s))\,\dot{x}(s))\,ds = a(t_1)\,\dot{\rho}(t_1)\left[\int_0^{x(\xi_2)} \psi(v)\,dv - \int_0^{x(t_1)} \psi(v)\,dv\right]$$
$$\leqslant -a(t_1)\,\dot{\rho}(t_1)\left(\int_0^{x(t_1)} \psi(v)\,dv\right).$$

So, for every $t \geqslant t_1$

$$u(t) \leqslant M - f(b)\int_{t_1}^{t} \rho(s)\,q(s)\,ds,$$

where $M = u(t_1) + p(t_1)\,\rho(t_1)\,x(t_1) - a(t_1)\,\dot{\rho}(t_1)(\int_0^{x(t_1)} \psi(v)\,dv)$. By assumptions of the theorem, there exists a $t_2 \geqslant t_1$ such that

$$u(t) \leqslant -\frac{f(b)}{2}\int_{t_1}^{t} \rho(s)\,q(s)\,ds \qquad \text{for} \quad t \geqslant t_2.$$

Thus,

$$\int_{t_2}^{t} \psi(x(s))\,\dot{x}(s)\,ds \leqslant -\frac{f(b)}{2}\int_{t_2}^{t} \frac{1}{a(s)\,\rho(s)}\int_{t_1}^{s} \rho(\tau)\,q(\tau)\,d\tau\,ds.$$

By condition (5) we get

$$\int_{x(t_2)}^{x(t)} \psi(v)\,dv \to -\infty \qquad \text{as} \quad t \to \infty,$$

a contradiction to the fact that $x(t) > 0$ for $t \geqslant t_0$. Thus $b = 0$ and $x(t) \to 0$.

The following examples are illustrative.

EXAMPLE 1. Consider the differential equations

$$(t^3\dot{x})^{\cdot} + \frac{1}{t}\,\dot{x} + q(t)\,f(x[g(t)]) = 0, \tag{8}$$

$$\left(t(1+t^2)\frac{1}{1+x^2}\,\dot{x}\right)^{\cdot} + \frac{1}{t}\,\dot{x} + q(t)\,f(x[g(t)]) = 0 \tag{9}$$

and

$$\left(\frac{t^5}{1+t^2}(1+x^2)\dot{x}\right)^{\cdot}+\frac{1}{t}\dot{x}+q(t)\,f(x[g(t)])=0, \tag{10}$$

where $g(t)$ is a continuous and nondecreasing function for $t \geqslant t_0 = 1$ and $\lim_{t\to\infty} g(t)=\infty$, $q(t)=g^{\alpha}(t)(1+t^{-3})$, and $f(x)=|x|^{\alpha}\operatorname{sgn} x$, $\alpha>0$. We take $\rho(t)=1$. If

$$\int^{t} g^{\alpha}(s)\,ds = O(t^2),$$

then all the conditions of Theorem 2.1 are satisfied and hence every solution $x$ of Eqs. (8)–(10) is either oscillatory or $x(t)\to 0$ monotonically as $t\to\infty$. Each of Eqs. (8), (9), and (10) admits the nonoscillatory solution $x(t)=1/t\to 0$ monotonically as $t\to\infty$.

*Remark* 1. One is tempted to believe that if we replace condition (5) by the stronger condition

$$\int^{\infty} \frac{1}{a(s)\,\rho(s)}\,ds=\infty, \tag{11}$$

then conditions (4) and (11) may ensure the oscillation of Eq. (1). In fact, this is not enough, since, if we take $\alpha \geqslant 1$ and $g(t)=t$ in Eqs. (8)–(10) and let $\rho(t)=1/t^2$, the hypotheses of Theorem 2.1 and condition (11) are satisfied. Therefore, we need further restrictions on the functions in Eq. (1).

In the following theorem we study the oscillatory behavior of Eq. (1) subject to the conditions

$$\psi(x)\geqslant c>0 \qquad \text{for all } x \tag{12}$$

and

$$\int_{+0} \frac{\psi(u)}{f(u)}\,du<\infty \qquad \text{and} \qquad \int_{-0} \frac{\psi(u)}{f(u)}\,du<\infty. \tag{13}$$

THEOREM 2.2. *Let $g(t)\leqslant t$, $\dot{g}(t)\geqslant 0$ for $t\geqslant t_0$, conditions* (2), (12), *and* (13) *hold, and assume that there exists a function $\rho\in C^2[[t_0,\infty),(0,\infty)]$ such that conditions* (3), (4), *and* (11) *hold. Then Eq.* (1) *is oscillatory.*

*Proof.* Let $x(t)$ be a nonoscillatory solution of Eq. (1). As in the proof of Theorem 2.1, three cases arise. The proof of Cases 1 and 2 is similar to the corresponding cases of Theorem 2.1. Hence we consider Case 3. By conditions (4) and (11) we conclude that $x(t)\to 0$ as $t\to\infty$. Let $x(t)>0$

and $x[g(t)]>0$ for $t \geqslant t_1 \geqslant t_0$ and consider the function $w$ defined earlier in the proof of Theorem 2.1 (Case 2). Then for every $t \geqslant t_1$ we obtain

$$\dot{w}(t) = -\rho(t)\,q(t) - p(t)\,\rho(t)\,\frac{\dot{x}(t)}{f(x[g(t)])} + a(t)\,\dot{\rho}(t)\,\frac{\psi(x(t))\,\dot{x}(t)}{f(x[g(t)])}$$
$$- w(t)\,\frac{(d/dt)\,f(x[g(t)])}{f(x[g(t)])}. \tag{14}$$

Using conditions (2) and (3) and the fact that $g(t) \leqslant t$ for $t \geqslant t_1$ we get

$$w(t) \leqslant w(t_1) - \int_{t_1}^{t} \rho(s)\,q(s)\,ds - \int_{t_1}^{t} \frac{1}{C}\,p(s)\,\rho(s)\left(\frac{\psi(x(s))\,\dot{x}(s)}{f(x(s))}\right) ds$$
$$+ \int_{t_1}^{t} (a(s)\,\dot{\rho}(s))\,\frac{\psi(x(s))\,\dot{x}(s)}{f(x(s))}\,ds - \int_{t_1}^{t} w(s)\,\frac{ds(x[g(s)])}{f(x[g(s)])}.$$

By the Bonnet theorem, for any $t \geqslant t_1$, there exist $\xi_1, \xi_2 \in [t_1, t]$ such that

$$-\int_{t_1}^{t} \frac{1}{C}\,p(s)\,\rho(s)\,\frac{\psi(x(s))\,\dot{x}(s)}{f(x(s))}\,ds = \frac{1}{C}\,p(t_1)\,\rho(t_1) \int_{x(t_1)}^{x(\xi_1)} \frac{\psi(u)}{f(u)}\,du$$
$$\leqslant -\frac{1}{C}\,p(t_1)\,\rho(t_1) \int_{x(t_1)}^{\infty} \frac{\psi(u)}{f(u)}\,du = M_1$$

and

$$\int_{t_1}^{t} (a(s)\,\dot{\rho}(s))\,\frac{\psi(x(s))\,\dot{x}(s)}{f(x(s))}\,ds = a(t_1)\,\dot{\rho}(t_1) \int_{x(t_1)}^{x(\xi_2)} \frac{\psi(u)}{f(u)}\,du$$
$$\leqslant a(t_1)\,\dot{\rho}(t_1) \int_{x(t_1)}^{\infty} \frac{\psi(u)}{f(u)}\,du = M_2.$$

So, for every $t \geqslant t_1$

$$w(t) \leqslant M - \int_{t_1}^{t} \rho(s)\,q(s)\,ds - \int_{t_1}^{t} w(s)\,\frac{df(x[g(t)])}{f(x[g(t)])},$$

where $M = w(t_1) + M_1 + M_2$, and hence by condition (4) we derive

$$-w(t) \geqslant C + \int_{t_1}^{t} w(s)\,\frac{df(x[g(s)])}{f(x[g(s)])}, \tag{15}$$

where $C$ is a positive constant. So, for every $t \geqslant t_1$

$$\left(w(t)\,\frac{d}{dt}\frac{f(x[g(t)])}{f(x[g(t)])}\right)\left(C + \int_{t_1}^{t} w(s)\,\frac{df(x[g(s)])}{f(x[g(s)])}\right)^{-1} \geqslant -\frac{d}{dt}\frac{f(x[g(t)])}{f(x[g(t)])},$$

and hence by integrating over $[t_1, t]$ we obtain

$$\ln \frac{1}{C}\left[C + \int_{t_1}^{t} w(s) \frac{df(x[g(s)])}{f(x[g(s)])}\right] \geqslant \ln \frac{f(x[g(t_1)])}{f(x[g(t)])}.$$

Thus,

$$C + \int_{t_1}^{t} w(s) \frac{df(x[g(s)])}{f(x[g(s)])} \geqslant \frac{Cf(x[g(t_1)])}{f(x[g(t)])} \qquad \text{for all} \quad t \geqslant t_1,$$

so (15) yields

$$\psi(x(t))\, \dot{x}(t) \leqslant -C_1 \frac{1}{a(t)\, \rho(t)} \qquad \text{for every} \quad t \geqslant t_1,$$

where $C_1 = Cf(x[g(t_1)])$, and consequently we have

$$\int_{x(t_1)}^{x(t)} \psi(u)\, du \leqslant -C_1 \int_{t_1}^{t} \frac{1}{a(s)\, \rho(s)}\, ds \to -\infty \qquad \text{as} \quad t \to \infty,$$

a contradiction to the fact that $x(t) > 0$ for $t \geqslant t_1$. This completes the proof.

The following two corollaries are immediate. We omit the proofs.

COROLLARY 2.3. *Let conditions* (12) *and* (13) *of Theorem* 2.2 *be replaced by*

$$0 < \psi(x) \leqslant C_1 \qquad \textit{for} \quad x \neq 0 \tag{16}$$

*and*

$$\int_{+0} \frac{du}{f(u)} < \infty \qquad \textit{and} \qquad \int_{-0} \frac{du}{f(u)} < \infty, \tag{17}$$

*respectively, then the conclusion of Theorem* 2.2 *holds.*

COROLLARY 2.4. *Let condition* (12) *of Theorem* 2.2 *be replaced by condition* (17), *then the conclusion of Theorem* 2.2 *holds.*

EXAMPLE 2. Consider the differential equations (8)–(10) with $q(t) = t$ and $0 \leqslant \alpha < 1$. We let $\rho(t) = 1/t^2$, $t \geqslant t_0 > 0$. It is easy to check that Eq. (8) is oscillatory by Theorem 2.2, while Eq. (9) is oscillatory by Corollary 2.3. Also, using Corollary 2.3 one can conclude that all bounded solutions of Eq. (10) are oscillatory.

*Remark* 2. We note that Theorem 2.2 is concerned only with the oscillatory behavior of ordinary and retarded sublinear differential equations of the form of Eq. (1). It fails to apply to other cases. To illustrate this point we consider two special cases of Eq. (8), namely, the ordinary differential equation

$$(t^3\dot{x}(t))^{\cdot} + \frac{1}{t}\dot{x}(t) + \frac{(1+t^3)t^\alpha}{t^3}|x(t)|^\alpha \operatorname{sgn} x(t) = 0, \qquad \alpha > 1,\ t \geqslant t_0 = 1 \tag{18}$$

and the advanced sublinear equation

$$(t^3\dot{x}(t))^{\cdot} + \frac{1}{t}\dot{x}(t) + \left(t^2 + \frac{1}{t}\right)x^{1/3}[t^6] = 0, \qquad t \geqslant t_0 = 1. \tag{19}$$

Each of these equations admits the nonoscillatory solution $x(t) = 1/t \to 0$ monotonically as $t \to \infty$. It is easy to check that the hypotheses of Theorem 2.2 are satisfied, using $\rho(t) = t^{-2}$, except condition (13) in the case of Eq. (18), and the condition on function $g$ in the case of Eq. (19).

The case when Eq. (1) is of advanced typed is covered in:

THEOREM 2.5. *Let* $g(t) \geqslant t$, $\dot{g}(t) > 0$ *for* $t \geqslant t_0$, *and conditions* (2), (12), *and* (13) *hold. Suppose that there exists a function* $\rho \in C^2[[t_0, \infty), (0, \infty)]$ *such that*

$$\dot{\rho}(t) \leqslant 0, \quad \left(\frac{a[g(t)]\,\rho[g(t)]\,p(t)}{a(t)\,\dot{g}(t)}\right)^{\cdot} \leqslant 0, \quad (a(t)\,\dot{\rho}(t))^{\cdot} \geqslant 0 \quad \textit{for} \quad t \geqslant t_0. \tag{20}$$

*If*

$$\int^{\infty} \rho[g(s)]\,q(s)\,ds = \infty \tag{21}$$

*and*

$$\int^{\infty} \frac{1}{a(s)\,\rho[g(s)]}\,ds = \infty, \tag{22}$$

*then Eq.* (1) *is oscillatory.*

*Proof.* Let $x(t)$ be a nonoscillatory solution of Eq. (1), say $x(t) > 0$ and $x[g(t)] > 0$ for $t \geqslant t_0$. As in the proof of Theorem 2.1 we consider three cases for the behavior of $\dot{x}$. The case when $\dot{x}$ is oscillatory is similar to

Case 1 in Theorem 2.1. Now, consider Case 2. Suppose $\dot{x}(t)>0$, for $t \geqslant t_1$ for some $t_1 \geqslant t_0$. Define

$$v(t)=\frac{a(t)\,\psi(x(t))\,\dot{x}(t)}{f(x(t))}\,\rho[g(t)], \qquad \text{for} \quad t \geqslant t_1.$$

Then for $t \geqslant t_1$

$$\dot{v}(t)= -\rho[g(t)]\,q(t)\frac{f(x[g(t)])}{f(x(t))}-p(t)\,\rho[g(t)]\,\frac{\dot{x}(t)}{f(x(t))}+a(t)\,\dot{\rho}[g(t)]\,\dot{g}(t)$$
$$\times\frac{\psi(x(t))\,\dot{x}(t)}{f(x(t))}-a(t)\,\rho[g(t)]\,\frac{\psi(x(t))\,f'(x(t))\,\dot{x}(t)}{f^2(x(t))}. \tag{23}$$

Using conditions (2) and (20) and the fact that $\dot{g}(t)>0$ for $t \geqslant t_0$ we obtain

$$\dot{v}(t) \leqslant -\rho[g(t)]\,q(t) \qquad \text{for} \quad t \geqslant t_1.$$

Integrating this inequality from $t_1$ to $t$ we get

$$v(t) \leqslant v(t_1)-\int_{t_1}^{t} \rho[g(s)]\,q(s)\,ds.$$

In view of condition (21)

$$v(t)<0 \qquad \text{for all large } t,$$

which is a contradiction.

*Case* 3. $\dot{x}(t)<0$ for $t \geqslant t_1 \geqslant t_0$. We let

$$V(t)=\frac{a(t)\,\psi(x(t))\,\dot{x}(t)}{f(x[g(t)])}\,\rho[g(t)], \qquad t \geqslant t_1.$$

Thus,

$$\dot{V}(t)= -\rho[g(t)]\,q(t)-p(t)\,\rho[g(t)]\,\frac{\dot{x}(t)}{f(x[g(t)])}+a(t)\,\dot{\rho}[g(t)]\,\dot{g}(t)$$
$$\times\frac{\psi(x(t))\,\dot{x}(t)}{f(x[g(t)])}-\frac{V(t)}{f(x[g(t)])}\frac{d}{dt}f(x[g(t)]). \tag{24}$$

Since $g(t) \geqslant t$ for $t \geqslant t_0$ and $a(t)\,\psi(x(t))\,\dot{x}(t)$ is nondecreasing for $t \geqslant t_1$ we obtain

$$a[g(t)]\,\psi(x[g(t)])\,\dot{x}[g(t)] \leqslant a(t)\,\psi(x(t))\,\dot{x}(t) \qquad \text{for} \quad t \geqslant t_1. \tag{25}$$

Thus, (24) becomes

$$\dot{V}(t) \leqslant -\rho[g(t)]\, q(t) - \frac{1}{C}\left(\frac{p(t)\,\rho[g(t)]\, a[g(t)]}{a(t)\,\dot{g}(t)}\right)\frac{\psi(x[g(t)])\,\dot{x}[g(t)]\,\dot{g}(t)}{f(x[g(t)])}$$
$$+ (a[g(t)]\,\dot{\rho}[g(t)])\frac{\psi(x[g(t)])\,\dot{x}[g(t)]\,\dot{g}(t)}{f(x[g(t)])}$$
$$-\frac{V(t)}{f(x[g(t)])}\frac{d}{dt} f(x[g(t)]).$$

The rest of the proof is similar to that of Theorem 2.2 and hence is omitted.

*Remark* 3. In [5–8], we studied the oscillatory behavior of Eq. (1) subject to conditions of the form

$$\psi(x) \geqslant c > 0 \quad \text{for} \quad x \neq 0 \quad \text{and} \quad \int^{\infty} \frac{1}{a(s)} \exp\left(\int_{t_0}^{s} -\frac{p(u)}{ca(u)}\, du\right) ds = \infty$$

and when $p(t) \equiv 0$, we required that

$$\int^{\infty} \frac{1}{a(s)}\, ds = \infty.$$

Such conditions are not required in the present paper. Moreover those conditions are not satisfied in case of Eqs. (8)–(10) given in Example 1 and hence our earlier results in [5–8] are not applicable to Eqs. (8)–(10) when $0 \leqslant \alpha < 1$ and $q(t) = t$. Thus, the results of this paper are stronger than those in [5–8].

The following results are concerned with the oscillatory and asymptotic behavior of strongly superlinear equations of the type (1). Note that the differential Eq. (1) is said to be strongly superlinear if the functions $f$ and $\psi$ are such that

$$\int^{+\infty} \frac{\psi(u)}{f(u)}\, du < \infty \qquad \text{and} \qquad \int^{-\infty} \frac{\psi(u)}{f(u)}\, du < \infty. \tag{26}$$

THEOREM 2.6. *Let* $g(t) \leqslant t$, $\dot{g}(t) > 0$ *for* $t \geqslant t_0$, *conditions* (2) *and* (26) *hold, and assume that there exist a function* $\rho \in C^2[[t_0, \infty), (0, \infty)]$ *such that*

$$\dot{\rho}(t) \geqslant 0, \qquad (p(t)\,\rho(t))^{\cdot} \leqslant 0, \qquad \left(\frac{a[g(t)]\,\dot{\rho}(t)}{\dot{g}(t)}\right)^{\cdot} \leqslant 0 \qquad \textit{for} \quad t \geqslant t_0. \tag{27}$$

*If conditions* (4) *and* (5) *hold, then every solution* $x$ *of Eq.* (1) *is either oscillatory or* $x(t) \to 0$ *monotonically as* $t \to \infty$.

*Proof.* Let $x(t)$ be a nonoscillatory solution of Eq. (1). Assume that $x(t)>0$ and $x[g(t)]>0$ for $t \geqslant t_0$. As in the proof of Theorem 2.1, three cases are considered. The case when $\dot{x}$ is oscillatory is similar to Case 1 in Theorem 2.1. For the Case 2, i.e., when $\dot{x}(t)>0$ for $t \geqslant t_1 \geqslant 0$, we consider the function $w$ defined in the proof of Theorem 2.1 (Case 2). Then for every $t \geqslant t_1$ we obtain (6) and using conditions (2) and (27) we get

$$\dot{w}(t) \leqslant -\rho(t)\, q(t) + \dot{\rho}(t)\, a(t) \frac{\psi(x(t))\, \dot{x}(t)}{f(x[g(t)])}.$$

It is easy to check that the function $(a(t)\, \psi(x(t))\, \dot{x}(t))$ is nonincreasing for $t \geqslant t_1$, and, since $g(t) \leqslant t$ for $t \geqslant t_1$, we have

$$a(t)\, \psi(x(t))\, \dot{x}(t) \leqslant a[g(t)]\, \psi(x[g(t)])\, \dot{x}[g(t)], \qquad t \geqslant t_1. \tag{28}$$

Thus,

$$\dot{w}(t) \leqslant -\rho(t)\, q(t) + \frac{a[g(t)]\, \dot{\rho}(t)}{\dot{g}(t)} \frac{\psi(x[g(t)])\, \dot{x}[g(t)]\, \dot{g}(t)}{f(x[g(t)])},$$

and hence

$$\begin{aligned} w(t) \leqslant w(t_1) - \int_{t_1}^{t} \rho(s)\, q(s)\, ds + \int_{t_1}^{t} \frac{a[g(s)]\, \dot{\rho}(s)}{\dot{g}(s)} \\ \times \left[ \frac{\psi(x[g(s)])\, \dot{x}[g(s)]\, \dot{g}(s)}{f(x[g(s)])} \right] ds. \end{aligned} \tag{29}$$

By the Bonnet theorem, for every $t \geqslant t_1$, there exists $\xi \in [t_1, t]$ so that

$$\begin{aligned} &\int_{t_1}^{t} \frac{a[g(s)]\, \dot{\rho}(s)}{\dot{g}(s)} \left[ \frac{\psi(x[g(s)])\, \dot{x}[g(s)]\, \dot{g}(s)}{f(x[g(s)])} \right] ds \\ &\quad = \frac{a[g(t_1)]\, \dot{\rho}(t_1)}{\dot{g}(t_1)} \int_{x[g(t_1)]}^{x[g(\xi)]} \frac{\psi(u)}{f(u)}\, du \\ &\quad \leqslant \frac{a[g(t_1)]\, \dot{\rho}(t_1)}{\dot{g}(t_1)} \int_{x[g(t_1)]}^{\infty} \frac{\psi(u)}{f(u)}\, du = M_1 < \infty. \end{aligned}$$

Consequently (29) becomes

$$w(t) \leqslant w(t_1) + M_1 - \int_{t_1}^{t} \rho(s)\, q(s)\, ds,$$

and by condition (3) we obtain the desired contradiction. Next, for the

Case 3 we use the function $u$ considered in the proof of Theorem 2.1 (Case 3), and obtain (7). Then by condition (27) we get

$$\dot{u}(t) \leqslant -\rho(t)\, q(t)\, f(x[g(t)]) - \rho(t)\, p(t)\, \dot{x}(t), \qquad t \geqslant t_1.$$

The rest of the proof is similar so that of Theorem 2.1 (Case 3) and hence is omitted.

THEOREM 2.7. *Let $g(t) \leqslant t$, $\dot{g}(t) \geqslant 0$ for $t \geqslant t_0$, and conditions* (2) *and* (26) *hold. Suppose there exists $\rho \in C^2[[t_0, \infty), (0, \infty)]$ such that*

$$\dot{\rho}(t) \geqslant 0, \qquad (p(t)\, \rho(t))^{\cdot} \leqslant 0, \qquad (a(t)\, \dot{\rho}(t))^{\cdot} \leqslant 0 \qquad \textit{for} \quad t \geqslant t_0, \tag{30}$$

*and that conditions* (5) *and* (21) *are satisfied. Then the conclusion of Theorem* 2.6 *holds.*

*Proof.* The proof is similar so that of Theorem 2.6 except that the function $w$ defined in Case 2 is replaced by the function $V$ considered in the proof of Theorem 2.5 (Case 3). The details are omitted.

THEOREM 2.8. *Let condition* (5) *in Theorem* 2.6 (*respectively Theorem* 2.7) *be replaced by condition* (11). *Then Eq.* (1) *is oscillatory.*

*Proof.* Let $x(t)$ be a nonoscillatory solution of Eq. (1), say $x(t) > 0$ and $x[g(t)] > 0$ for $t \geqslant t_0$. As in the proof of Theorem 2.6, three cases are considered for the behavior of $\dot{x}$. The proof for the first two cases when $\dot{x}$ is oscillatory and when $\dot{x}(t) > 0$, respectively, for $t \geqslant t_1$ is similar to the proof of Cases 1 and 2 in Theorem 2.6 (respectively Theorem 2.7). We consider the third case where $\dot{x}(t) < 0$ for $t \geqslant t_1$ and use the function $u$ considered in the proof of Theorem 2.1 (Case 3). Then for $t \geqslant t_1$ we obtain equality (7). Integrating (7) and using the hypotheses of Theorem 2.6 (respectively Theorem 2.7) we obtain

$$a(t)\, \rho(t)\, \psi(x(t))\, \dot{x}(t) \leqslant u(t_1) < 0 \qquad \text{for} \quad t \geqslant t_1,$$

or

$$\int_{x(t_1)}^{x(t)} \psi(v)\, dv \leqslant u(t_1) \int_{t_1}^{t} \frac{1}{a(s)\, \rho(s)}\, ds \to -\infty \qquad \text{as} \quad t \to \infty,$$

a contradiction to the fact that $x(t) > 0$ for $t \geqslant t_1$. This completes the proof.

Next, we consider Eq. (1) with advanced argument and obtain the following criteria for its behavior.

THEOREM 2.9. *Let $g(t) \geqslant t$ and $\dot{g}(t) \geqslant 0$ for $t \geqslant t_0$, let conditions* (2) *and* (26) *hold, and let there exist $\rho \in C^2[[t_0, \infty), (0, \infty)]$ such that conditions*

(30), (4), *and* (5) *are satisfied. Then every solution* $x$ *of Eq.* (1) *is either oscillatory or* $x(t) \to 0$ *monotonically as* $t \to \infty$.

*Proof.* The proof is similar to that of Theorem 2.6 except that we make use of the advanced argument instead of the retarded one. The details are omitted.

THEOREM 2.10. *Let condition* (5) *in Theorem* 2.9 *be replaced by condition* (11), *then Eq.* (1) *is oscillatory.*

*Proof.* The proof is similar to that of Theorems 2.8 and 2.9 and hence is omitted.

In the following results we discuss the oscillatory and asymptotic behavior of Eq. (1) with advanced argument and nonpositive damping coefficient $p(t)$, $t \geqslant t_0$.

THEOREM 2.11. *Let* $p(t) \leqslant 0$, $g(t) \geqslant t$, *and* $\dot{g}(t) \geqslant 0$ *for* $t \geqslant t_0$, *condition* (2) *hold, and*

$$\int^{+\infty} \frac{du}{f(u)} < \infty \quad \text{and} \quad \int^{-\infty} \frac{du}{f(u)} < \infty. \tag{31}$$

*Assume that there exists a function* $\rho \in C^2[[t_0, \infty), (0, \infty)]$ *such that*

$$\dot{\rho}(t) \leqslant 0, \qquad (p(t)\,\rho(t))^{\cdot} \geqslant 0, \qquad (a(t)\,\dot{\rho}(t))^{\cdot} \geqslant 0 \qquad \text{for} \quad t \geqslant t_0. \tag{32}$$

*If conditions* (4) *and* (5) *hold, then every solution* $x$ *of Eq.* (1) *is either oscillatory or* $\lim_{t \to \infty} x(t) = 0$ *monotonically.*

*Proof.* Let $x(t)$ be a nonoscillatory solution of Eq. (1), say $x(t) > 0$ and $x[g(t)] > 0$ for $t \geqslant t_0$. As in the proof of Theorem 2.1, three cases arise. Since the first one coincides with Case 1 of Theorem 2.1, we consider the other two cases. Now, assume that Case 2 holds, i.e., suppose that $\dot{x}(t) > 0$ for $t \geqslant t_1$ and define the function $w$ as in the proof for Case 2 of Theorem 2.1 to get (6). In view of the hypotheses of the theorem we obtain

$$\dot{w}(t) \leqslant -\rho(t)\,q(t) - p(t)\,\rho(t)\,\frac{\dot{x}(t)}{f(x(t))}.$$

Using the Bonnet theorem and conditions (31), (32), and (4) we obtain the desired contradiction. Next, we consider Case 3, i.e., $\dot{x}(t) < 0$ for $t \geqslant t_1$, and define the function $u$ as in the proof of Case 3 of Theorem 2.1 to obtain (7), which by conditions of the theorem reduces to

$$\dot{u}(t) \leqslant -\rho(t)\,q(t)\,f(x[g(t)]) + a(t)\,\dot{\rho}(t)\,\psi(x(t)\,\dot{x}(t)).$$

The rest of the proof is similar to that of Theorem 2.1 and hence is omitted.

THEOREM 2.12. *Let condition* (5) *in Theorem* 2.11 *be replaced by condition* (11). *Then Eq.* (1) *is oscillatory.*

*Proof.* The proof is similar to that of Theorem 2.11 and Theorem 2.8 (Case 3) and hence is omitted.

The following examples are illustrative:

EXAMPLE 3. The differential equation

$$(t^2(1+x^2(t))\,\dot{x}(t))^{\cdot}+\frac{3}{t}\,\dot{x}(t)+x^3(t)=0, \qquad t\geqslant t_0>0, \tag{33}$$

has a nonoscillatory solution $x(t)=1/t\to 0$ as $t\to\infty$. The conditions of Theorem 2.1 are satisfied if we choose $\rho(t)=1$. We note that Theorem 2.6 is not applicable to Eq. (33), since condition (26) is violated.

EXAMPLE 4. Consider the differential equation

$$(t(1+x^2(t))\,\dot{x}(t))^{\cdot}-(1+7t)\,\dot{x}(t)+\frac{1}{t}\,x^3(t)=0, \qquad t\geqslant t_0>0. \tag{34}$$

The conditions of Theorem 2.11 are satisfied for $\rho(t)=1$ except that $(p(t)\,\rho(t))^{\cdot}=-7\leqslant 0$ for $t\geqslant t_0$. Equation (34) has a nonoscillatory solution $x(t)=\sqrt{t}\not\to 0$ as $t\to\infty$. On the other hand, we see that the equation

$$(t(1+x^2(t))\,\dot{x}(t))^{\cdot}-\left(1+\frac{7}{t}\right)\dot{x}(t)+\frac{1}{t}\,x^3(t)=0, \qquad t\geqslant t_0>0 \tag{35}$$

is oscillatory by Theorem 2.12 with $\rho(t)=1$.

We note that Theorems 3 and 4 in [13] fail to apply to Eq. (35) since $\psi(x)\neq 1$ and $p(t)\neq 0$.

EXAMPLE 5. Consider the differential equation

$$((1+x^2(t))\,\dot{x}(t))^{\cdot}-(2t+g^{\alpha}(t))\,\dot{x}(t)+|x[g(t)]|^{\alpha}\operatorname{sgn}x[g(t)]=0, \tag{36}$$

where $g(t)$ is a continuous, nondecreasing function for $t\geqslant t_0\geqslant 0$, $g(t)\geqslant t$ for $t\geqslant t_0$, and $\alpha>0$. Here we let $\rho(t)=1$.

We observe the following:

(i) Theorem 2.11 is not applicable to Eq. (36) if $g(t)=t$ and $\alpha=1$, since condition (31) is violated.

(ii) Again Theorem 2.11 fails to apply to Eq. (36) if $g(t)=t$ and $\alpha=5$, since the condition that $(p(t)\,\rho(t))^{\cdot}\geqslant 0$ for $t\geqslant t_0$ is violated.

(iii) If $g(t)=t$, and $\alpha=\frac{1}{3}$, we note that Theorem 2.1 fails to apply to

Eq. (36) since the condition on the sign of the damping coefficient $p$ is violated.

Note that Eq. (36) has $x(t)=t$, as a solution which is nonoscillatory.

## 3. Further Results

THEOREM 3.1. *Let $p(t)\geqslant 0$, $g(t)\leqslant t$, and $\dot{g}(t)>0$ for $t\geqslant t_0$,*

$$\frac{f'(x)}{\psi(x)}\geqslant k>0 \qquad \text{for} \quad x\neq 0, \tag{37}$$

*and suppose that there exists a function $\rho\in C^1[[t_0,\infty),(0,\infty)]$ such that*

$$\dot{\rho}(t)\geqslant 0 \qquad \text{and} \qquad (p(t)\,\rho(t))^{\cdot}<0 \qquad \text{for} \quad t\geqslant t_0. \tag{38}$$

*If condition (5) holds and*

$$\int^{\infty}\left[\rho(s)\,q(s)-\frac{\alpha[g(s)]\,\dot{\rho}^2(s)}{4k\rho(s)\,\dot{g}(s)}\right]ds=\infty, \tag{39}$$

*then every solution $x$ of Eq. (1) is either oscillatory or $\lim_{t\to\infty}x(t)=0$ monotonically.*

*Proof.* Let $x(t)$ be a nonoscillatory solution of Eq. (1) without loss of generality, and assume that $x(t)>0$ and $x[g(t)]>0$ for $t\geqslant t_0$. As in the proof of Theorem 2.1 (Case 1), $\dot{x}(t)$ cannot oscillate for all large $t$, so we consider the other two cases. Let $\dot{x}(t)>0$ for $t\geqslant t_1\geqslant t_0$, and define the function $W$ as in the proof of Theorem 2.1 (Case 2). The for $t\geqslant t_1$ we obtain (6). Since $p(t)\geqslant 0$ and $g(t)\leqslant t$ for $t\geqslant t_0$ we get (28). Thus, (6) becomes

$$\dot{w}(t)\leqslant-\rho(t)\,q(t)+\frac{\dot{\rho}(t)}{\rho(t)}\,w(t)-\frac{f'(x[g(t)])}{\psi(x[g(t)])}\,\frac{\dot{g}(t)}{a[g(t)]\,\rho(t)}\,w^2(t).$$

Using condition (37) we have

$$\begin{aligned}
\dot{w}(t)&\leqslant-\rho(t)\,q(t)+\frac{\dot{\rho}(t)}{\rho(t)}\,w(t)-\frac{k\dot{g}(t)}{a[g(t)]\,\rho(t)}\,w^2(t)\\
&=-\left[\rho(t)\,q(t)-\frac{a[g(t)]\,\dot{\rho}^2(t)}{4k\rho(t)\,\dot{g}(t)}\right]\\
&\quad-\left[\sqrt{\frac{k\dot{g}(t)}{a[g(t)]\,\rho(t)}}\,w(t)-\frac{\sqrt{a[g(t)]}\,\dot{\rho}(t)}{2\sqrt{k\rho(t)\,\dot{g}(t)}}\right]^2\\
&\leqslant-\left[\rho(t)\,q(t)-\frac{a[g(t)]\,\dot{\rho}^2(t)}{4k\rho(t)\,\dot{g}(t)}\right], \qquad t\geqslant t_1.
\end{aligned}$$

Integrating the above inequality from $t_1$ to $t$ we obtain

$$\int_{t_1}^{t}\left[\rho(s)\,q(s)-\frac{a[g(s)]\,\dot{g}^2(s)}{4k\rho(s)\,\dot{g}(s)}\right]ds\leqslant w(t_1)-w(t)\leqslant w(t_1)\leqslant\infty,$$

which contradicts (39). The proof of the case when $\dot{x}(t)<0$ for $t\geqslant t_1\geqslant t_0$ is similar to the proof of Case 3 in Theorem 2.6.

The proof of the following theorems is immediate.

THEOREM 3.2. *Let condition* (38) *in Theorem* 3.1 *be replaced by*

$$a(t)\,\dot{\rho}(t)-\frac{1}{c}\,p(t)\,\rho(t)\geqslant 0 \qquad \textit{for} \quad t\geqslant t_0. \tag{40}$$

*If we assume* (12) *then the conclusion of Theorem* 3.1 *holds.*

THEOREM 3.3. *Let condition* (5) *in Theorem* 3.1 (*respectively Theorem* 3.2) *be replaced by condition* (11), *then Eq.* (1) *is oscillatory.*

We consider the following:

EXAMPLE 6. In the differential equation

$$\left(t^2\frac{1}{1+x^2(t)}\,\dot{x}(t)\right)^{\boldsymbol{\cdot}}+\frac{1}{t}\,\dot{x}(t)+x(t)=0, \qquad t\geqslant t_0>0, \tag{41}$$

the conditions of Theorem 3.1 are satisfied for $\rho(1)=t$ and hence every solution $x$ of Eq. (41) is either oscillatory or $x(t)\to 0$ monotonically as $t\to\infty$. We note that Theorem 3.2 fails to apply to Eq. (41), since condition (12) is violated. On the other hand, for the differential equation

$$\left(t\frac{1}{2-\sin x(t)}\,\dot{x}(t)\right)^{\boldsymbol{\cdot}}+\frac{1}{2}\,t\dot{x}(t)+x(t)=0, \qquad t\geqslant t_0\geqslant 0, \tag{42}$$

all the conditions of Theorem 3.2 are satisfied for $\rho(t)=1$ and hence every solution $x$ of Eq. (42) is either oscillatory or $x(t)\to 0$ monotonically as $t\to\infty$. One can easily check that Theorem 3.1 fails to apply to Eq. (42) because the condition (38) is violated.

Next, we have the following:

THEOREM 3.4. *Let condition* (38) *in Theorem* 3.1 (*respectively Theorem* 3.3) *be replaced by*

$$\left(a(t)\,\dot{\rho}(t)-\frac{1}{c}\,p(t)\,\rho(t)\right)=\gamma(t)\leqslant 0 \qquad \textit{and} \qquad \dot{\gamma}(t)\geqslant 0 \qquad \textit{for} \quad t\geqslant t_0, \tag{43}$$

*then the conclusion of Theorem* 3.1 (*respectively Theorem* 3.3) *holds.*

*Proof.* The proof is immediate and hence is deleted.

*In the following results we let* $\rho(t) \leqslant 0$ *and* $g(t) = t$ *for* $t \geqslant t_0$.

THEOREM 3.5. *Suppose that*

$$f'(x)\,\psi(x) \geqslant k_1 > 0 \qquad \text{for} \quad x \neq 0, \tag{44}$$

*and let* $\rho \in C^1[[t_0, \infty), (0, \infty)]$ *such that*

$$\dot{\rho}(t) \leqslant 0 \qquad \text{and} \qquad (a(t)\,\dot{\rho}(t))^{\cdot} \geqslant 0 \qquad \text{for} \quad t \geqslant t_0. \tag{45}$$

*If condition* (5) *holds and*

$$\int^{\infty} \rho(s)\left[q(s) - \frac{p^2(s)}{4k_1 a(s)}\right] ds = \infty, \tag{46}$$

*then every solution* $x$ *of Eq.* (1) *is either oscillatory or* $x(t) \to 0$ *monotonically as* $t \to \infty$.

*Proof.* Let $x(t)$ be a nonoscillatory solution of Eq. (1) and assume that $x(t) > 0$ and $x[g(t)] > 0$ for $t \geqslant t_0$. Here, we distinguish three cases of the behavior of $\dot{x}$. The proofs of the cases when $\dot{x}$ is oscillatory and when $\dot{x}(t) < 0$ for $t \geqslant t_1$ are similar to those in Theorem 2.11. We consider only the case when $\dot{x}(t) > 0$ for $t \geqslant t_1 \geqslant t_0$.

Define the function

$$w(t) = \frac{a(t)\,\psi(x(t))\,\dot{x}(t)}{f(x(t))}\,\rho(t), \qquad t \geqslant t_1.$$

It is easy to check that

$$\begin{aligned}
\dot{w}(t) &\leqslant -\left[\rho(t)\,q(t) - \frac{p^2(t)\,\rho(t)}{4k_1 a(t)}\right] \\
&\quad - \left[\sqrt{k_1 a(t)\,\rho(t)}\,\frac{\dot{x}(t)}{f(x(t))} - \frac{p(t)\,\rho(t)}{2\sqrt{k_1 a(t)\,\rho(t)}}\right]^2 \\
&\leqslant -\left[\rho(t)\,q(t) - \frac{p^2(t)\,\rho(t)}{4k_1 a(t)}\right].
\end{aligned}$$

The rest of the proof is similar to the one in Theorem 3.1 and hence is omitted.

By using the same technique as above we have the following theorem:

THEOREM 3.6. *Let conditions* (12) *and* (37) *hold and assume that there exists a function* $\rho \in C^1[[t_0, \infty), (0, \infty)]$ *such that* $\dot{\rho}(t) \geqslant 0$ *for* $t \geqslant t_0$. *If, in addition to condition* (5),

$$\int^{\infty} \left[\rho(s)\, q(s) - \frac{(a(s)\dot{\rho} - (1/c)\, p(s)\, \rho(s))^2}{4k\rho(s)\, a(s)}\right] ds = \infty, \tag{47}$$

*then every solution* $x$ *of Eq.* (1) *is either oscillatory or* $x(t) \to 0$ *monotonically as* $t \to \infty$.

*Proof.* The proof is immediate and hence is omitted.

THEOREM 3.7. *Let conditions* (12), (37), *and* (47) *in Theorem* 3.5 *be replaced by conditions* (16), (44), *and*

$$\int^{\infty} \left[\rho(s)\, q(s) - \frac{(c_1 a(s)\, \dot{\rho}(s) - p(s)\, \rho(s))^2}{4k_1 a(s)\, \rho(s)}\right] ds = \infty, \tag{48}$$

*respectively, then the conclusion of Theorem* 3.5 *holds.*

THEOREM 3.8. *Let condition* (5) *in Theorems* 3.5–3.7 *be replaced by condition* (11), *then Eq.* (1) *is oscillatory.*

The following examples are illustrative.

EXAMPLE 7. The differential equation

$$\left(\frac{t^4}{t^2 + \ln^2 t}(1 + x^2(t))\, \dot{x}(t)\right)^{\cdot} - \frac{1}{2(\ln t - 1)}\dot{x}(t) + \frac{1}{2 \ln t} x(t) = 0, \qquad t \geqslant t_0 > e, \tag{49}$$

has a nonoscillatory solution $x(t) = (\ln t)/t \to 0$ monotonically as $t \to \infty$. We not that all the conditions of Theorem 3.5 are satisfied if we choose $\rho(t) = 1$.

EXAMPLE 8. Consider the differential equation

$$(t(1 + x^2(t))\, \dot{x}(t))^{\cdot} - \frac{1}{2(\ln t - 1)}\dot{x}(t) + \frac{1}{2 \ln t} x(t) = 0, \qquad t \geqslant t_0 > e. \tag{50}$$

The hypotheses of Theorem 3.8 are satisfied for $\rho(t) = 1$ and hence all the solutions of Eq. (50) are oscillatory. We believe that none of the criteria in [1–21] can be applied to Eq. (50).

Finally, we consider the forced equation of the form

$$(a(t)\, \psi(x(t))\, \dot{x}(t))^{\cdot} + p(t)\, \dot{x}(t) + q(t)\, f(x[g(t)]) = e(t), \tag{51}$$

where the functions $a$, $g$, $p$, $\psi$, and $f$ are as in Eq. (1), and $e, q\colon [t_0, \infty) \to R$ are continuous. In fact there is nothing known regarding the oscillatory and asymptotic behavior of Eq. (51) when $q$ is of varying sign. Therefore, the purpose of the following results is to investigate Eq. (51) when $q$ is of varying sign.

THEOREM 3.9. *In addition to the hypotheses Theorem* 2.1, *assume that*

$$\int^{\infty} \rho(s)\, |e(s)|\, dx < \infty. \tag{52}$$

*Let $x(t)$ be any solution of Eq.* (51). *Then either $\dot{x}(t)$ is oscillatory or else $x(t) \to 0$ monotonically as $t \to \infty$.*

*Proof.* Let $x(t)$ be a nonoscillatory solution of Eq. (51). Assume that $x(t) > 0$ and $x[g(t)] > 0$ for $t \geqslant t_0$. Here we need to consider two cases of behavior of $\dot{x}$.

*Case* 1. $\dot{x}(t) > 0$ for $t \geqslant t_1$, for some $t_1 \geqslant t_0$. We define the function $w$ as in the proof of Theorem 2.1 (Case 2) and get

$$\begin{aligned} \dot{w}(t) = {} & -\rho(t)\, q(t) + \rho(t) \frac{e(t)}{f(x[g(t)])} - p(t)\, \rho(t) \frac{\dot{x}(t)}{f(x[g(t)])} \\ & + a(t)\, \dot{\rho}(t) \frac{\psi(x(t))\, \dot{x}(t)}{f(x[g(t)])} \\ & - a(t)\, \rho(t)\, \dot{g}(t) \frac{\dot{x}(t)\, \dot{x}[g(t)]\, \psi(x(t))\, f'(x[g(t)])}{f^2(x[g(t)])}. \end{aligned}$$

Using the conditions of the theorem we obtain

$$\dot{w}(t) \leqslant -\rho(t)\, q(t) + \rho(t) \frac{|e(t)|}{f(x[g(t)])}, \qquad t \geqslant t_1.$$

Since $x(t)$ is an increasing function for $t \geqslant t_1$, there exist a $t_2 \geqslant t_1$ and a constant $b > 0$ such that $x[g(t)] \geqslant b$ for $t \geqslant t_2$. Thus

$$\dot{w}(t) \leqslant -\rho(t)\, q(t) + \frac{1}{f(b)} \rho(t)\, |e(t)|, \qquad t > t_2$$

or

$$w(t) \leqslant M - \int_{t_2}^{t} \rho(s)\, q(s)\, ds,$$

where $M = w(t_2) + (1/f(b)) \int_{t_2}^{\infty} \rho(s)\, |e(s)|\, ds$. The rest of the proof is similar to that of Theorem 2.1 (Case 2) and hence is omitted.

*Case* 2. $\dot{x}(t)<0$ for $t \geqslant t_1$ for some $t_1 \geqslant t_0$. Consider the function $u$ as in the proof of Theorem 2.1 (Case 3), then we get

$$\begin{aligned}\dot{u}(t) = & -\rho(t)\, q(t)\, f(x[g(t)]) + \rho(t)\, e(t) - p(t)\, \rho(t)\, \dot{x}(t) \\ & + a(t)\, \dot{\rho}(t)\, \psi(x(t))\, \dot{x}(t).\end{aligned}$$

Condition (4) implies that there exists $T > t_1$ such that

$$\int_{t_1}^{T} \rho(s)\, q(s)\, ds = 0 \qquad \text{and} \qquad \int_{T}^{t} \rho(s)\, q(s)\, ds \geqslant 0 \qquad \text{for} \quad t \geqslant T.$$

As in the proof of Theorem 2.1 (Case 3) we obtain

$$\begin{aligned}u(t) &< u(T) + \int_{T}^{t} \rho(s)\, |e(s)|\, ds - f(x[q(t)]) \int_{T}^{t} \rho(s)\, q(s)\, ds \\ &\leqslant u(T) + \int_{T}^{\infty} \rho(s)\, |e(t)|\, ds - f(b) \int_{T}^{t} \rho(s)\, q(s)\, ds,\end{aligned}$$

where $b = \lim_{t \to \infty} x(t)$. By conditions (4) and (52), there exists a $T_1 \geqslant T$ so that

$$u(t) \leqslant -\frac{f(b)}{2} \int_{T}^{t} \rho(s)\, q(s)\, ds \qquad \text{for} \quad t \geqslant T_1.$$

The rest of the proof is similar to that of Theorem 2.1 (Case 3) and hence is omitted.

THEOREM 3.10. *Assume the conditions of Theorem* 3.9 *except conditions* (3) *and* (5), *and let the function* $\rho$ *be nonincreasing on* $[t_0, \infty)$. *If* $x(t)$ *is any solution of Eq.* (52), *then either* $\dot{x}(t)$ *is oscillatory or* $|x(t)|$ *decreases monotonically to a limit as* $t \to \infty$.

*Proof.* The proof is similar to that of Theorem 3.9 (Case 1) and hence is omitted.

For illustration we consider the following examples.

EXAMPLE 9. Consider the differential equation

$$\begin{aligned}&(t\dot{x}(t))^{\cdot} + \left(\frac{1}{\sqrt{t}} + \cos t\right) |x[g(t)]|^{\alpha} \operatorname{sgn} x[g(t)] \\ &\quad = e^{-\alpha g(t)} \left(\frac{1}{\sqrt{t}} + \cos t\right) + te^{-t} - e^{-t}, \qquad t \geqslant t_0 = \pi/2, \qquad (53)\end{aligned}$$

where $g(t)$ is any nondecreasing continuous function on $[\pi/2, \infty)$, $g(t) \to \infty$ as $t \to \infty$, and $\alpha > 0$. Let $\rho(t) = 1$. Then all the conditions of Theorem 3.9 are satisfied and hence the conclusion of Theorem 3.9 holds. Equation (54) has the nonoscillatory solution $x(t) = e^{-t} \to 0$ monotonically as $t \to \infty$.

EXAMPLE 10. The differential equation

$$\begin{aligned}(t^3\dot{x}(t))^{\cdot} + 2e^t(1+\cos t)\,\dot{x}(t) + (1+\cos t)\,x[\sqrt{t}]\\ = e^{-\sqrt{t}}(1+\cos t) + t^3e^{-t} - 3t^2e^{-t}, \qquad t \geq t_0 = \pi/2, \qquad (54)\end{aligned}$$

has the nonoscillatory solution $x(t) = 2 + e^{-t} \to 2$ monotonically as $t \to \infty$. All the conditions of Theorem 3.10 are satisfied with $\rho(t) = 1$. It is easy to check that Theorem 3.9 is not applicable to Eq. (54) since conditions (3) and (5) are violated.

EXAMPLE 11. The differential equation

$$\begin{aligned}\ddot{x}(t) + \frac{1}{t}\dot{x}(t) + \left(\frac{1}{t} + \sin t\right)x(t)\\ = e^{-t}\left[2 + \frac{\cos t}{t} - (1+\cos t)^2\right], \qquad t \geq t_0 > 0, \qquad (55)\end{aligned}$$

has the oscillatory solution $x(t) = e^{-t}\sin t$. All the conditions of Theorem 3.9 are satisfied with $\rho(t) = 1$.

EXAMPLE 12. The differential equation

$$\ddot{x}(t) + \frac{1}{t^2}\dot{x}(t) + \frac{\sin t}{2 - \sin t}x(t-\pi] = \frac{\cos t}{t^2}, \qquad t \geq t_0 > 0, \qquad (56)$$

has the nonoscillatory solution $x(t) = 2 + \sin t$ and $\dot{x}(t) = \cos t$ which is oscillatory. The hypotheses of Theorem 3.9 (or Theorem 3.10) are satisfied with $\rho(t) = 1$.

We believe that none of the known criteria can describe the behavioral properties of Eqs. (53)–(56).

The following corollaries are immediate:

COROLLARY 3.11. *Let $e(t) = 0$ in Theorem 3.9 or Theorem 3.10 and let $x(t)$ be any solution of Eq. (51). Then $\dot{x}(t)$ is oscillatory.*

COROLLARY 3.12. *Let condition* (5) *in Theorem* 3.9 *be replaced by condition* (11). *Furthermore, assume that*

$$\int^{\infty} \frac{1}{a(s)\,\rho(s)} \int_{t_0}^{s} \rho(u)\,|e(u)|\,du\,ds < \infty. \tag{57}$$

*Then the conclusion of Corollary* 3.11 *holds.*

The following example is illustrative.

EXAMPLE 13. The differential equation

$$\ddot{x}(t) + \frac{\sin t}{2 - \sin t}\, x[t-\pi] = 0, \qquad t \geqslant t_0 > 0, \tag{58}$$

has a nonoscillatory solution $x(t) = 2 + \sin t$ with $\dot{x}(t) = \cos t$. All the conditions of Corollary 3.11 are satisfied for $\rho(t) = 1$.

*Remark.* It is easy to check that Theorem 3.10 can be applied to more general equations of the form

$$(a(t)\,\psi(x(t))\,\dot{x}(t))^{\cdot} + p(t)\,\gamma(\dot{x}(t))\,\dot{x}(t) + q(t)\,f(x[g(t)]) = e(t), \tag{59}$$

where $a$, $e$, $g$, $p$, $f$, and $\psi$ are as in Eq. (51) and $\gamma: R \to R$ is continuous and $\gamma(y) \geqslant 0$ for all $y$.

For illustration we consider the following example.

EXAMPLE 14. Consider the differential equation

$$\ddot{x}(t) + \frac{1}{t^2}(\cosh \dot{x}(t))\,\dot{x}(t) + \frac{\sin t}{2-\sin t}\, x[t-\pi] = \frac{\cos t \cosh(\cos t)}{t^2} \quad \text{for} \quad t \geqslant t_0 > 0. \tag{60}$$

Here $\gamma(x) = \cosh(x) > 0$ for all $x$, and hence by using the above remark we can apply Theorem 3.10 for $\rho(t) = 1$ and conclude that if $x(t)$ is any solution of Eq. (60), then either $\dot{x}(t)$ is oscillatory or $|x(t)|$ monotonically decreases to a finite limit as $t \to \infty$. Equation (60) has the nonoscillatory solution $x(t) = 2 + \sin t$, satisfying the above conclusion.

*Some Remarks.* 1. The deviating argument $g(t)$ is chosen to be either retarded or advanced and hence our results are applicable to ordinary, retarded, as well as advanced differential equations. As indicated earlier the deviating argument $g(t)$ plays an important role in the study of the behavioral properties of Eq. (1) (see examples given above).

2. Our results can be applied to Eq. (1) when the damping coefficient "$p$" is either nonnegative or nonpositive.

3. If $\psi(x) \equiv 1$ and $p(t) \equiv 0$, then Theorems 2.6–2.8 are related to Theorems 3 and 4 in [13].

4. The results of this paper extend and unify some of the results in [5–8]. In the case when the functions $a$, $p$, and $\psi$ in Eq. (1) satisfy condition (7) given in [8], the extra conditions imposed in our results here can be discarded and hence our results become similar to the corresponding ones in [5–8] as well as in [1–4] and [9–21]. We also mention that the results of this paper are quite general and can be applied to a larger class of nonlinear differential equations when some of the known criteria in [1–21] may fail to apply (see above examples).

5. In Theorems 3.9 and 3.10 we investigate the behavioral properties of Eq. (51) where the function $q(t)$ is allowed to change sign on $[t_0, \infty)$. The forcing term "$e$" need not be oscillatory as is usually required. We impose no condition on $e$ other than condition (52) and since the weight* function $\rho(t)$ is nonincreasing on $[t_0, \infty)$, the forcing term $e$ need not be small. The forcing term $e$ in Eq. (51) can either preserve or destroy the oscillatory character of Eq. (1) as is the case in the following equations:

The differential equation

$$\ddot{x}(t) + \frac{2}{t}\dot{x}(t) + x(t) = 0, \qquad t > 0 \tag{61}$$

has the oscillatory solutions $(\sin t)/t$ and $(\cos t)/t$. All the conditions of Theorem 3.3 are satisfied for $\rho(t) = 1$. Next, the differential equation

$$\ddot{x}(t) + \frac{2}{t}\dot{x}(t) + x(t) = \frac{1}{2t}, \qquad t > 0 \tag{62}$$

has the oscillatory solution $x(t) = (1 + 2\sin t)/2t$. All the conditions of Theorem 3.9 are satisfied with $\rho(t) = 1/t$. Finally, the differential equation

$$\ddot{x}(t) + \frac{2}{t}\dot{x}(t) + x(t) = \frac{1}{\sqrt{t}} - \frac{1}{4t^2\sqrt{t}}, \qquad t > 0 \tag{63}$$

has the nonoscillatory solution $x(t) = 1/\sqrt{t} \to 0$ as $t \to \infty$. The hypotheses of Theorem 3.9 are satisfied with $\rho(t) = 1/t$.

It remains an open question to the authors whether the results of this paper remain true for Eq. (1) (or Eq. (52)) when $q\colon [t_0, \infty) \to R$ is a continuous function and is of varying signs on $[t_0, \infty)$.

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