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Multiplicative Properties of Dual Canonical Bases of Quantum Groups

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1. INTRODUCTION

Let $C = (a_{i,j})_{i,j \in I}$ be a Cartan matrix of finite simply laced type, let $\mathscr{U}$ be the quantized enveloping algebra over $\mathbf{Q}(v)$ associated with C, and let $\mathscr{U}^+$ be its positive part. Let $\mathscr{B}$ be the canonical basis of $\mathscr{U}^+$, and let $\mathscr{B}^*$ be the dual of $\mathscr{B}$ with respect to the standard inner product of $\mathscr{U}^+$ (see Section 2 for precise definitions).

In [BZ], Berenstein and Zelevinsky studied multiplicative properties of $\mathscr{B}^*$ for type A, centering around the following notions: two elements of $\mathscr{B}^*$ are said to quasi-commute if they commute up to a twist (i.e., a power of v) and to be multiplicative if their product equals an element of $\mathscr{B}^*$ up to a twist. Berenstein and Zelevinsky conjectured that two elements of $\mathscr{B}^*$ quasi-commute if and only if they are multiplicative. The validity of this conjecture implies the existence of a subset $\mathscr{P}$ of $\mathscr{B}^*$ such that each element of $\mathscr{B}^*$ equals a quasi-commuting product of elements of $\mathscr{P}$ up to a twist.

This conjecture was verified in [BZ] for types A_2 and A_3; furthermore, in these cases $\mathscr{P}$ can be chosen as the set of so-called quantum minors (see Section 4).

The aim of this paper is to develop some further tools for the study of $\mathscr{B}^*$ (Section 4) and to prove parts of the conjecture:

We prove that two quasi-commuting elements of $\mathscr{B}^*$ are multiplicative provided one of the elements is a "small" quantum minor (Theorem 5.4).

Concerning the existence of a subset $\mathscr{P}$ as above, we show that "a lot of" elements of $\mathscr{B}^*$ (i.e., subjected to some linear inequalities if parame-

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terized by functions on the positive root system) can be written as a quasi-commuting product of quantum minors (Theorem 6.1).

But in contrast to the cases considered in [BZ], one cannot expect to get all elements of $\mathscr{B}^*$ as products of quantum minors for general A_n; we give an example for type A_4 (Section 7).

Our main working tool is Ringel's Hall algebra approach (see Section 3). In particular, we use the connection between $\mathscr{B}$ and degenerations of representations of quivers already used in the construction of $\mathscr{B}$ in [L1] and the description of the colored graph structure of $\mathscr{B}$ in terms of representations of quivers given in [R].

These methods provide a partial explanation for why, for some questions, $\mathscr{B}^*$ is more accessible than $\mathscr{B}$ (see the remark following Proposition 4.2).

The results of this work show a main advantage of the Hall algebra approach as pointed out by Ringel: the parameterization of elements of $\mathscr{B}^*$ by representations of quivers gives them some extra structure, which makes it possible to control the combinatorics involved.

2. RECOLLECTIONS FROM [BZ]

In this section we first recall some general facts concerning $\mathscr{U}^+$ (see [L]), thereby fixing some notations. Then we define the dual canonical basis and collect some results from [BZ].

We denote by E_i for $i \in I$ the generators of $\mathscr{U}^+$ subjected to the quantized Serre relations. Then $\mathscr{U}^+$ is $\mathbf{N}^I$-graded by setting the degree of E_i equal to e_i; for homogeneous elements x of $\mathscr{U}^+$ we denote the degree of x by $|x|$. The Cartan matrix C defines an inner product $(d, e) \mapsto d \cdot e$ on $\mathbf{N}^I$.

Let $\Delta: \mathscr{U}^+ \to \mathscr{U}^+ \otimes_{\mathbf{Q}(v)} \mathscr{U}^+$ be the $\mathbf{Q}(v)$-algebra map given by $\Delta(E_i) = E_i \otimes 1 + 1 \otimes E_i$ for all i, where the algebra structure on $\mathscr{U}^+ \otimes \mathscr{U}^+$ is defined by $(x \otimes x') \cdot (y \otimes y') = v^{|x'| \cdot |y|}(xy) \otimes (x'y')$ on homogeneous elements. Then there exists a unique $\mathbf{Q}(v)$-bilinear form (,) on $\mathscr{U}^+$ satisfying $(E_i, E_j) = \delta_{i,j}(1 - v^{-2})^{-1}$ and $(x, yy') = (\Delta(x), y \otimes y')$, where (,) is extended to the tensor product by the rule $(x \otimes x', y \otimes y') = (x, y) \cdot (x', y')$.

For all $i \in I$, there exists an operator L_i on $\mathscr{U}^+$ adjoint to left multiplication by E_i with respect to (,) (see [L], Sect. 1.2.13); for $r \geq 0$ we set $L_i^{(r)} = ([r]!)^{-1} L_i^r$. The operator L_i is locally nilpotent, so for all $x \in \mathscr{U}^+$, we can define $l_i(x)$ as the maximal l such that $L_i^{(l)}(x) \neq 0$. Since L_i fulfills a quantized Leibniz rule, $L_i^{(l)}(xy)$ is a sum of expressions of the form $v^D L_i^{(t)}(x) L_i^{(l-t)}(y)$ for $t = 0 \ldots l$ and $D \in \mathbf{Z}$ on homogeneous elements. Thus the function l_i is additive on products of homogeneous elements.

Let $\mathcal{B}$ be Lusztig's canonical basis of $\mathcal{U}^+$ and let $\tilde{E}_i, \tilde{F}_i$: $\mathcal{U}^+ \to \mathcal{U}^+$ be Kashiwara's operators (see [L]). For $b \in \mathcal{B}$, we denote by $\rho_i(b)$ the maximal r such that $b \in E_i^r \mathcal{U}^+$. It is known that $\tilde{E}_i(b)$ (resp. $\tilde{F}_i(b)$) equals some b_0 in $\mathcal{B}$ (resp. in $\mathcal{B} \cup \{0\}$) modulo $v^{-1}\mathbf{Z}[v^{-1}]\mathcal{B}$ for all $b \in \mathcal{B}$; we denote this b_0 by $\tilde{e}_i b$ (resp. $\tilde{f}_i b$). This defines operators $\tilde{e}_i, \tilde{f}_i$ on $\mathcal{B} \cup \{0\}$, which are mutually inverse provided $\tilde{f}_i b \neq 0$. The value $\rho_i(b)$ equals the maximal r such that $\tilde{f}_i^r b \neq 0$; we set $\tilde{f}_i^{\max} b = \tilde{f}_i^{\rho_i(b)} b$.

We define the dual $\mathcal{B}^*$ of Lusztig's canonical basis $\mathcal{B}$ as the set of elements b^* for $b \in \mathcal{B}$, given by $(b^*, b') = \delta_{b,b'}$. (This coincides with the definition of [BZ] up to a twist.)

By dualizing ([L], Sect. 14.3.2 and 22.1.7) it is easy to see that for all $b \in \mathcal{B}$ and $r \geq 0$ we have

1. $L_i^{(r)} b^* \in \mathbf{N}[v, v^{-1}]\mathcal{B}^*$.
2. $L_i^{(r)} b^* = 0$ for $r > \rho_i(b)$.
3. $L_i^{(r)} b^* = (\tilde{f}_i^r b)^*$ for $r = \rho_i(b)$.

As a consequence, $l_i(b^*)$ equals $\rho_i(b)$ for all $b \in \mathcal{B}$.

Combining these statements, we see that if $x \in \mathbf{N}[v, v^{-1}]\mathcal{B}^*$, $x = \sum_{b \in \mathcal{B}} c_b b^*$, then $c_b \neq 0$ implies $\rho_i(b) \leq l_i(x)$ for all $b \in \mathcal{B}$. Dualizing the positivity property of $\mathcal{B}$ with respect to comultiplication ([L], Sect. 14.4.13), it follows that for $b_1, b_2 \in \mathcal{B}$ we have $b_1^* b_2^* \in \mathbf{N}[v, v^{-1}]\mathcal{B}^*$.

Note that in particular we can apply the previous statements to get

LEMMA 2.1. *For $b_1 \dots b_k \in \mathcal{B}$ write $b_1^* \dots b_k^* = \sum_{b \in \mathcal{B}} \gamma_{b_1 \dots b_k}^b b^*$. Then $\gamma_{b_1 \dots b_k}^b \neq 0$ implies $\rho_i(b) \leq \rho_i(b_1) + \cdots + \rho_i(b_k)$.*

We finish the recollection from [BZ] by defining the notions of multiplicative properties of $\mathcal{B}^*$ we are interested in:

DEFINITION 2.2. Two elements b_1^* and b_2^* of $\mathcal{B}^*$ are called

1. quasicommuting if $b_2^* b_1^* = v^D b_1^* b_2^*$ for some $D \in \mathbf{Z}$
2. multiplicative if $b_1^* b_2^* = v^D b^*$ for some $b^* \in \mathcal{B}^*$ and $D \in \mathbf{Z}$.

3. HALL ALGEBRAS

In this section we briefly recall some basic facts on representations of quivers (see, e.g., [Ri2]) and the results of Ringel and Green on Hall algebras (see, e.g., [Ri1], [Ri3]).

Let Q be a quiver with an underlying unoriented graph of type C, and let k be an algebraically closed field; we consider the category mod kQ of k-representations of Q. For M, N in mod kQ, we denote by $\langle M, N \rangle$ (resp. $\langle M, N \rangle^1$) the k-dimension of $Hom(M, N)$ (resp. $Ext^1(M, N)$). The function $R(M, N) := \langle M, N\rangle - \langle M, N\rangle^1$ only depends on the dimension types $\underline{\dim} M, \underline{\dim} N$ of M and N.

Since all representations M can already be defined over $\mathbf{Z}$ (and thus in particular over finite fields $\mathbf{F}_q$), we can define the following functions on square roots of prime powers:

$a_M = a_M(v^2)$ equals the number of automorphisms of M as a representation over $\mathbf{F}_{v^2}$;

$F^X_{M,N} = F^X_{M,N}(v^2)$ equals the number of submodules of X isomorphic to N with quotient isomorphic to M (as representations over $\mathbf{F}_{v^2}$).

In fact, these define polynomials in $\mathbf{Z}[v^2]$ (the latter is called the Hall polynomial).

Ringel's Hall algebra approach (see, e.g., [Ri1]) provides $\mathcal{U}^+$ with bases $B_Q = \{E_{[M]}: [M]$ an isoclass in $kQ\}$ such that the structure constants $c^X_{M,N}$ of $\mathcal{U}^+$ with respect to B_Q defined by $E_{[M]}E_{[N]} = \Sigma_{[X]} c^X_{M,N} E_{[X]}$ are given as

$$c^X_{M,N} = v^{\langle M,M\rangle + \langle N,N\rangle + R(M,N) - \langle X,X\rangle} F^X_{M,N}(v^2).$$

Green's description of the (twisted) bialgebra structure of $\mathcal{U}^+$ (see [Ri3]) shows that the bases B_Q are orthogonal:

$$\left(E_{[M]}, E_{[N]}\right) = \delta_{[M],[N]} \underbrace{v^{2\langle M,M\rangle} a_M(v^2)^{-1}}_{=: b_M^{-1}}.$$

Lusztig's construction of $\mathcal{B}$ in [L1] gives a parameterization of $\mathcal{B}$ (and thus of $\mathcal{B}^*$) by kQ-representations; we write $\mathcal{B} = \{\mathcal{E}_{[M]}: [M]$ an isoclass in $kQ\}$.

In the following, we will frequently use the degeneration order on the set of isomorphism classes of representations of a fixed dimension type: $M \leq N$ if and only if the orbit of N in the variety of all representations of a fixed dimension is contained in the closure of the orbit of M (see, e.g., [B] for a precise definition and a representation-theoretic characterization of this order).

4. GENERAL FACTS ON MULTIPLICATION IN $\mathcal{B}^*$

In this section, we use the orthogonality of the bases B_Q to dualize some well-known facts on $\mathcal{B}$ to statements for $\mathcal{B}^*$. These provide a frame for the more detailed examinations of the following sections.

It is known from [L1] that

$$\mathcal{E}_{[M]} = E_{[M]} + \sum_{[M']:\, M < M'} \zeta^M_{M'} E_{[M']}$$

for certain coefficients $\zeta_{M'}^{M}$ fulfilling

$$\zeta_{M'}^{M} \in v^{-1}\mathbf{Z}[v^{-1}], \quad v^{\langle M',M'\rangle - \langle M,M\rangle}\zeta_{M'}^{M} \in \mathbf{Z}[v^{2}].$$

This fact dualizes to the following statement for $\mathscr{B}^*$:

LEMMA 4.1. *For any module M,*

$$\mathscr{E}_{[M]}^{*} = b_M E_{[M]} + \sum_{[M']:\, M'<M} b_{M'}\mu_{M'}^{M} E_{[M']}$$

for coefficients $\mu_{M'}^{M}$ satisfying

$$\mu_{M'}^{M} \in v^{-1}\mathbf{Z}[v^{-1}] \quad \text{and} \quad v^{\langle M,M\rangle - \langle M',M'\rangle}\mu_{M'}^{M} \in \mathbf{Z}[v^{2}].$$

Proof. Writing $\mathscr{E}_{[M]}^{*} = \sum_{[M']} b_{M'}\mu_{M'}^{M} E_{[M']}$ and evaluating the scalar product $(\mathscr{E}_{[M]}^{*}, \mathscr{E}_{[N]})$ we get

$$\delta_{[M],[N]} = \sum_{\substack{M',N':\\ N\le N'}} b_{M'}\mu_{M'}^{M}\zeta_{N'}^{N}(E_{[M']}, E_{[N']}) = \sum_{M':\, N\le M'} \mu_{M'}^{M}\zeta_{M'}^{N}.$$

Since every module degenerates to a semisimple one, we can prove the claimed properties of the coefficients μ_N^M by a downward induction on the degeneration ordering: in the case where N is $\le$ -maximal, i.e., semisimple, the above calculation simplifies to $\delta_{[M],[N]} = \mu_N^M\zeta_N^N = \mu_N^M$, so we are done.

Now suppose the properties are proved for all $\mu_{N'}^{M}$ such that $N < N'$. Then

$$\delta_{[M],[N]} = \mu_N^M + \sum_{M':\, N<M'} \mu_{M'}^{M}\zeta_{M'}^{N} = \mu_N^M + \sum_{\substack{M':\\ N<M'\le M}} \mu_{M'}^{M}\zeta_{M'}^{N}.$$

All statements follow immediately from this formula and the inductive assumption: if $N \not\le M$, then $N \not\le M'$ for all $M' \le M$, so $\zeta_{M'}^{N} = 0$ and thus $\mu_N^M = 0$. For $N = M$ we find $1 = \delta_{[M],[N]} = \mu_N^M$. For $N < M$,

$$v^{\langle M,M\rangle - \langle N,N\rangle}\mu_N^M = -\sum_{M'} \underbrace{v^{\langle M,M\rangle - \langle M',M'\rangle}\mu_{M'}^{M}}_{\in \mathbf{Z}[v^2]}\, \underbrace{v^{\langle M',M'\rangle - \langle N,N\rangle}\zeta_{M'}^{N}}_{\in \mathbf{Z}[v^2]} \in \mathbf{Z}[v^{2}].$$

∎

PROPOSITION 4.2. *For modules M, N we have*

$$\mathscr{E}^*_{[M]}\mathscr{E}^*_{[N]} = v^{\langle N, M\rangle - \langle M, N\rangle^{!}}\left(\mathscr{E}^*_{[M\oplus N]} + \sum_{[X]:\, X < M\oplus N} \tilde{\gamma}^X_{M,N}\mathscr{E}^*_{[X]}\right)$$

for coefficients $\tilde{\gamma}^X_{M,N}$ satisfying

$$v^{\langle M\oplus N, M\oplus N\rangle - \langle X, X\rangle}\tilde{\gamma}^X_{M,N} \in \mathbf{Z}[v^2].$$

Proof. First we write both sides of the expression $\mathscr{E}^*_{[M]}\mathscr{E}^*_{[N]} = \sum_{[X]} g_X \mathscr{E}^*_{[X]}$ in terms of the basis of PBW type with the aid of the previous lemma; then we compare coefficients and get for all X'

$$\sum_{\substack{M' \le M\\ N' \le N}} b_{M'} b_{N'} \mu^M_{M'} \mu^N_{N'} c^{X'}_{M', N'} = \sum_{X' \le X} b_{X'} \mu^X_{X'} g_X. \qquad (*)$$

Let X_0 be $\le$-maximal such that $g_{X_0} \neq 0$. Then the right-hand side of $(*)$ for $X' = X_0$ reduces to $b_{X_0} g_{X_0}$, so $c^{X_0}_{M', N'} \neq 0$ for some deformations M', N' of M, N, respectively. This provides us with an exact sequence,

$$0 \to N' \to X_0 \to M' \to 0.$$

But the middle term of an exact sequence always degenerates to the direct sum of the end terms, so

$$X_0 \le M' \oplus N' \le M \oplus N.$$

Next we compute the coefficient of $\mathscr{E}^*_{[M\oplus N]}$ in the product. Setting $X' = M \oplus N$ in $(*)$, a pair (M', N') contributes to the sum on the left-hand side if and only if there exists an exact sequence

$$0 \to N' \to M \oplus N \to M' \to 0.$$

Again we use the fact that the middle term degenerates to the direct sum of the end terms, so $M \oplus N \le M' \oplus N'$. But on the other hand, $M' \oplus N' \le M' \oplus N \le M \oplus N$ since M' (resp. N') is a deformation of M (resp. N). Since degeneration is a partial order, we find $M' \oplus N = M \oplus N$ and thus $M' = M$. Applying the same argument again gives $N' = N$.

So we see that the identity $(*)$ reduces to

$$b_M b_N c^{M\oplus N}_{M,N} = b_{M\oplus N} g_{M\oplus N}.$$

Using the formula for $c_{M,N}^{M\oplus N}$ of Section 3 we get

$$g_{M\oplus N} = v^{R(M,N)+\langle M,N\rangle+\langle N,M\rangle} a_M a_N a_{M\oplus N}^{-1} F_{M,N}^{M\oplus N}(v^2);$$

thus it remains to compute this Hall polynomial.

So assume that M and N are defined over a finite field of order v^2. Note that

$$F_{M,N}^{M\oplus N} = W\cdot(a_M a_N)^{-1},$$

where W counts the number of split exact sequences, i.e., it equals the cardinality of the set

$$\mathscr{W} = \{(\alpha: N \to M\oplus N,\ \beta: M\oplus N \to M):\ \alpha \text{ injective},\ \beta \text{ surjective},\ \beta\alpha = 0\}.$$

Denoting by ι (resp. ρ) the natural inclusion (resp. projection), we can define a map

$$Aut(M\oplus N) \to \mathscr{W}, \quad g \mapsto (g\iota, \rho g^{-1}).$$

This map is easily seen to be surjective with fibers of cardinality equal to the number of homomorphisms from M to N, i.e., $v^{2\langle M,N\rangle}$.

This yields the formula for the coefficient of $\mathscr{E}_{[M\oplus N]}^{*}$.

To prove the formula for the degree of the coefficients $\tilde{\gamma}_{M,N}^{X}$, we multiply both sides of $(*)$ by

$$v^{\langle M\oplus N, M\oplus N\rangle+\langle X',X'\rangle-\langle N,M\rangle+\langle M,N\rangle^1};$$

after a short calculation this can be written as the following identity for all X':

$$\sum_{\substack{M'\le M\\ N'\le N}} \left(v^{\langle M,M\rangle-\langle M',M'\rangle}\mu_{M'}^{M}\right)\left(v^{\langle N,N\rangle-\langle N',N'\rangle}\mu_{N'}^{N}\right)(a_{M'})(a_{N'})$$
$$\times\left(v^{2\langle M,N\rangle}\right)\left(F_{M',N'}^{X}(v^2)\right)$$
$$= \sum_{X'\le X}\left(v^{\langle X,X\rangle-\langle X',X'\rangle}\mu_{X'}^{X}\right)(a_{X'})$$
$$\times v^{\langle M\oplus N, M\oplus N\rangle-\langle X,X\rangle}\underbrace{v^{-\langle N,M\rangle+\langle M,N\rangle^1} g_X}_{=\tilde{\gamma}_{M,N}^{X}}$$

(all embraced terms belong to $\mathbf{Z}[v^2]$).

Proceeding by a downward induction on X' with respect to the ordering $\le$, we see that the formula is proved. ∎

Remark. A similar statement can be proved for $\mathscr{B}$: if $\mathscr{E}_{[X]}$ appears in the product $\mathscr{E}_{[M]}\mathscr{E}_{[N]}$ with a nonzero coefficient, then $G \leq X$ for G the generic extension of M by N, i.e., the uniquely determined extension with minimal dimension of its space of endomorphisms.

This gives a partial explanation for the conjecturally better multiplicative properties of $\mathscr{B}^*$ compared to $\mathscr{B}$: "good" properties of modules (for example, vanishing of spaces of homomorphisms or extensions) get lost by degenerating. But they are often preserved under deformation, which, for example, allows us to estimate the degree of the coefficients $\tilde{\gamma}_{M,N}^{X}$ in some "good" situations. The result in the next section is based on these considerations.

The **Q**-automorphism $\bar{\ }$ of $\mathscr{U}^+$ defined by $\overline{E_\iota} = E_i$ and $\bar{v} = v^{-1}$ fixes each element of $\mathscr{B}$ ([L], Sect. 14.2.3). We will now prove a dual result for $\mathscr{B}^*$.

Besides the **Q**-involution $\bar{\ }$ of $\mathscr{U}^+$, we also have the **Q**(v)-anti-involution σ of $\mathscr{U}^+$ given by $\sigma(E_\iota) = E_i$. For $d \in \mathbf{N}^I$, we write $tr\,d = \sum_{i \in I} d_\iota$.

LEMMA 4.3. *For all $b^* \in \mathscr{B}^*$, we have* $\overline{b^*} = (-v)^{tr|b|} v^{|b|\cdot|b|/2} \sigma(b^*)$.

Proof. In ([L], Sect. 1.2.10), Lusztig defines a bilinear form {,} on $\mathscr{U}^+$ by

$$\{x, y\} = \overline{(\bar{x}, \bar{y})}.$$

It satisfies the following formula ([L], Sect. 1.2.11 b)):

$$\{x, y\} = (-v)^{-tr|x|} v^{-|x|\cdot|y|/2} (x, \sigma(y)).$$

(The original formula simplifies slightly for finite simply laced types.) Using the fact that all elements of $\mathscr{B}$ are fixed under $\bar{\ }$ we can compute

$$(\overline{b^*}, \sigma(b')) = (-v)^{tr|b^*|} v^{|b^*|\cdot|b'|/2} \{\overline{b^*}, b'\} = (-v)^{tr|b^*|} v^{|b^*|\cdot|b'|/2}\, \overline{(b^*, b')}$$
$$= (-v)^{tr|b^*|} v^{|b^*|\cdot|b'|/2} \delta_{b,b'}.$$

Since $(\sigma(x), \sigma(y)) = (x, y)$ we get $(\sigma(\overline{b^*}), b') = (-v)^{tr|b^*|} v^{|b^*|\cdot|b'|/2} \delta_{b,b'}$ and thus $\sigma(\overline{b^*}) = (-v)^{tr|b|} v^{|b|\cdot|b|/2} b^*$. ∎

Now we apply this lemma to products of elements of $\mathscr{B}^*$:

PROPOSITION 4.4. *Writing* $b_1^* b_2^* = \sum_{b \in \mathscr{B}} \gamma_{b_1,b_2}^{b} b^*$, *we have*

$$\gamma_{b_2,b_1}^{b} = v^{|b_1|\cdot|b_2|} \overline{\gamma_{b_1,b_2}^{b}}.$$

Proof. Application of $\bar{\ }$ yields $\overline{b_1^*} \cdot \overline{b_2^*} = \sum_b \overline{\gamma_{b_1,b_2}^{b}}\, \overline{b^*}$, which by the above lemma is equivalent to

$$v^{-|b_1|\,|b_2|} \sigma(b_1^*) \sigma(b_2^*) = \sum_b \overline{\gamma_{b_1,b_2}^{b}} \sigma(b^*),$$

and thus by applying σ,

$$\sum_b v^{-|b_1|\cdot|b_2|}\gamma^b_{b_2,b_1}b^* = v^{-|b_1|\cdot|b_2|}b_2^*b_1^* = \sum_b \overline{\gamma^b_{b_1,b_2}}b^*.$$

Comparing coefficients, we are done. ∎

We derive

COROLLARY 4.5. *If b_1^* and b_2^* in $\mathscr{B}^*$ are multiplicative, then they quasi-commute.*

Proof. If b_1^* and b_2^* are multiplicative, we have $\gamma^b_{b_1,b_2} = v^N\delta_{b,b_0}$ for some $b_0 \in \mathscr{B}$ and some $N \in \mathbf{Z}$. But then the same holds for $\gamma^b_{b_2,b_1}$ by Proposition 4.4. ∎

Using the results of this section, we obtain some necessary conditions for the multiplicative properties of $\mathscr{B}^*$ we are interested in.

LEMMA 4.6. 1. *If $\mathscr{E}^*_{[M]}$ and $\mathscr{E}^*_{[N]}$ are multiplicative, then their product equals*

$$v^{\langle N,M\rangle - \langle M,N\rangle^1}\mathscr{E}^*_{[M\oplus N]}$$

and $\rho_i(M \oplus N) = \rho_i(M) + \rho_i(N)$ for all $i \in I$.

2. *If $\mathscr{E}^*_{[M]}$ and $\mathscr{E}^*_{[N]}$ quasi-commute, then $\tilde{\gamma}^X_{M,N} = \overline{\tilde{\gamma}^X_{M,N}}$ for all X.*

Proof. The first part of 1 is obvious by Proposition 4.2; the second part follows from the additivity of the function l_i on products of homogeneous elements and the fact that $l_i(b^*) = \rho_i(b)$.

For 2, we first compare the coefficients of $\mathscr{E}^*_{[M\oplus N]}$ in both products, using Proposition 4.2. Noting that

$$\gamma^{\mathscr{E}^*_{[X]}}_{\mathscr{E}^*_{[M]},\mathscr{E}^*_{[N]}} = v^{\langle N,M\rangle - \langle M,N\rangle^1}\tilde{\gamma}^X_{M,N}$$

and $|\mathscr{E}^*_{[M]}|\cdot|\mathscr{E}^*_{[N]}| = R(M,N) + R(N,M)$, the statement follows by a small calculation. ∎

To apply the facts on the behavior of ρ_i to products of elements of $\mathscr{B}^*$ in the Hall algebra setting, we need a description of these functions (and of the operators $\tilde{e}_i$) in terms of representations of quivers. This was done in [R]; we only need this description in the case of the quiver Q_n: $1 \to 2 \to \cdots \to n$, where it reads as follows: denote by E_{jk} for $1 \le j \le k \le n$ the unique indecomposable representation of Q_n supported by the subquiver $j \to \cdots \to k$. These exhaust all indecomposables, so an arbitrary module M can be written as $\oplus_{j\le k} E_{jk}^{m_{jk}}$. We call $(m_{jk})_{jk}$ the tuple corresponding to M (and vice versa).

For all $j \leq k$ we set

$$f_{jk} = \sum_{l=k}^{n} m_{jl} - \sum_{l=k+1}^{n} m_{j+1,l}.$$

For fixed i, let k_0 be minimal such that $f_{ik_0} = \max_k f_{ik}$ and set

$$m'_{jk} = m_{jk} + \begin{cases} 1, & j = i, \quad k = k_0, \\ -1, & j = i+1, \quad k = k_0, \\ 0, & \text{else}. \end{cases}$$

Then $\tilde{e}_\iota \mathscr{E}_{[M]} = \mathscr{E}_{[M']}$, where M' corresponds to the tuple (m'_{jk}) and $\rho_\iota(\mathscr{E}_{[M]}) = \max_k f_{ik}$.

In the following we will have to perform several simple calculations with this description; unless otherwise stated, they are left to the reader.

Next we define special elements of $\mathscr{B}^*$ called quantum minors, already mentioned in the Introduction (we use the same notation as in [BZ]): for sequences $I = (i_1 < \cdots < i_r)$, $J = (j_1 < \cdots < j_r)$ in $\{1, 2, \ldots, n+1\}$ such that $i_k \leq j_k$ for all $k = 1 \ldots r$, we define $\Delta(I, J)) = \mathscr{E}^*_{[M(I,J)]}$ where

$$M(I, J) = \oplus_{k=1}^{r} E_{i_k, j_k - 1}.$$

Actually this notion of quantum minor coincides with the definition in [BZ] up to a twist: just as the operators $\tilde{e}_\iota, \tilde{f}_\iota$ are defined by means of left multiplication with E_ι in $\mathscr{U}^+$, we can define $\tilde{e}_\iota^\vee, \tilde{f}_\iota^\vee$ by means of right multiplication. For type A_n, these operators can be described on Q_n-modules as

$$\tilde{e}_\iota^\vee(M) = \eta \tilde{e}_{n+1-\iota} \eta M$$

(same for $\tilde{f}_\iota^\vee$), where η sends $E_{\iota j}$ to $E_{n+1-j, n+1-\iota}$ and is additive with respect to direct sums (see [R]).

In [BZ] it is proved that all $\Delta(I, J)$ belong to $\mathscr{B}^*$ and that

$$\tilde{f}_\iota^\vee \Delta(I, J) = \begin{cases} \Delta(I, J \cup \{i\} \setminus \{i+1\}), & i \notin J, i+1 \in J, \\ 0, & \text{else}. \end{cases}$$

A short calculation using the description of $\tilde{e}_\iota$ above shows that the same holds if $\Delta(I, J)$ is replaced by $\mathscr{E}^*_{[M(I,J)]}$ for $M(I, J)$ as defined above. Induction on the dimension shows that these elements of $\mathscr{B}^*$ are in fact the same. ∎

5. SMALL QUANTUM MINORS

In this section we show how the general statements proved so far can be combined to prove part of the conjecture of Berenstein and Zelevinsky.

Using the definitions of the coefficients $\tilde{\gamma}_{M,N}^{X}$ and $\mu_{M'}^{M}$ we can write down a recursive formula for $\tilde{\gamma}_{M,N}^{X}$ as follows (see Proposition 4.2):

$$\tilde{\gamma}_{M,N}^{X} = \sum_{\substack{M' \leq M \\ N' \leq N}} \mu_{M'}^{M} \mu_{N'}^{N} m_{M',N'}^{X} - \sum_{X < X' \leq M \oplus N} \mu_{X}^{X'} \tilde{\gamma}_{M,N}^{X'},$$

where $m_{M',N'}^{X}$ equals

$$v^{\langle M,N \rangle^1 - \langle N,M \rangle} \frac{b_{M'} b_{N'}}{b_X} c_{M',N'}^{X}.$$

Note that as a consequence of Green's calculation of the comultiplication in $\mathscr{U}^+$ (see [Ri3]) we see that $m_{M',N'}^{X}$ belongs to $\mathbf{Z}[v, v^{-1}]$.

THEOREM 5.1. *Assume that $\mathscr{E}_{[M]}^{*}$ and $\mathscr{E}_{[N]}^{*}$ quasi-commute and that $\langle M, N \rangle^1$ equals 0. Then these two elements are multiplicative.*

Proof. Since vanishing of Ext^1 is preserved under deformations (see [B]), we have $\langle M', N' \rangle^1 = 0$ for all deformations $M' \leq M$ and $N' \leq N$. So $c_{M',N'}^{X} \neq 0$ implies the existence of an exact sequence $0 \to N' \to X \to M' \to 0$, yielding $X \simeq M' \oplus N'$. Calculating the Hall polynomial as in the proof of Proposition 4.2, we get

$$\begin{aligned} \deg c_{M',N'}^{X} &= \langle M', M' \rangle + \langle N', N' \rangle + R(M', N') \\ &\qquad - \langle M' \oplus N', M' \oplus N' \rangle + 2 \cdot \langle N', M' \rangle \\ &= \langle N', M' \rangle - \langle M', N' \rangle^1 = \langle N', M' \rangle \leq \langle N, M \rangle \\ &= \langle N, M \rangle - \langle M, N \rangle^1. \end{aligned}$$

But this means $\deg m_{M',N'}^{X} \leq 0$. Proceeding by a downward induction, using the recursive formula for $\tilde{\gamma}_{M,N}^{X}$, we get $\tilde{\gamma}_{M,N}^{X} \in v^{-1}\mathbf{Z}[v^{-1}]$, and thus $\tilde{\gamma}_{M,N}^{X} = 0$ for all $X \neq M \oplus N$, since $\overline{\tilde{\gamma}_{M,N}^{X}} = \tilde{\gamma}_{M,N}^{X}$ in the quasi-commuting case. ∎

COROLLARY 5.2. *If two elements $\mathscr{E}_{[M]}^{*}$ and $\mathscr{E}_{[N]}^{*}$ quasi-commute where M or N is projective or injective, then they are multiplicative.*

Note that this statement holds independently of the orientation of the quiver; so we can consider projective (resp. injective) modules for different orientations. This yields the following result.

DEFINITION 5.3. A quantum minor $\Delta(I, J)$ for type A_n is called small if there exist numbers $1 \leq i \leq j \leq k \leq n$ such that

$$I = (i, j+1, j+2, \dots, k), \quad J = (j+1, j+2, \dots, k+1) \text{ or}$$
$$I = (i, i+1, \dots, j), \quad J = (i+1, i+2, \dots, j, k+1).$$

THEOREM 5.4. *Assume we are in type A. If two elements b_1^* and b_2^* of $\mathscr{B}^*$ quasi-commute where one of them is a small quantum minor, then they are multiplicative.*

Proof. We only treat the first case of a small quantum minor from the definition; the second case is dual. So assume $b_1^* = \Delta(I, J)$, where

$$I = (i, j+1, j+2, \dots, k), \quad J = (j+1, j+2, \dots, k+1)$$

for $i \leq j \leq k$. Let Q be the orientation

$$1 \leftarrow \cdots \leftarrow i \rightarrow \cdots \rightarrow j \leftarrow \cdots \leftarrow k \rightarrow \cdots \rightarrow n.$$

By [Ri1], the product

$$E_k \dots E_{j+1} E_i \dots E_j$$

equals $\mathscr{E}^Q_{[M]}$, where M is the unique module of dimension vector $e_i + \cdots + e_k$ such that $\langle M, M \rangle^1 = 0$; thus M is the indecomposable injective module I_j associated with the vertex j. By the definition of Kashiwara's operators, $\mathscr{E}^Q_{[M]}$ can be written as

$$\tilde{e}_k \dots \tilde{e}_{j+1} \tilde{e}_i \dots \tilde{e}_j \mathscr{E}_{[0]}.$$

But for the orientation Q_n, this equals $\mathscr{E}^{Q_n}_{[M]}$, where $M = E_{ij} \oplus E_{j+1} \oplus \cdots \oplus E_k$. Thus $(\mathscr{E}^Q_{[I_j]})^* = (\mathscr{E}^{Q_n}_{[M]})^* = \Delta(I, J)$, and we can apply the previous corollary. ∎

6. THE "STANDARD CHAMBER"

In this section we construct "a lot of" elements of $\mathscr{B}^*$ for type A by studying the relation between the order given by the values ρ_i and the degeneration order on modules for the quiver Q_n.

THEOREM 6.1. *Let M be a representation of the quiver Q_n corresponding to a tuple $(m_{ij})_{1 \leq i \leq j \leq n}$ satisfying*

$$m_{ij} \geq m_{i-1, j-1} \qquad \textit{for all } 1 < i \leq j \leq n.$$

Denote by $B_{ij}^{(n)}$ *the module*

$$E_{i-j+n,n} \oplus \cdots \oplus E_{i+1,j+1} \oplus E_{ij}$$

for all $1 \le i \le j \le n$; *thus*

$$\mathscr{E}^*_{[B_{ij}^{(n)}]} = \Delta((i, i+1, \ldots, i-j+n), (j+1, j+2, \ldots, n+1)).$$

Then for some $D \in \mathbf{Z}$,

$$\mathscr{E}^*_{[M]} = v^D \prod_{1 \le i \le j \le n} \left(\mathscr{E}^*_{[B_{ij}^{(n)}]}\right)^{m_{ij} - m_{i-1,j-1}}$$

(*where* $m_{0,j}$ *is defined as* 0).

(We say that elements $\mathscr{E}^*_{[M]}$ for M satisfying the conditions above belong to the 'standard chamber.' This is reminiscent of the results of Berenstein and Zelevinsky for type A_2 and A_3, where $\mathscr{B}^*$ can be partitioned into chambers given as quasicommuting products of a fixed subset of the set of quantum minors.)

Proof. We proceed by an induction on the number of points n of the quiver Q_n; for $n = 1$ there is nothing to prove.

First note that indeed,

$$M = \oplus_{i \le j} \left(B_{ij}^{(n)}\right)^{m_{ij} - m_{i-1,j-1}}.$$

We have $\rho_i(B_{jk}) = \delta_{i,j-k+n}$ and $\rho_i(M) = m_{in}$, since $m_{ij} \ge m_{i-1,j-1}$ for all j, so we calculate

$$\sum_{j \le k} (m_{jk} - m_{j-1,k-1}) \cdot \rho_i\left(B_{jk}^{(n)}\right)$$
$$= \sum_{\substack{j \le k, \\ k-j=n-i}} (m_{jk} - m_{j-1,k-1}) = m_{in} = \rho_i(M).$$

Now assume some $\mathscr{E}^*_{[N]}$ appears in the above product of quantum minors with nonzero coefficient. Then by Lemma 2.1, we have for all $i \in I$,

$$\rho_i(N) \le \sum_{j \le k} (m_{jk} - m_{j-1,k-1}) \cdot \rho_i\left(B_{jk}^{(n)}\right) = \rho_i(M)$$

and by Proposition 4.2,

$$N \le \oplus_{i \le j} \left(B_{ij}^{(n)}\right)^{m_{ij} - m_{i-1,j-1}} = M.$$

To make this condition more explicit, we use the fact that if $N \leq M$, then $\langle N, U\rangle \leq \langle M, U\rangle$ for all indecomposables U (see [B]). By [Rie], this is equivalent to the existence of nonnegative integers x_U indexed by the nonprojective indecomposables U such that

$$N \oplus \bigoplus_U (U \oplus \tau U)^{x_U} = M \oplus \bigoplus_U B(U)^{x_U},$$

where the direct sum runs over all nonprojective indecomposables U, and

$$0 \to \tau U \to B(U) \to U \to 0$$

denotes the Auslander–Reiten sequence ending in U. Since for Q_n these sequences are of the form

$$0 \to E_{i+1,j+1} \to E_{i,j+1} \oplus E_{i+1,j} \to E_{ij} \to 0$$

(where $E_{i+1,i}$ is defined as 0), we see that the condition $N \leq M$ is equivalent to the existence of nonnegative integers $(x_{ij})_{1 \leq i \leq j < n}$ such that

$$n_{ij} = m_{ij} + x_{i-1,j} + x_{i,j-1} - x_{i-1,j-1} - x_{ij},$$

where N corresponds to the tuple $(n_{ij})_{ij}$ (we set $x_{ij} = 0$ for $i > j$ or $i \leq 0$ or $j \geq n$).

By the formula for ρ_i, we get for all $i = 1 \dots n$

$$m_{in} + x_{i,n-1} - x_{i-1,n-1} = n_{in} \leq \rho_i(N) \leq \rho_i(M) = m_{in},$$

and thus

$$x_{n-1,n-1} \leq \cdots \leq x_{1,n-1} \leq 0,$$

yielding $x_{i,n-1} = 0$ for $i = 1 \dots n-1$.

It follows that $\rho_i(N) = \rho_i(M)$ for all i. Now we apply the operators $\tilde{f}_1^{\max}, \dots, \tilde{f}_n^{\max}$. By an easy induction using the formulas for $\tilde{e}_i$ and ρ_i we see that the following holds for all i:

$$\tilde{f}_i^{\max} \dots \tilde{f}_1^{\max}(N) \text{ corresponds to a tuple } \left(\tilde{n}_{jk}\right)_{jk}, \text{ where}$$

$$\tilde{n}_{jk} = \begin{cases} n_{jk}, & k < n \text{ or } j = n \text{ and } j > i+1, \\ n_{1n} + \cdots + n_{in}, & k = n \text{ and } j = i+1, \\ 0, & k = n \text{ and } j \leq i. \end{cases}$$

(and analogously for M). So $\tilde{f}_n^{\max} \dots \tilde{f}_1^{\max}(N)$ (resp. applied to M) equals the module N' given by $\oplus_{1 \leq i \leq j < n} E_{ij}^{n_{ij}}$ (same for M).

This allows us to descend to the subquiver Q_{n-1} of Q_n: We set $L_i^{\max}(x) := L_i^{(l_i(x))}(x)$ and apply the operator $L_n^{(\max)} \dots L_1^{(\max)}$ to both sides of the equation,

$$\prod_{1 \le i \le j \le n} \left(\mathscr{E}^*_{[B_{ij}^{(n)}]}\right)^{m_{ij}-m_{i-1,j-1}} = \sum_{\substack{N: \\ \rho_i(N) \le \rho_i(M),\, i=1\dots n}} c_N \mathscr{E}^*_{[N]}.$$

By the properties of the operators L_i stated in Section 2, we get on the left-hand side

$$\prod_{1 \le i \le j < n} \left(\mathscr{E}^*_{[B_{ij}^{(n-1)}]}\right)^{m_{ij}-m_{i-1,j-1}}.$$

(Note that $\tilde{f}_n^{\max} \dots \tilde{f}_1^{\max} B_{ij}^{(n)} = B_{ij}^{(n-1)}$.)

But now we can apply the inductive assumption, so this expression equals $v^D \mathscr{E}_{[M']}$. On the right-hand side we get

$$\sum_{\substack{N: \\ \rho_i(N) \le \rho_i(M),\, i=1\dots n}} c_N \mathscr{E}^*_{[\tilde{f}_n^{\max} \dots \tilde{f}_1^{\max} N]}.$$

By the description of $\tilde{f}_n^{\max} \dots \tilde{f}_1^{\max} N$ above we see that the only module N that can appear with nonzero coefficient c_N is the module M, so we are done. ∎

Remark. By applying the involution η, we see that we also get the "opposite standard chamber"; i.e., products of quantum minors of the form

$$\Delta((1,2,\dots,i),(j-i+2, j-i+1,\dots,j+1)) \qquad \text{for } 1 \le i \le j \le n$$

belong to $\mathscr{B}^*$ up to a twist.

7. AN EXAMPLE

Let M be the module $E_{34} \oplus E_{13} \oplus E_2$ for Q_4. We will show that $\mathscr{E}^*_{[M]}$ cannot be written as a nontrivial product of elements of $\mathscr{B}^*$: By Lemma 4.6, modules N_1, N_2 such that $\mathscr{E}^*_{[M]} = v^D \mathscr{E}^*_{[N_1]} \mathscr{E}^*_{[N_2]}$ have to fulfill $N_1 \oplus N_2 = M$ and $\rho_i(N_1) + \rho_i(N_2) = \rho_i(M)$; the same holds for the function $\sigma_i := \rho_i \eta$ by duality (see Section 4). So there are three cases; we will find contradictions to the additivity of the values ρ_i (resp. σ_i):

- If $N_1 = E_{34}$ and $N_2 = E_{13} \oplus E_2$, then $\rho_2(N_2) = 1$, but $\rho_2(M) = 0$.
- If $N_1 = E_{13}$ and $N_2 = E_{34} \oplus E_2$, then $\sigma_4(N_2) = 1$, but $\sigma_4(M) = 0$.
- If $N_1 = E_2$ and $N_2 = E_{34} \oplus E_{13}$, then $\rho_2(N_1) = 0$, but $\rho_2(M) = 0$.

So we see that there exist elements of $\mathscr{B}^*$ that are not a product of quantum minors. As A. Berenstein and A. Zelevinsky pointed out to me, this element also appears in ([BFZ], Sect. 3.3.9).

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