g-Functions and some related spaces

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ABSTRACT

We present some conditions which guarantee that a topological space is metrizable in terms of g-functions and we characterize some generalized metric spaces in various ways which are different from those appeared in the literature.

1. Introduction

The notion of g-functions was introduced by Heath [1] to characterize developable spaces. It turned out later that g-functions are useful tools in characterizing generalized metric spaces. Moreover, we can see more clearly the interrelations between some topological spaces when they are characterized with g-function.

g-Functions are also useful tools in defining new classes of topological spaces. In [5], Hodel introduced several classes of generalized metric spaces, such as γ-spaces, wN-spaces and wγ-spaces with g-functions. These spaces were shown to play important roles in the metrizability of topological spaces. However, the metrization theorems appeared in the literature which are formulated with g-functions all take the form that a space X is metrizable if and only if there is a g-function for X satisfying condition (A) and there is another g-function for X satisfying condition (B). For example, Martin [7] proved that a Hausdorff space X is metrizable if and only if it is a quasi-Nagata, γ-space. In this paper, we shall characterize metric spaces with g-functions in a direct way.

Consider the following conditions imposed on a g-function:

(1) if \( g(n, x) \cap g(n, x_n) \neq \emptyset \) for all \( n \in \mathbb{N} \), then \( x \) is a cluster point of \( \langle x_n \rangle \);
(2) if \( \{ x, x_n \} \subset g(n, y_n) \) for all \( n \in \mathbb{N} \), then \( x \) is a cluster point of \( \langle x_n \rangle \);
(3) if \( y_n \in g(n, x) \) and \( x_n \in g(n, y_n) \) for all \( n \in \mathbb{N} \), then \( x \) is a cluster point of \( \langle x_n \rangle \).

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It is known that Nagata spaces (resp. developable spaces) can be characterized with a $g$-function satisfying condition (1) (resp. condition (2)) and a space $X$ has a $g$-function satisfying condition (3) is precisely a $\gamma$-space.

Now consider the following conditions:

(a) if $g^2(n,x) \cap g(n,x_n) \neq \emptyset$ for all $n \in \mathbb{N}$, then $x$ is a cluster point of $(x_n)$;

(b) if $g(n,x) \cap g^2(n,x_n) \neq \emptyset$ for all $n \in \mathbb{N}$, then $x$ is a cluster point of $(x_n)$;

(c) if $g(n,x) \cap g(n,x_n) \neq \emptyset$ for all $n \in \mathbb{N}$, then $x$ is a cluster point of $(x_n)$;

(d) if $[x, z_n] \subseteq g(n, y_n)$ and $x_n \in g(n, z_n)$ for all $n \in \mathbb{N}$, then $x$ is a cluster point of $(x_n)$;

(e) if $y_n \in g(n,x)$ and $x_n \in g(n,y_n)$ for all $n \in \mathbb{N}$, then $x$ is a cluster point of $(x_n)$.

In this paper we consider the following natural question: If a space $X$ has a $g$-function satisfying one of the above conditions, what space will $X$ be?

Throughout, a space means a topological space and the set of all positive integers is denoted by $\mathbb{N}$ while $(x_n)$ denotes a sequence. $\tau$ and $C(X)$ denote the topology of $X$ and the family of all compact subsets of $X$, respectively.

**Definition 1.1.** ([9]) A sequence of open covers $\{G_n\}_{n \in \mathbb{N}}$ of a space $X$ is called a strong development for the space $X$ if for every point $x \in X$ and any neighborhood $U$ of $x$ there exists a neighborhood $V$ of $x$ and $n \in \mathbb{N}$ such that $st(V, G_n) \subseteq U$.

It is easy to verify that a sequence of open covers $\{G_n\}_{n \in \mathbb{N}}$ of $X$ is a strong development for $X$ if and only if for each $x \in X$, $(st^2(x, G_n), n \in \mathbb{N})$ constitutes a base for $X$ at the point $x$. Let $k \geq 2$; then by induction on $k$, one readily shows that a sequence of open covers $\{G_n\}_{n \in \mathbb{N}}$ of $X$ is a strong development for $X$ if and only if when $x_n \in st^k(x, G_n)$ for all $n \in \mathbb{N}$, then $x$ is a cluster point of $(x_n)$.

**Lemma 1.2.** ([9]) A $T_0$ space $X$ is metrizable if and only if it has a strong development.

A $g$-function for a space $X$ is a map $g : \mathbb{N} \times X \to \tau$ such that for every $x \in X$ and $n \in \mathbb{N}$, $x \in g(n,x)$ and $g(n+1,x) \subseteq g(n,x)$.

Let $g$ be a $g$-function for $X$ and $A \subseteq X$. Define $g(n,A) = \bigcup\{g(n,y) : y \in A\}$ and $g^2(n,x) = g(n,g(n,x))$.

Consider the following conditions imposed on a $g$-function $g$:

($\gamma$) if $y_n \in g(n,x)$ and $x_n \in g(n,y_n)$ for all $n \in \mathbb{N}$, then $x$ is a cluster point of $(x_n)$;

(wN) if $g(n,x) \cap g(n,x_n) \neq \emptyset$ for all $n \in \mathbb{N}$, then $(x_n)$ has a cluster point;

(Developable) if $[x, x_n] \subseteq g(n, y_n)$ for all $n \in \mathbb{N}$, then $x$ is a cluster point of $(x_n)$;

(Nagata) if $g(n,x) \cap g(n,x_n) \neq \emptyset$ for all $n \in \mathbb{N}$, then $x$ is a cluster point of $(x_n)$;

($\sigma$) if $x \in g(n,y_n)$ and $y_n \in g(n,x_n)$ for all $n \in \mathbb{N}$, then $x$ is a cluster point of $(x_n)$;

($ss$) if $x \in g(n, x_n)$ for all $n \in \mathbb{N}$, then $x$ is a cluster point of $(x_n)$;

($ks$) if $y_n \in g(n,x_n)$ for all $n \in \mathbb{N}$ and $y_n \to x$, then $x$ is a cluster point of $(x_n)$.

A space that has a $g$-function satisfying condition ($\gamma$) (resp. (wN)) is called a $\gamma$-space [5] (resp. $wN$-space [5]). It was proved respectively in [1–4,10] that developable (resp. Nagata, $\sigma$- assume regularity), semi-stratifiable, $k$-semi-stratifiable spaces can be characterized by a function $g$ satisfying condition (developable) (resp. (Nagata), ($\sigma$), ($ss$), ($ks$)).

A function $g$ that satisfies condition ($\gamma$) is called a $\gamma$-function. The others are defined analogously.

**Remark 1.3.** In conditions ($\gamma$), (developable), (Nagata), ($\sigma$), ($ss$) and ($ks$), it is equivalent to say that $x_n \to x$ (see [4]).

**2. Metrizability conditions**

In this section, we shall present some criteria for the metrizability of a topological space in terms of $g$-functions. To begin, we need the following lemma.

**Lemma 2.1.** ([5]) Every Hausdorff developable $wN$-space is metrizable.

Let $X$ be a metric space, by letting $g(n,x) = B(x, \frac{1}{n})$ for each $x \in X$ and $n \in \mathbb{N}$, we get a $g$-function $g$ for $X$ which satisfies the conditions of the following Theorem 2.2 and Theorem 2.3. So the necessity of Theorem 2.2 and Theorem 2.3 is clear.

**Theorem 2.2.** A $T_0$ space $X$ is metrizable if and only if there is a $g$-function $g$ for $X$ such that if $[x, z_n] \subseteq g(n, y_n)$ and $z_n \in g(n, x_n)$ for all $n \in \mathbb{N}$, then $x$ is a cluster point of $(x_n)$.
Proof. Let $g$ be the function in the theorem. We show that if $x \neq y$ then $g(m(x)) \cap g(m(y)) = \emptyset$ for some $m \in \mathbb{N}$ which will then imply that $X$ is Hausdorff. Assume that $g(n(x)) \cap g(n(y)) \neq \emptyset$ for all $n \in \mathbb{N}$; then $x$ is a cluster point of $\langle x_n \rangle$ which follows that $x \in [y]$ and $y \in [x]$. This contradicts the fact that $X$ is a $T_0$ space. It is clear that $X$ is developable. Now suppose that $z_n \in g(n(x)) \cap g(n(x_n))$ for all $n \in \mathbb{N}$. It follows from $x \in g(n(x))$ that $x$ is a cluster point of $\langle x_n \rangle$. Thus $X$ is a $wN$-space. By Lemma 2.1, $X$ is metrizable. \hfill $\Box$

**Theorem 2.3.** A Hausdorff space $X$ is metrizable if and only if there is a $g$-function $g$ for $X$ such that for each $K \in C(X)$, if $K \cap g(n, y_n) \neq \emptyset$ and $x_n \in g(n, y_n)$ for all $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point in $K$.\hfill $\Box$

**Proof.** Let $g$ be the function in the theorem. Suppose that $\langle x, x_n \rangle \in g(n, y_n)$ for all $n \in \mathbb{N}$. By letting $K = \{x\}$, we see that $\langle x_n \rangle$ has a cluster point in $K$. That is, $x$ is a cluster point of $\langle x_n \rangle$. Thus $X$ is developable. Now suppose that $y_n \in g(n, x) \cap g(n, x_n)$ for all $n \in \mathbb{N}$. Since $y_n \in g(n, x)$ and $g$ is a developable function, by Remark 1.3, $y_n \rightarrow x$. Let $K = \{y_n: n \in \mathbb{N}\} \cup \{x\}$; then $K \in C(X)$ and $K \cap g(n, x_n) \neq \emptyset$ for all $n \in \mathbb{N}$. Thus $\langle x_n \rangle$ has a cluster point in $K$ which implies that $X$ is a $wN$-space. By Lemma 2.1, $X$ is metrizable. \hfill $\Box$

**Corollary 2.4.** A Hausdorff space $X$ is metrizable if and only if there is a sequence $\{U_n\}_{n \in \mathbb{N}}$ of open covers of $X$ such that for each $K \in C(X)$, if $x_n \in st(K, U_n)$ for all $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point in $K$.\hfill $\Box$

**Proof.** Necessity. Let $g$ be a function satisfying the condition of Theorem 2.3. For each $n \in \mathbb{N}$, put $U_n = g(n(x)) \subseteq X$. Suppose that $K \in C(X)$ and $x_n \in st(K, U_n)$ for all $n \in \mathbb{N}$; then there exists $y_n \in X$ such that $K \cap g(n, y_n) \neq \emptyset$ and $x_n \in g(n, y_n)$ for each $n \in \mathbb{N}$. By Theorem 2.3, $\langle x_n \rangle$ has a cluster point in $K$.\hfill $\Box$

**Proposition 2.5.** A space $X$ has a strong development if and only if there is a $g$-function $g$ for $X$ such that if $g^2(n, x) \cap g(n, x_n) \neq \emptyset$ for all $n \in \mathbb{N}$, then $x$ is a cluster point of $\langle x_n \rangle$.\hfill $\Box$

**Proof.** Let $\{G_n\}_{n \in \mathbb{N}}$ be a strong development for $X$ with $G_{n+1} \subseteq G_n$ for all $n \in \mathbb{N}$. For each $x \in X$ and $n \in \mathbb{N}$, put $g(n, x) = st(x, G_n)$; then $g$ is a $g$-function for $X$. If $g^2(n, x) \cap g(n, x_n) \neq \emptyset$ for all $n \in \mathbb{N}$, then $x_n \in st^2(x, G_n)$. Thus $x$ is a cluster point of $\langle x_n \rangle$.\hfill $\Box$

**Theorem 2.6.** A $T_0$ space $X$ is metrizable if and only if there is a $g$-function $g$ for $X$ such that if $g^2(n, x) \cap g(n, x_n) \neq \emptyset$ for all $n \in \mathbb{N}$, then $x$ is a cluster point of $\langle x_n \rangle$.\hfill $\Box$

**Proof.** Follows directly from Lemma 1.2 and Proposition 2.5. \hfill $\Box$

The referee reminded the author that there is a direct proof of the sufficiency of Theorem 2.6. We sketch it as follows.\hfill $\Box$

**Proof.** Suppose that there is a $g$-function $g$ for $X$ satisfying: if $g^2(n, x) \cap g(n, x_n) \neq \emptyset$ for all $n \in \mathbb{N}$, then $x$ is a cluster point of $\langle x_n \rangle$. Clearly $X$ is a Nagata space. We now show that $X$ is also a $\gamma$-space and therefore $X$ is metrizable (see [5]). Suppose that $x_n \in g(n, y_n)$ and $y_n \in g(n, x)$ for all $n \in \mathbb{N}$. Then $x_0 \in g^2(n, x) \cap g(n, x_n)$ for all $n \in \mathbb{N}$ and therefore $x$ is a cluster point of $\langle x_n \rangle$. \hfill $\Box$

**Lemma 2.7.** ([8]) A $T_0$ space $X$ is metrizable if and only if there is a sequence $\{F_n\}_{n \in \mathbb{N}}$ of locally finite closed covers of $X$ such that for each $x \in X$ and any open neighborhood $U$ of $x$, there is $n \in \mathbb{N}$ such that $st(x, F_n) \subseteq U$.\hfill $\Box$

**Theorem 2.8.** A $T_0$ space $X$ is metrizable if and only if there is a $g$-function $g$ for $X$ satisfying the following conditions:

1. If $y \in g(n, x)$, then $g(n, y) \subseteq g(n, x)$;
2. If $g(n, x) \cap g(n, x_n) \neq \emptyset$ for all $n \in \mathbb{N}$, then $x$ is a cluster point of $\langle x_n \rangle$.\hfill $\Box$
3. Some related spaces

In this section, we shall give characterizations of some generalized metric spaces which are different from those in the literature.

Proposition 3.1. X is a γ-space if and only if there is a g-function g for X such that for each K ∈ C(X), if y_n ∈ g(n, K) and x_n ∈ g(n,y_n) for all n ∈ N, then <x_n> has a cluster point in K.

Proof. Sufficiency is clear.

Let g be a γ-function for X and K ∈ C(X). Suppose that y_n ∈ g(n, K) and x_n ∈ g(n, y_n) for all n ∈ N; then there is z_n ∈ K such that y_n ∈ g(n, z_n). From K ∈ C(X) it follows that {z_n} has a cluster point x ∈ K. Since y_n ∈ g(n, z_n), x_n ∈ g(n, y_n) and g is a γ-function, x is a cluster point of <x_n>. □

Proposition 3.2. A Hausdorff space X is a k-semi-stratifiable if and only if there is a g-function g for X such that for each K ∈ C(X), if K ∩ g(n, y_n) ≠ ∅ and y_n ∈ g(n, x_n) for all n ∈ N, then <x_n> has a cluster point in K.

Proof. Let g be a k-semi-stratifiable function for X and K ∈ C(X). Suppose that K ∩ g(n, y_n) ≠ ∅ and y_n ∈ g(n, x_n) for all n ∈ N. Choose z_n ∈ K ∩ g(n, y_n). Since K ∈ C(X) and each point of X is a G_{δ}, there is a subsequence {z_{n_k}} of {z_n} converging to some point x ∈ K. Since z_{n_k} ∈ g(k, y_{n_k}), y_{n_k} ∈ g(k, x_{n_k}) and g is a k-semi-stratifiable function, we have that x is a cluster point of <x_n>.

Now let g be the function in the theorem and suppose that y_n ∈ g(n, x_n) for all n ∈ N and y_n → x. If x is not a cluster point of <x_n>, then there is m ∈ N such that x ∈ X \ {x_n: n ≥ m} = U. Since y_n → x, there is k ≥ m such that {y_n: n ≥ k} ⊂ U. Let K = {y_n: n ≥ k} ∪ {x}; then K ∈ C(X) and K ∩ g(n, y_n) ≠ ∅ for all n ≥ k. Since y_n ∈ g(n, x_n), it follows that <x_{n_k}> has a cluster point in K ⊂ U, a contradiction. Therefore x is a cluster point of <x_n>. □

Proposition 3.3. A T_{0} space X is a Nagata space if and only if there is a g-function g for X such that if g(n, x) ∩ g^{2}(n, x_n) ≠ ∅ for all n ∈ N, then x is a cluster point of <x_n>.

Proof. Sufficiency is clear.

To prove the necessity, let g be a Nagata function for X and suppose that g(n, x) ∩ g^{2}(n, x_n) ≠ ∅ for all n ∈ N. Choose y_n ∈ g(n, x) ∩ g^{2}(n, x_n); then y_n ∈ g(n, x) and there is z_n ∈ X such that y_n ∈ g(n, z_n) and z_n ∈ g(n, x_n). Thus g(n, x) ∩ g(n, z_n) ≠ ∅ which follows that x is a cluster point of <x_n>. There exists a subsequence {z_{n_k}} of {z_n} such that z_{n_k} ∈ g(k, x) for all k ∈ N. Since z_{n_k} ∈ g(n_k, x_{n_k}) ⊂ g(k, x_{n_k}), we have g(k, x) ∩ g(k, x_{n_k}) ≠ ∅. Hence x is a cluster point of <x_{n_k}> and so of <x_n>. □

Let X be a set, a function d : X × X → R^{+} (where R^{+} denotes the set of all nonnegative real numbers) is called a quasi-metric on X if:

(i) d(x, y) = 0 if and only if x = y;
(ii) for all x, y, z ∈ X, d(x, z) ≤ d(x, y) + d(y, z).

If d is a quasi-metric on X, let d^{-1}(x, y) = d(y, x) for all x, y ∈ X. Then d^{-1} is also a quasi-metric on X. Denote by τ_d and τ_{d^{-1}} the topologies on X generated by d and d^{-1}, respectively. If τ_d ⊂ τ_{d^{-1}}, then d is called a strong quasi-metric.

A space (X, τ) is called strongly quasi-metrizable [6] if there exists a strong quasi-metric d on X such that τ = τ_d.

Lemma 3.4. ([6]) A T_{1} space X is strongly quasi-metrizable if and only if it is a semi-stratifiable γ-space.

Theorem 3.5. A T_{1} space X is strongly quasi-metrizable if and only if there is a g-function g for X such that if {x, z_n} ⊂ g(n, y_n) and x_n ∈ g(n, z_n) for all n ∈ N, then x is a cluster point of <x_n>.
Proof. Suppose that $X$ is strongly quasi-metrizable and let $h$ be a $\gamma$-function and $l$ a semi-stratifiable function for $X$ respectively. For each $x \in X$ and $n \in \mathbb{N}$, put $g(n, x) = h(n, x) \cap l(n, x)$. Suppose that $\{x, z_n\} \subset g(n, y_n)$ and $x_n \in g(n, z_n)$ for all $n \in \mathbb{N}$. Since $x \in g(n, y_n) \subset h(n, y_n)$ and $l$ is a semi-stratifiable function, $x$ is a cluster point of $\langle y_n \rangle$. There is a subsequence $\langle y_{n_k} \rangle$ of $\langle y_n \rangle$ such that $y_{n_k} \in h(k, x)$ for all $k \in \mathbb{N}$. Since $z_{n_k} \in g(n_k, y_{n_k}) \subset h(k, y_{n_k})$ and $h$ is a $\gamma$-function, $x$ is a cluster point of $\langle z_{n_k} \rangle$. There is a subsequence $\langle z_{n_{k_j}} \rangle$ of $\langle z_{n_k} \rangle$ such that $z_{n_{k_j}} \in h(j, x)$ for all $j \in \mathbb{N}$. With a similar argument, we see that $x$ is a cluster point of $\langle x_{n_{k_j}} \rangle$ and thus of $\langle x_n \rangle$.

Now, if $X$ has a $g$-function $g$ satisfies the condition of the theorem, then it is clear that $g$ is both a semi-stratifiable function and a $\gamma$-function. Therefore $X$ is strongly quasi-metrizable by Lemma 3.4. □

It is easy to verify that if a $g$-function $g$ for $X$ satisfies conditions $\langle \gamma \rangle$ and $\langle \sigma \rangle$, then it also satisfies the following condition: if $\{x, x_n\} \subset g^2(n, y_n)$ for all $n \in \mathbb{N}$, then $x$ is a cluster point of $\langle x_n \rangle$, and vice versa. It is known that a developable space has a $\sigma$-locally finite closed network and thus has a $g$-function satisfying condition $\langle \sigma \rangle$. Since a strongly quasi-metrizable space is developable, we have the following:

Theorem 3.6. A $T_1$ space $X$ is strongly quasi-metrizable if and only if there is a $g$-function $g$ for $X$ such that if $\{x, x_n\} \subset g^2(n, y_n)$ for all $n \in \mathbb{N}$, then $x$ is a cluster point of $\langle x_n \rangle$.

Lemma 3.7. Let $X$ be a regular space and $h$ a $g$-function for $X$; then there is a $g$-function $g$ for $X$ such that $\overline{g(n, x)} \subset h(n, x)$ for each $x \in X$ and $n \in \mathbb{N}$.

Proof. Since $X$ is regular, there is an open neighborhood $U_n(x)$ of $x$ such that $\overline{U_n(x)} \subset h(n, x)$ for each $x \in X$ and $n \in \mathbb{N}$. Let $g(n, x) = \bigcap_{1 \leq n} U_l(x)$; then $g$ is a $g$-function for $X$ with $g(n, x) \subset \overline{U_n(x)} \subset h(n, x)$ for each $x \in X$ and $n \in \mathbb{N}$. □

Proposition 3.8. A $T_0$ space $X$ is a Nagata space if and only if there is a $g$-function $g$ for $X$ such that if $\overline{g(n, x)} \cap \overline{g(n, x_n)} \neq \emptyset$ for all $n \in \mathbb{N}$, then $x$ is a cluster point of $\langle x_n \rangle$.

Proof. Sufficiency is obvious.

Let $h$ be a Nagata function for $X$. Since $X$ is regular, by Lemma 3.7, there is a $g$-function $g$ for $X$ such that $\overline{g(n, x)} \subset h(n, x)$ for each $x \in X$ and $n \in \mathbb{N}$. Suppose that $\overline{g(n, x)} \cap \overline{g(n, x_n)} \neq \emptyset$ for all $n \in \mathbb{N}$; then $h(n, x) \cap h(n, x_n) \neq \emptyset$. Since $h$ is a Nagata function, $x$ is a cluster point of $\langle x_n \rangle$. □

Proposition 3.9. For a space $X$, the following are equivalent:

(a) $X$ is a regular $\gamma$-space;
(b) there is a $g$-function $g$ for $X$ such that if $y_n \in g(n, x)$ and $x_n \in g(n, y_n)$ for all $n \in \mathbb{N}$, then $x$ is a cluster point of $\langle x_n \rangle$;
(c) there is a $g$-function $g$ for $X$ such that if $x$ is a cluster point of $\langle y_n \rangle$ (or $y_n \rightarrow x$) and $x_n \in g(n, y_n)$ for all $n \in \mathbb{N}$, then $x$ is a cluster point of $\langle x_n \rangle$.

Proof. (a) $\Rightarrow$ (b) Similar to the proof of the necessity of Proposition 3.8.

(b) $\Rightarrow$ (c) Let $g$ be the function in (b) and suppose that $x$ is a cluster point of $\langle y_n \rangle$ and $x_n \in g(n, y_n)$ for all $n \in \mathbb{N}$. Then there is a subsequence $\langle y_{n_k} \rangle$ of $\langle y_n \rangle$ such that $y_{n_k} \in g(k, x) \subset \overline{g(k, x)}$ for all $k \in \mathbb{N}$. Since $x_{n_k} \in g(n_k, y_{n_k}) \subset \overline{g(k, y_{n_k})}$, $x$ is a cluster point of $\langle x_{n_k} \rangle$ and thus of $\langle x_n \rangle$.

(c) $\Rightarrow$ (a) Let $g$ be the function in (c); then it is clear that $X$ is a $\gamma$-space. Next, we show that for each $x \in X$ and any open neighborhood $U$ of $x$ there is a $\sigma$-locally finite closed network such that $\overline{g(m, x)} \subset U$ which will then imply that $X$ is regular. Assume that there exist $x \in X$ and an open neighborhood $U$ of $x$ such that $\overline{g(m, x)} \setminus U \neq \emptyset$ for all $n \in \mathbb{N}$. Choose $x_n \in \overline{g(n, x)} \setminus U$ for each $n \in \mathbb{N}$. Since $x$ is a cluster point of $\langle x \rangle$, by (c), $x$ is a cluster point of $\langle x_n \rangle$, a contradiction. □

Theorem 3.10. For a space $X$, the following are equivalent:

(1) $X$ is a regular $\gamma$-space;
(2) there is a $g$-function $g$ for $X$ such that for each $K \in \mathcal{C}(X)$ and $U \in \tau$ with $K \subset U$, there is $m \in \mathbb{N}$ such that $\overline{g(m, K)} \subset U$;
(3) there is a $g$-function $g$ for $X$ such that for each $K \in \mathcal{C}(X)$, if $x_n \in \overline{g(m, K)}$ for all $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point in $K$;
(4) there is a $g$-function $g$ for $X$ such that if $x_n \rightarrow x$, then for any open neighborhood $U$ of $x$ and each $n \in \mathbb{N}$, there is $m \geq n$ such that $g(m, x_n) \subset U$.

Proof. (1) $\Rightarrow$ (2) Let $g$ be a $\gamma$-function for $X$. Let $K \in \mathcal{C}(X)$ and $U \in \tau$ with $K \subset U$. Since $X$ is regular, there exists an open subset $V$ of $X$ such that $K \subset V \subset \overline{V} \subset U$. We shall show that there is $m \in \mathbb{N}$ such that $g(m, K) \subset V$. Assume that $g(m, K) \setminus V \neq \emptyset$ for all $n \in \mathbb{N}$ and choose $x_n \in g(n, K) \setminus V$; then there is $y_n \in K$ such that $x_n \in g(n, y_n)$ for each $n \in \mathbb{N}$. Then
\( \langle y_n \rangle \) has a cluster point \( p \in K \subset V \). Since \( g \) is a \( \gamma \)-function, \( p \) is a cluster point of \( \langle x_n \rangle \), a contradiction. Now \( g(m, K) \subset V \subset U \) as required.

(2) \( \Rightarrow \) (3) Let \( g \) be the function in (2) and suppose that \( K \in C(X) \) and \( x_n \in g(n, K) \) for all \( n \in \mathbb{N} \). Assume that \( \langle x_n \rangle \) has no cluster point in \( K \); then there is \( m \in \mathbb{N} \) such that \( K \subset X \setminus \{ x_n : n \geq m \} = U \). From (2) it follows that there is \( j \geq m \) such that \( g(j, K) \subset U \). Hence \( x_j \in g(j, K) \subset U \), a contradiction. Therefore, \( \langle x_n \rangle \) has a cluster point in \( K \).

(3) \( \Rightarrow \) (4) Let \( g \) be the function in (3) and suppose that \( x_n \rightarrow x \). Assume that there exist an open neighborhood \( U \) of \( x \) and \( m \in \mathbb{N} \) such that \( g(n, x_n) \setminus U \neq \emptyset \) for all \( n \geq m \). Since \( x_n \rightarrow x \), there is \( j \geq m \) such that \( \{ x_j : n \geq j \} \subset U \). Let \( K = \{ x_i : n \geq j \} \cup \{ x \} \); then \( K \in C(X) \) and \( K \subset U \). For each \( n \geq j \), choose \( y_n \in g(n, x_n) \setminus U \); then \( y_n \in g(n, K) \) for all \( n \geq j \).

By (3), \( \langle x_n \rangle \) has a cluster point in \( K \subset U \), a contradiction.

(4) \( \Rightarrow \) (1) Let \( g \) be the function in (4) and suppose that \( y_n \rightarrow x \) and \( x_n \in g(n, y_n) \) for all \( n \in \mathbb{N} \). Assume that \( x \) is not a cluster point of \( \langle x_n \rangle \); then there is \( m \in \mathbb{N} \) such that \( x \in X \setminus \{ x_n : n \geq m \} = U \). Since \( y_n \rightarrow x \), by (4), there is \( k \geq m \) such that \( g(k, y_k) \subset U \). From \( x_k \in g(k, y_k) \) it follows that \( x_k \in U \), a contradiction. By Proposition 3.9, \( X \) is a regular \( \gamma \)-space.

**Corollary 3.11.** For a space \( X \), the following are equivalent:

1. \( X \) is a regular \( \gamma \)-space;
2. there is a \( g \)-function \( g \) for \( X \) such that if \( x \notin F \) with \( F \) closed and \( x_n \rightarrow x \), then for each \( n \in \mathbb{N} \) there is \( m \geq n \) such that \( F \cap g(n, x_m) = \emptyset \);
3. there is a \( g \)-function \( g \) for \( X \) such that for each closed subset \( F \subset X \) and \( K \in C(X) \) with \( F \cap K = \emptyset \), there exists \( m \in \mathbb{N} \) such that \( F \cap g(m, K) = \emptyset \).

**Proof.** Follows directly from Theorem 3.10.

**Proposition 3.12.** \( X \) is a Moore space (i.e. regular developable space) if and only if there is a \( g \)-function \( g \) for \( X \) such that if \( \{ x, x_n \} \subset g(n, y_n) \) for all \( n \in \mathbb{N} \), then \( x \) is a cluster point of \( \langle x_n \rangle \).

**Proof.** Similar to the proof of Proposition 3.9.

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**References**