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Note

Number of models and satisfiability of sets of clauses

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Abstract

We present a way of calculating the number of models of propositional formulas represented by sets of clauses. The complexity of such a procedure is $O(\psi_k^n)$, where k is the length of clauses and n is the number of variables in the clauses. The value of ψ_2 is approximately 1.619, value of ψ_3 is approximately 1.840 and the value of ψ_k approaches 2 when k is large. Further we apply the theory on satisfiability problems, especially on the 3-SAT problems. The complexity of the 3-SAT problems is $O(\varphi^n)$, where n is the number of variables in the clauses. The value of φ is approximately 1.571 which is better than the results in Schiermeyer (1993) and Monien and Schiermeyer (1985).

1. Introduction

We consider propositional formulas represented by sets of clauses. A clause is a set of literals. A literal is either an atomic formula or the negation of an atomic formula. An atomic formula and the negation of it forms a complementary pair. We consider an atomic formula as a variable which can be interpreted as either *true* or *false*. The number of variables in a set is the number of distinct atomic formulas in the set. A complementary literal of an atomic formula is interpreted as the opposite of the atomic formula. A clause is interpreted as the disjunction of the literals in the clause (it is sometimes represented explicitly as a disjunction of the literals and sometimes represented as a set of literals). An empty clause is interpreted as *false*. We assume that a clause does not contain a complementary pair. A set of clauses is interpreted as a conjunction of the clauses of the set. An empty set of clauses is interpreted as *true*. Following are some conventions of writing formulas and sets of formulas:

- A, B, $\neg A$, $\neg B$ are literals.
- F, G are clauses.

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- $\Gamma, \Gamma_0, \Gamma_1$ are sets of clauses.
- We write Γ_0, Γ_1 instead of $\Gamma_0 \cup \Gamma_1$, and Γ, F or $\{\Gamma, F\}$ instead of $\Gamma \cup \{F\}$.

Definition 1.1. Let Γ be a set of clauses and A be a literal. $\Gamma|A$ is the set of clauses with the property: $F \in \Gamma|A$ iff

- $\neg A, A \notin F$ and
- $F \in \Gamma$ or $F \cup \{\neg A\} \in \Gamma$.

The order of the A_i's in $\Gamma|A_1|A_2|\cdots|A_n$ does not matter, since $\Gamma|A|B$ and $\Gamma|B|A$ are equivalent. Both of them are equivalent to the set of formulas of which F is an element if and only if:

$$\neg A, A \notin F,$$

$$\neg B, B \notin F,$$

$$F \in \Gamma, F \cup \{\neg A\} \in \Gamma, F \cup \{\neg B\} \in \Gamma \text{ or } F \cup \{\neg A, \neg B\} \in \Gamma.$$

We write $\Gamma|A_1|A_2|\cdots|A_n$ as $\Gamma|A_1 \wedge A_2 \wedge \cdots \wedge A_n$. If F is the clause $A_1 \vee A_2 \vee \cdots \vee A_n$, we use $\Gamma|\neg F$ to represent $\Gamma|\neg A_1 \wedge \neg A_2 \wedge \cdots \wedge \neg A_n$ and we use $\Gamma, \neg F$ to represent $\Gamma, \neg A_1, \neg A_2, \ldots, \neg A_n$ (where $\neg A_i$'s are unit clauses).

Theorem 1.1. *$A_1 \wedge A_2 \wedge \cdots \wedge A_n$ implies that Γ and $\Gamma|A_1 \wedge A_2 \wedge \cdots \wedge A_n$ are equivalent.*

Corollary 1.1. *$\{\Gamma|A_1 \wedge A_2 \wedge \cdots \wedge A_n, A_1, A_2, \ldots, A_n\}$ and $\{\Gamma, A_1, A_2, \ldots, A_n\}$ are equivalent.*

2. Number of models of formulas

A variable can be interpreted as either *true* or *false*. A truth-table of n variables contains 2^n interpretations and the number of models of a formula is the number of interpretations in which the formula has value *true*.

Definition 2.1. Let Γ be the set of clauses $\{F_1, F_2, \ldots, F_n\}$. Γ forms a base if Γ is inconsistent and $F_i \cup F_j$ contains a complementary pair for $1 \leqslant i < j \leqslant k$.

Note that a clause does not contain a complementary pair.

Lemma 2.1. *If Γ forms a base with k variables, there is exactly one of the clauses of Γ which is false for any interpretation which contains the k variables.*

Proof. There must be one *false* clause for any interpretation, since Γ is inconsistent. There cannot be two *false* clauses simultaneously, since they contain a complementary pair. □

Definition 2.2. Let $m(S)$ be the number of members of the set S. Let 2^n be the number of combinations of all possible interpretations of atomic formulas.

- $p_n(\{\ \}) = 2^n$.
- $p_n(\{\{\ \}\}) = 0$.
- $p_n(\Gamma) = \sum_{i=1}^{k} p_n(\Gamma \mid \neg F_i)/2^{m(F_i)}$ where $\{F_1, F_2, \ldots, F_k\}$ is any chosen set of clauses which forms a base.

Theorem 2.1. *$p_n(\Gamma)$ is equal to the number of interpretations in which Γ is evaluated to true.*

Proof. The first item means that every interpretation satisfies the empty set. The second item means that no interpretation satisfies a set consisting of the empty clause. In the following, we show that the third item holds. Let $\Gamma_0 = \{F_1, F_2, \ldots, F_k\}$ be a set which forms a base, and $\mathcal{I}_i$ be the set of interpretations which satisfy $\neg F_i$. Let $\mathcal{I}$ be the set of interpretations which satisfy Γ. $\mathcal{I} \cap \mathcal{I}_i$ is the set of interpretations which satisfy both $\neg F_i$ and Γ. We obtain $\bigcup_{i=1}^{k}(\mathcal{I} \cap \mathcal{I}_i) = \mathcal{I}$ and $(\mathcal{I} \cap \mathcal{I}_i) \cap (\mathcal{I} \cap \mathcal{I}_j) = \emptyset$ if $i \neq j$ according to lemma 2.1. Hence $p_n(\Gamma) = m(\mathcal{I}) = \sum_{i=1}^{k} m(\mathcal{I} \cap \mathcal{I}_i) = \sum_{i=1}^{k} p_n(\Gamma, \neg F_i)$. Further, we have $p_n(\Gamma \mid \neg F) = p_n(\Gamma \mid \neg F, \neg F) \cdot 2^{m(F)} = p_n(\Gamma, \neg F) \cdot 2^{m(F)}$. Hence $p_n(\Gamma, \neg F) = p_n(\Gamma \mid \neg F)/2^{m(F)}$ and $p_n(\Gamma) = \sum_{i=1}^{k} p_n(\Gamma \mid \neg F_i)/2^{m(F_i)}$. □

Note that $\Gamma \mid \neg F_i$ is not necessary simpler than Γ. If Γ does not contain any variable which appears in F_i, we obtain $\Gamma \mid \neg F_i = \Gamma$ and $p_n(\Gamma) = \sum_{i=1}^{k} p_n(\Gamma)/2^{m(F_i)}$. If Γ is valid, we obtain $p_n(\Gamma) = p_n(\Gamma \mid F_i) = 2^n$ and $2^n = \sum_{i=1}^{k} 2^n/2^{m(F_i)}$. Both lead to the following equation.

Corollary 2.1. *If $\{F_1, \ldots, F_k\}$ forms a base, then $\sum_{i=1}^{k} 1/2^{m(F_i)} = 1$.*

Definition 2.3. To avoid the number n in the calculation, we define p_0 as follows:

- $p_0(\{\ \}) = 1$.
- $p_0(\{\{\ \}\}) = 0$.
- $p_0(\Gamma) = \sum_{i=1}^{k} p_0(\Gamma \mid \neg F_i)/2^{m(F_i)}$ where $\{F_1, F_2, \ldots, F_k\}$ is any chosen set of clauses which forms a base.

Theorem 2.2. *If 2^n is the number of combinations of all possible interpretations of atomic formulas, then $p_0(\Gamma) = p_n(\Gamma)/2^n$.*

Proof. This theorem follows from Definitions 2.2, 2.3 and Theorem 2.1. We may say that $p_0(\Gamma)$ (or $p_0(\Gamma) \cdot 100$) is the percentage of the interpretations in which Γ is true. The advantage with this definition is that we can calculate $p_0(\Gamma)$ without mentioning the total number of interpretations. □

Corollary 2.2. *p_0 has the following properties:*

- $p_0(\Gamma \cup \{\{\ \}\}) = 0$.
- $p_0(A) = 1/2$, *if A is a literal.*

- $p_0(\Gamma, \Delta) = p_0(\Gamma) \cdot p_0(\Delta)$, *if* Γ *and* Δ *have no variables in common.*
- $p_0(\Gamma) = \sum_{i=1}^{k} p_0(\Gamma, \neg F_i)$, *if* $\{F_1, F_2, \dots, F_k\}$ *forms a base.*
- $p_0(\Gamma, F \vee G) = p_0(\Gamma, F) + p_0(\Gamma, \neg F, G)$.

Theorem 2.3. $p_0(\Gamma, A_1 \vee A_2 \vee \cdots \vee A_k) = \frac{1}{2} p_0(\Gamma | A_1) + \frac{1}{2^2} p_0(\Gamma | A_2 \wedge \neg A_1) + \frac{1}{2^3} p_0(\Gamma | A_3 \wedge \neg A_2 \wedge \neg A_1) + \cdots + \frac{1}{2^k} p_0(\Gamma | A_k \wedge \neg A_{k-1} \wedge \cdots \wedge \neg A_1)$.

Proof. Let $F_i = A_1 \vee A_2 \vee \cdots \vee A_{k-1} \vee \neg A_k$ for $i = 1, \dots, k$ and let $F_{k+1} = A_1 \vee A_2 \vee \cdots \vee A_{k-1} \vee A_k$. Since $\{F_1, \dots, F_{k+1}\}$ forms a base, we obtain

$$\begin{aligned} & p_0(\Gamma, A_1 \vee A_2 \vee \cdots \vee A_k) \\ &= \sum_{i=1}^{k+1} p_0(\{\Gamma, A_1 \vee A_2 \vee \cdots \vee A_k\} | \neg F_i)/2^{m(F_i)} \\ &= \sum_{i=1}^{k} p_0(\{\Gamma, A_1 \vee A_2 \vee \cdots \vee A_k\} | \neg F_i)/2^{m(F_i)} \\ &= \sum_{i=1}^{k} p_0(\Gamma | \neg F_i)/2^{m(F_i)} \\ &= \frac{1}{2} p_0(\Gamma | A_1) + \frac{1}{2^2} p_0(\Gamma | A_2 \wedge \neg A_1) + \frac{1}{2^3} p_0(\Gamma | A_3 \wedge \neg A_2 \wedge \neg A_1) \\ &\quad + \cdots + \frac{1}{2^k} p_0(\Gamma | A_k \wedge \neg A_{k-1} \wedge \cdots \wedge \neg A_1). \quad \square \end{aligned}$$

2.1. Complexity

In the discussion of complexity, the time used in one step in the calculation is a polynomial function of the size of the set of clauses. We first look at the cases where clauses are restricted to be either unit clauses, 2-literal clauses or 3-literal clauses. We obtain:

$$p_0(\Gamma, A) = p_0(\Gamma | A)/2.$$
$$p_0(\Gamma, A \vee B) = p_0(\Gamma | A)/2 + p_0(\Gamma | \neg A \wedge B)/4.$$
$$p_0(\Gamma, A \vee B \vee C) = p_0(\Gamma | A)/2 + p_0(\Gamma | \neg A \wedge B)/4 + p_0(\Gamma | \neg A \wedge \neg B \wedge C)/8.$$

The complexity of calculating the number of models of a set of clauses with length greater than 1 is exponential. If we use the recurrence function $f(n) = f(n-1) + f(n-2) + f(n-3)$ as a starting point, we obtain $f(n) = O(\psi_3^n)$, where n is the number of variables in the set of clauses and ψ_3 is the largest root of $1 - 2 \cdot z^3 + z^4 = 0$ and it is approximately 1.839287.

Many problems can be solved by only using $p_0(\Gamma, A) = p_0(\Gamma | A)/2$ and $p_0(\Gamma, A \vee B) = p_0(\Gamma | A)/2 + p_0(\Gamma | \neg A, B)/4$. In these cases, the complexity function $f(n)$ will be $O(\psi_2^n)$ where ψ_2 is the largest root of $1 - 2 \cdot z^2 + z^3 = 0$ and it is equal to $(1 + \sqrt{5})/2$ which is approximately 1.618034. The pigeon-hole principle [2] is among these problems.

Generally, if the upper bound of the number of literals in a clause of a set is k, we only need using the equations $p_0(\Gamma, A_1 \vee A_2 \vee \cdots \vee A_i) = \frac{1}{2}p_0(\Gamma|A_1) + \frac{1}{2^2}p_0(\Gamma|A_2 \wedge \neg A_1) + \frac{1}{2^3}p_0(\Gamma|A_3 \wedge \neg A_2 \wedge \neg A_1) + \cdots + \frac{1}{2^i}p_0(\Gamma|A_i \wedge \neg A_{i-1} \wedge \cdots \wedge \neg A_1)$ for $i = 1, 2, \ldots, k$. By solving the recurrence $f(n) = f(n-1) + \cdots + f(n-k)$, we obtain an upper bound for the number of steps needed to calculate the number of models of sets of clauses where the length of any clause is bounded by k. Since $f(n) = O(\psi_k^n)$, where ψ_k is the largest root of $1 - 2 \cdot z^k + z^{k+1} = 0$, we obtain the following theorem.

Theorem 2.4. *The number of models of a set of clauses can be calculated within* $O(\psi_k^n)$ *steps, where* n *is the number of variables and* k *is the upper bound of the number of literals in a clause.*

Some approximate values for ψ_k are as follows: $\psi_2 = 1.618034$, $\psi_3 = 1.839287$, $\psi_4 = 1.927562$, $\psi_5 = 1.965948$ and ψ_k approaches 2 as k approaches infinity.

3. Satisfiability

Satisfiability is easier to calculate than the number of models. The former is NP-complete and the latter is #P-complete [3]. For satisfiability, we can cut away many branches of the calculation by using appropriate theorems. For instance, if we know $p_0(\Gamma) = p_0(\Gamma_0)/a + p_0(\Gamma_1)/b$ and $p_0(\Gamma_0) \geqslant p_0(\Gamma_1)$, we can conclude that $p_0(\Gamma) = 0$ iff $p_0(\Gamma_0) = 0$ and hence there is no need to calculate $p_0(\Gamma_1)$. We have some inequalities:

$p_0(\Gamma) \geqslant p_0(\Gamma, F)$.

$p_0(\Gamma, F \vee G) \geqslant p_0(\Gamma, F)$.

$1 \geqslant p_0(\Gamma) \geqslant 0$.

$p_0(\Gamma) > 0$ iff Γ is satisfiable.

We use the pigeon-hole principle as an example of reasoning about unsatisfiability. The pigeon-hole principle can be understood as that there is no injective mapping from a set with $n+1$ elements to a set with n elements [2]. We use P_{ij} to represent that the ith element in the first set maps to the jth element in the second set. Let Γ_n be the set of formulas $\{ P_{i1} \vee P_{i2} \vee \cdots \vee P_{in} \mid i = 1, \ldots, n+1\}$ and Δ_n be the set of formulas $\{P_{ik} \wedge P_{jk} \mid k = 1, \ldots, n \text{ and } 1 \leqslant i < j \leqslant n+1\}$. The pigeon-hole principle can then be represented by: $\Gamma_n \rightarrow \Delta_n$. Proving this formula is the same as proving the inconsistency of Γ_n, Δ_n', where Δ_n' is the set of formulas $\{\neg P_{ik} \vee \neg P_{jk} \mid k = 1, \ldots, n$ and $1 \leqslant i < j \leqslant n+1\}$. Let us denote Γ_n, Δ_n' by Π_n.

We obtain $p_0(\Pi_n) = p_0(\Pi_n, P_{11}) + p_0(\Pi_n, \neg P_{11}, P_{12}) + \cdots + p_0(\Pi_n, \neg P_{11}, \neg P_{12}, \ldots, P_{1,n})$. From the symmetry of the variables P_{ij} in Π_n, we conclude that $p_0(\Pi_n, P_{11}) = p_0(\Pi_n, P_{12}) = \cdots = p_0(\Pi_n, P_{1,n})$. Since $p_0(\Pi_n, P_{1i}, \Pi) \leqslant p_0(\Pi_n, P_{1i})$ for any set Π, we obtain $p_0(\Pi_n) \leqslant n \cdot p_0(\Pi_n, P_{11})$. Since we also have $p_0(\Pi_n, P_{11}) \leqslant p_0(\Pi_n)$, we obtain $p_0(\Pi_n) = 0$ if and only if $p_0(\Pi_n, P_{11}) = 0$. Further we have $p_0(\Pi_n, P_{11}) = 0 \Leftrightarrow p_0(\Pi_n|P_{11}) = 0$ and $p_0(\Pi_n|P_{11}) = 0 \Leftrightarrow p_0(\Pi_n|P_{11}, \neg P_{21}, \ldots, \neg P_{n+1,1}) = 0$, since,

$\neg P_{11} \vee \neg P_{21}, \ldots, \neg P_{11} \vee \neg P_{n+1,1} \in \Pi_n$. The last equation is equivalent to $p_0(\Pi_{n-1}) = 0$. Hence after $n-1$ steps, we obtain $p_0(\Pi_n) = 0$ if and only if $p_0(\Pi_1) = 0$. The validity of $p_0(\Pi_1) = 0$ is easy to prove. It is a simple way to reason the validity of the pigeon-hole formulas. For automatic reasoning, the main problem here is to detect the structural similarity of $p_0(\Pi_n, P_{1i})$ for $i = 1, \ldots, n$. If there is no such mechanism in an automatical proof procedure, it will carry out n such proofs and the number of steps will be an exponential of n.

In the rest of this section we present theorems about the relations between unsatisfiable formulas. We need the following notations for the theorems and the following analysis.

- $Inc(\Gamma)$ means $p_0(\Gamma) = 0$ (i.e. Γ is unsatisfiable).
- $\Gamma_1 \ominus \Gamma_2$ means the set of clauses which is in Γ_1 and not in Γ_2.
- $\Gamma_1 \subset \Gamma_2$ means any clause in Γ_1 is also in Γ_2 (and it implies that $\Gamma_1 \ominus \Gamma_2$ is empty).
- $\Gamma[F/A]$ means the result of substituting A by F in Γ ($\neg A$ is not substituted by $\neg F$ in this substitution).

Theorem 3.1. $Inc(\Gamma, A) \Leftrightarrow Inc(\Gamma \mid A)$.

Proof. Since $p_0(\Gamma, A) = p_0(\Gamma \mid A)/2$ by Theorem 2.2, we obtain $p_0(\Gamma, A) = 0$ if and only if $p_0(\Gamma \mid A) = 0$. This theorem corresponds to the unit clause rule of the Davis–Putnam procedure [1]. □

Theorem 3.2. $Inc(\Gamma, F \vee G) \Leftrightarrow Inc(\Gamma, F) \wedge Inc(\Gamma, \neg F, G)$.

Proof. Since $p_0(\Gamma, F \vee G) = p_0(\Gamma, F) + p_0(\Gamma, \neg F, G)$ and $p_0(\Gamma, F), p_0(\Gamma, \neg F, G) \geqslant 0$, we obtain $p_0(\Gamma, F \vee G) = 0$ if and only if $p_0(\Gamma, F) = 0$ and $p_0(\Gamma, \neg F, G) = 0$. Note that F may be a clause of more than one literal. In the special case where F is a literal, we obtain $\Gamma, F \vee G$ is unsatisfiable if and only if Γ, F and $\Gamma, \neg F, G$ are unsatisfiable and they are unsatisfiable if and only if $\Gamma \mid F$ and $\{\Gamma, G\} \mid \neg F$ are unsatisfiable. This special case corresponds to the split rule of the Davis–Putnam procedure. □

Corollary 3.1. $Inc(\Gamma, A \vee B, \neg A \vee C) \Leftrightarrow Inc(\Gamma \mid A \wedge C) \wedge Inc(\Gamma \mid \neg A \wedge B)$.

Note that B and C could be the same literal or a complementary pair and both of them should be different from A and $\neg A$.

Corollary 3.2. $Inc(\Gamma, A \vee B, A \vee C) \Leftrightarrow Inc(\Gamma \mid A) \wedge Inc(\Gamma \mid \neg A \wedge B \wedge C)$.

Corollary 3.3. $Inc(\Gamma, A \vee B, A \vee C, A \vee D) \Leftrightarrow Inc(\Gamma \mid A) \wedge Inc(\Gamma \mid \neg A \wedge B \wedge C \wedge D)$.

Theorem 3.3. *If* $\Gamma \mid A_1 \wedge A_2 \wedge \cdots \wedge A_n \subset \Gamma$, *then* $Inc(\Gamma) \Leftrightarrow Inc(\Gamma, A_1, A_2, \ldots, A_n)$.

Proof. Since $\Gamma \mid A_1 \wedge A_2 \wedge \cdots \wedge A_n \subset \Gamma$, we obtain $p_0(\Gamma) \geqslant p_0(\Gamma, A_1, A_2, \ldots, A_n) = p_0(\Gamma \mid A_1 \wedge A_2 \wedge \cdots \wedge A_n)/2^n \geqslant p_0(\Gamma)/2^n$. Hence $p_0(\Gamma) = 0$ if and only if $p_0(\Gamma, A_1, A_2, \ldots, A_n) = 0$. A special case of this theorem is that if the negation of a literal A does not appear in Γ, then Γ is unsatisfiable if and only if the set obtained by removing clauses containing A from Γ is unsatisfiable. This special case corresponds to the pure literal rule of the Davis–Putnam procedure. □

Theorem 3.4. *If A and B do not appear in Γ ($\neg A$ and $\neg B$ may appear in Γ), then* $Inc(\Gamma, A \vee B \vee C) \Leftrightarrow Inc(\Gamma \,|\, C \wedge \neg A \wedge \neg B) \wedge Inc(\Gamma \,|\, \neg C \wedge A \wedge \neg B) \wedge Inc(\Gamma \,|\, \neg C \wedge \neg A \wedge B)$.

Proof. Since $p_0(\Gamma, A \vee B \vee C) = p_0(\Gamma \,|\, C)/2 + p_0(\Gamma \,|\, \neg C \wedge A)/4 + p_0(\Gamma \,|\, \neg C \wedge \neg A \wedge B)/8$, $p_0(\Gamma, A \vee B \vee C) = 0$ if and only if $p_0(\Gamma \,|\, C) = 0$, $p_0(\Gamma \,|\, \neg C \wedge A) = 0$ and $p_0(\Gamma \,|\, \neg C \wedge \neg A \wedge B) = 0$. Since $\Gamma \,|\, C \,|\, \neg A \wedge \neg B \subset \Gamma \,|\, C$, we obtain $p_0(\Gamma \,|\, C) = 0$ if and only if $p_0(\Gamma \,|\, C \wedge \neg A \wedge \neg B) = 0$. Since $\Gamma \,|\, \neg C \wedge A \,|\, \neg B \subset \Gamma \,|\, \neg C \wedge A$, we obtain $p_0(\Gamma \,|\, \neg C \wedge A) = 0$ if and only if $p_0(\Gamma \,|\, \neg C \wedge A \wedge \neg B) = 0$. Hence we obtain the theorem. □

Theorem 3.5. *If A does not appear in Γ, then* $Inc(\Gamma, A \vee F) \Leftrightarrow Inc(\Gamma[F/\neg A])$.

Proof. Let Γ be $\Gamma', \neg A \vee G_1, \ldots, \neg A \vee G_k$ where A and $\neg A$ do not appear in Γ'. We obtain

$$
\begin{aligned}
p_0(\Gamma, A \vee F) &= p_0(\Gamma', \neg A \vee G_1, \ldots, \neg A \vee G_k, A \vee F) \\
&= p_0(\Gamma', \neg A \vee G_1, \ldots, \neg A \vee G_k, F) + p_0(\Gamma', \neg A \vee G_1, \ldots, \neg A \vee G_k, \neg F, A) \\
&= p_0(\Gamma', \neg A \vee G_1, \ldots, \neg A \vee G_{k-1}, F, \neg A) \\
&\quad + p_0(\Gamma', \neg A \vee G_1, \ldots, \neg A \vee G_{k-1}, F, A, G_k) + p_0(\Gamma', G_1, \ldots, G_k, \neg F, A) \\
&= p_0(\Gamma', F, \neg A) + p_0(\Gamma', G_1, \ldots, G_{k-1}, F, A, G_k) + p_0(\Gamma', G_1, \ldots, G_k, \neg F, A) \\
&= p_0(\Gamma', F)/2 + p_0(\Gamma', G_1, \ldots, G_{k-1}, F, G_k)/2 + p_0(\Gamma', G_1, \ldots, G_k, \neg F)/2.
\end{aligned}
$$

$$
\begin{aligned}
p_0(\Gamma[F/\neg A]) &= p_0(\Gamma', F \vee G_1, \ldots, F \vee G_k) \\
&= p_0(\Gamma', F \vee G_1, \ldots, F \vee G_{k-1}, F) + p_0(\Gamma', F \vee G_1, \ldots, F \vee G_{k-1}, \neg F, G_k) \\
&= p_0(\Gamma', F) + p_0(\Gamma', G_1, \ldots, G_{k-1}, \neg F, G_k).
\end{aligned}
$$

Since $p_0(\Gamma', F) = 0$ implies $p_0(\Gamma', G_1, \ldots, G_{k-1}, F, G_k) = 0$, we obtain $p_0(\Gamma, A \vee F) = 0$ if and only if $p_0(\Gamma[F/\neg A]) = 0$. □

Corollary 3.4. *If A appears only in $A \vee B \vee F$ of Γ, then* $Inc(\Gamma) \Leftrightarrow Inc(\Gamma \,|\, B \wedge \neg A) \wedge Inc(\Gamma[F/\neg A] \,|\, \neg B \wedge A)$.

If F is the empty clause, we obtain $\Gamma[F/\neg A] \,|\, \neg B \wedge A = \Gamma \,|\, \neg B \wedge A$.

4. 3-SAT problems

We divide a set Γ (which may contain unit clauses, 2-literal clauses and 3-literal clauses) into 3 parts $\Gamma_0, \Gamma_1, \Gamma_2$ and Γ_3. Γ_0 is a set of one-literal clauses, Γ_1 is a set of two-literal clauses and every variable appears only once in $\Gamma_1 \cup \Gamma_0$, Γ_2 is the set of formulas containing the one-literal and two-literal clauses not in $\Gamma_1 \cup \Gamma_0$ and Γ_3 is the set of three-literal clauses. Let n be the number of variables in Γ, a be the number of clauses in Γ_0, b be the number of clauses in Γ_1, c be 0 if Γ_2 is empty, c be 1 if Γ_2 contains one clause and c be 2 if Γ_2 contains more than one clause. We assume $a + b + c > 0$. We denote the complexity of the set Γ by $f(n,a,b,c)$ where $a,b,c,n \geqslant 0$. We use the number of branches of subproofs as the measure of the complexity. In the following, we try to find the properties of this function. Some desired properties of $f(n,a,b,c)$ are:

$$a < a' \to f(n,a,b,c) > f(n,a',b,c).$$
$$b < b' \to f(n,a,b,c) > f(n,a,b',c).$$
$$c < c' \to f(n,a,b,c) > f(n,a,b,c').$$
$$n > n' \to f(n,a,b,c) > f(n',a,b,c).$$

In addition to these inequalities, we need the following inequalities in the following discussions.

1. $f(n,a,b,c) \geqslant f(n-1,a-1,b,0)$. (case 1).
2. $f(n,1,0,c) \geqslant f(n-1,0,0,1) + f(n-4,0,0,1)$. (case 1).
3. $f(n,1,0,c) \geqslant f(n-3,1,0,0)$. (case 1).
4. $f(n,0,b,c) \geqslant f(n,1,b,c-1)$. (case 2, 3).
5. $f(n,0,b,c) \geqslant f(n,1,b-1,c)$. (case 2, 3).
6. $f(n,0,b,c) \geqslant f(n-1,m-1,b-m,0)$ for $m > 1$. (case 3).
7. $f(n,0,b,c) \geqslant 2 \cdot f(n-2,0,b-2,1)$. (case 4, 5).
8. $f(n,0,b,c) \geqslant 3 \cdot f(n-3,0,b-2,1)$. (case 4).
9. $f(n,0,b,c) \geqslant 2 \cdot f(n-3,0,b-3,1) + f(n-4,0,b-3,1)$. (case 4).
10. $f(n,0,b,c) \geqslant f(n-1,0,b-1,2) + f(n-3,0,b-2,1)$. (case 5).
11. $f(n,0,b,c) \geqslant f(n,0,b+1,c-1)$. (case 5).
12. $f(n,0,b,0) \geqslant f(n-1,0,b-1,2) + f(n-2,0,b-1,1)$. (case 6).

4.1. Case analysis

In the following discussion, we first remove all subsumed 3-literal clauses in Γ for simplifying the analysis. Subsumed 2-literal clauses are not removed for technical reasons, because removing them will affect the value of b and c.

Case 1: $a > 0$. We use Theorem 3.1 to reduce the number of the variables in Γ. In case the result of the reduction does not contain unit clauses or 2-literal clauses, Theorem 3.2 is used to split the set to two sets. Two subcases:

(i) $a > 1$ or $a = 1 \wedge b > 0$. We obtain $Inc(\Gamma_0 \cup \{A\}, \Gamma_1, \Gamma_2, \Gamma_3) \Leftrightarrow Inc(\Gamma_0, \Gamma_1 | A, \Gamma_2 | A$ $\Gamma_3 | A)$ by Theorem 3.1. We restructure $\{\Gamma_0, \Gamma_1 | A, \Gamma_2 | A, \Gamma_3 | A\}$ to $\{\Gamma_0', \Gamma_1', \Gamma_2', \Gamma_3'\}$. The number of variables in $\{\Gamma_0', \Gamma_1', \Gamma_2', \Gamma_3'\}$ is n' with $n' \leqslant n-1$, the number of clauses in Γ_0' is a' with $a' \geqslant a-1$, the number of clauses in Γ_1' is b' with $b' \geqslant b$, the number of clauses in Γ_2' is c' with $c' \geqslant 0$. With $a > 1$ or $a = 1 \wedge b > 0$, we obatin $a' + b' + c' \geqslant a - 1 + b > 0$. Hence by the induction hypothesis, we can prove $Inc(\Gamma_0', \Gamma_1', \Gamma_2', \Gamma_3')$ within $f(n', a', b', c')$ branches and $f(n', a', b', c') \leqslant f(n-1, a-1, b, 0)$. Hence we can prove $Inc(\Gamma)$ within $f(n-1, a-1, b, 0)$ branches. In the following, we omit this kind of details in the proofs.

(ii) $a = 1$ and $b = 0$. Assume $A \in \Gamma_0$ and $B \vee C \vee D \in \Gamma_3$. We obtain $Inc(\Gamma) \Leftrightarrow Inc(\Gamma | A, B \vee C) \wedge Inc(\Gamma | A \wedge \neg B \wedge \neg C \wedge D)$ by Theorems 3.1 and 3.2. If $\Gamma | A \wedge \neg B \wedge \neg C \wedge D \subset \Gamma$, we obtain $Inc(\Gamma) \Leftrightarrow Inc(\Gamma, A, \neg B, \neg C, D) \Leftrightarrow Inc(\Gamma | A \wedge \neg B \wedge \neg C, D)$ by Theorem 3.3. Hence we can prove $Inc(\Gamma)$ within $f(n-3, 1, 0, 0)$ branches. Otherwise we can prove $Inc(\Gamma)$ within $f(n-1, 0, 0, 1) + f(n-4, 0, 0, 1)$ branches.

Case 2: $a = 0$ *and there is a unit clause in* Γ_2. We move one of the unit clauses from Γ_2 to Γ_0 and make sure that the variables of Γ_1 and that of Γ_0 are different by possibly move a 2-literal clause from Γ_1 to Γ_2. Hence we can prove $Inc(\Gamma)$ within either $f(n, 1, b, c-1)$ branches or within $f(n, 1, b-1, c)$ branches.

Case 3: $a = 0$, $\Gamma | B_1 \wedge B_2 \wedge \cdots \wedge B_m \subset \Gamma$ *and either* B_i, $\neg B_i$ *or both of them appear in* Γ *for* $i = 1, \ldots, m$ $(m \geqslant 1)$. We obtain $Inc(\Gamma) \Leftrightarrow Inc(\Gamma, B_1, \cdots, B_m)$ by Theorem 3.3. If $m = 1$, we can prove $Inc(\Gamma)$ within either $f(n, 1, b, c-1)$ branches or within $f(n, 1, b-1, c)$ branches. If $m > 1$, we can prove $Inc(\Gamma)$ within $f(n-1, m-1, b-m, 0)$ branches.

Case 4: $\Gamma | A \ominus \Gamma = \{F\}$ *and there is no unit clause in* Γ. F is either a unit clause or a 2-literal clause. (i) If F is a unit clause, let F be B. $\neg A$ appears only in $\neg A \vee B$. We obtain $Inc(\Gamma) \Leftrightarrow Inc(\Gamma | B \wedge A) \wedge Inc(\Gamma | \neg B \wedge \neg A)$ by Corollary 3.4. Either we can reduce it to case 3 (when Theorem 3.3 is applicable) or we can prove it within $f(n-2, 0, b-2, 1) + f(n-2, 0, b-2, 1)$ branches. (ii) If F is a 2-literal clause, let F be $B \vee C$. $\neg A$ appears only in $\neg A \vee B \vee C$ of Γ. We obtain $Inc(\Gamma) \Leftrightarrow Inc(\Gamma | B \wedge A) \wedge Inc(\Gamma[C/A] | \neg B \wedge \neg A)$ by Corollary 3.4.

(a) If $\Gamma | B \wedge A \subset \Gamma$, this case is reduced to case 3.

(b) If $\Gamma[C/A] | \neg B \wedge \neg A \subset \Gamma$, B appears only in $\neg A \vee B \vee C$. We obtain $Inc(\Gamma) \Leftrightarrow Inc(\Gamma | C \wedge A \wedge \neg B) \wedge Inc(\Gamma | \neg C \wedge \neg A \wedge \neg B) \wedge Inc(\Gamma | \neg C \wedge A \wedge B)$ by Theorem 3.4. Either we can reduce this case to case 3 or we have the following cases:

- None of B and $\neg B$ appears in Γ_1.
  We can prove $Inc(\Gamma)$ within $f(n-3, 0, b-2, 1) + f(n-3, 0, b-2, 1) + f(n-3, 0, b-2, 1)$ branches.
- $\neg B \vee D$ is in Γ_1 and D is one of $A, \neg A, C, \neg C$.
  We can prove $Inc(\Gamma)$ within $f(n-3, 0, b-2, 1) + f(n-3, 0, b-2, 1) + f(n-3, 0, b-2, 1)$ branches.
- $\neg B \vee D$ is in Γ_1 and D is different from any of $A, \neg A, C, \neg C$.

We can prove $Inc(\Gamma)$ within $f(n-3, 0, b-3, 1) + f(n-3, 0, b-3, 1) + f(n-4, 0, b-3, 1)$ branches.

(c) None of $\Gamma|B\wedge A\subset\Gamma$ and $\Gamma[C/A]|\neg B\wedge\neg A\subset\Gamma$.

There must be at least one new clause in each of $\Gamma|B\wedge A$ and $\Gamma[C/A]|\neg B\wedge\neg A$ and we can prove $Inc(\Gamma)$ within $f(n-2,0,b-2,1)+f(n-2,0,b-2,1)$ branches.

Case 5: $c\geqslant 1$ *and there is no unit clause in* Γ.

(a) $A\vee B$ in Γ_2 and $\neg A\vee C$ in Γ_1. We obtain $Inc(\Gamma)\Leftrightarrow Inc(\Gamma|A\wedge C)\wedge Inc(\Gamma|\neg A\wedge B)$ by Corollary 3.1.

We can either reduce this case to case 3 or we can prove $Inc(\Gamma)$ within $f(n-2,0,b-2,1)+f(n-2,0,b-2,1)$ branches.

(b) $A\vee B$ in Γ_2 and $A\vee C$ in Γ_1. We obtain $Inc(\Gamma)\Leftrightarrow Inc(\Gamma|A)\wedge Inc(\Gamma|\neg A\wedge B\wedge C)$ by Corollary 3.2.

We can either reduce this case to case 3, case 4 or we can prove $Inc(\Gamma)$ within $f(n-1,0,b-1,2)+f(n-3,0,b-2,1)$ branches.

(c) $A\vee B$ in Γ_2 and none of the literals $A,\neg A,B,\neg B$ appears in Γ_1. We move $A\vee B$ from Γ_2 to Γ_1 and obtain that we can prove $Inc(\Gamma)$ within $f(n,0,b+1,c-1)$ branches.

Case 6: $a=c=0$. Γ_1 must be nonempty. Assume that $A\vee B$ is in Γ_1. We obtain $Inc(\Gamma)\Leftrightarrow Inc(\Gamma|A)\wedge Inc(\Gamma|\neg A\wedge B)$ by Theorem 3.2. We can either reduce this case to case 3, case 4 or we can prove $Inc(\Gamma)$ within $f(n-1,0,b-1,2)+f(n-2,0,b-1,1)$ branches.

4.2. Complexity

To begin with, we write $f(n,a,b,c)$ as an exponential function $\varphi^{n-x\cdot a-y\cdot b-z\cdot c}$, where x,y,z are numbers between 0 and 1 (which are meant to be the weights of a,b,c) and $x\geqslant y\geqslant z$. For simplicity we set $x=y=z$. We shall find a φ that satisfies the set of inequalities listed at the beginning of this section. By replacing $f(n,a,b,c)$ with $\varphi^{n-x\cdot a-y\cdot b-z\cdot c}$, we obtain the following inequalities:

1. $\varphi^{n-x(a+b+c)}\geqslant\varphi^{n-1-x(a-1+b)}$.
2. $\varphi^{n-x(1+c)}\geqslant\varphi^{n-1-x}+\varphi^{n-4-x}$.
3. $\varphi^{n-x(1+c)}\geqslant\varphi^{n-3-x}$.
4. $\varphi^{n-x(b+c)}\geqslant\varphi^{n-x(1+b+c-1)}$.
5. $\varphi^{n-x(b+c)}\geqslant\varphi^{n-x(1+b-1+c)}$.
6. $\varphi^{n-x(b+c)}\geqslant\varphi^{n-1-x(m-1+b-m)}$ for $m>1$.
7. $\varphi^{n-x(b+c)}\geqslant 2\cdot\varphi^{n-2-x(b-2+1)}$.
8. $\varphi^{n-x(b+c)}\geqslant 3\cdot\varphi^{n-3-x(b-2+1)}$.
9. $\varphi^{n-x(b+c)}\geqslant 2\cdot\varphi^{n-3-x(b-3+1)}+\varphi^{n-4-x(b-3+1)}$.
10. $\varphi^{n-x(b+c)}\geqslant\varphi^{n-1-x(b-1+2)}+\varphi^{n-3-x(b-2+1)}$.
11. $\varphi^{n-x(b+c)}\geqslant\varphi^{n-x(b-1+c+1)}$.
12. $\varphi^{n-x\cdot b}\geqslant\varphi^{n-1-x(b-1+2)}+\varphi^{n-2-x(b-1+1)}$.

First, we remove inequalities 4, 5 and 11, since the left-hand side and the right-hand side are equal. Second, since $c\leqslant 2$ and $\varphi^{n-x\cdot t}\geqslant\varphi^{n-x\cdot s}$ if $s\geqslant t$, φ satisfies the inequalities, if φ satisfies the inequalites with c replaced by 2 in the left-hand side terms. It results in the following 9 inequalities:

1. $\varphi^{n-x(a+b+2)} \geqslant \varphi^{n-1-x(a-1+b)}$.
2. $\varphi^{n-3x} \geqslant \varphi^{n-1-x} + \varphi^{n-4-x}$.
3. $\varphi^{n-3x} \geqslant \varphi^{n-3-x}$.
4. $\varphi^{n-x(b+2)} \geqslant \varphi^{n-1-x(b-1)}$.
5. $\varphi^{n-x(b+2)} \geqslant 2 \cdot \varphi^{n-2-x(b-1)}$.
6. $\varphi^{n-x(b+2)} \geqslant 3 \cdot \varphi^{n-3-x(b-1)}$.
7. $\varphi^{n-x(b+2)} \geqslant 2 \cdot \varphi^{n-3-x(b-2)} + \varphi^{n-4-x(b-2)}$.
8. $\varphi^{n-x(b+2)} \geqslant \varphi^{n-1-x(b+1)} + \varphi^{n-3-x(b-1)}$.
9. $\varphi^{n-x\cdot b} \geqslant \varphi^{n-1-x(b+1)} + \varphi^{n-2-x\cdot b}$.

First, we remove item 3 by assuming $x < 1$ and $\varphi > 1$. Second, we remove item 4, since it is the same as item 1. Third, we assume that φ is between 1.5 and 2. By this assumption, we remove item 1 and item 6, since both are consequences of item 5. By simplifying the remaining inequalities, we obtain:

1. $1 \geqslant \varphi^{2x-1} + \varphi^{2x-4}$.
2. $1 \geqslant 2 \cdot \varphi^{3x-2}$.
3. $1 \geqslant 2 \cdot \varphi^{4x-3} + \varphi^{4x-4}$.
4. $1 \geqslant \varphi^{x-1} + \varphi^{3x-3}$.
5. $1 \geqslant \varphi^{-1-x} + \varphi^{-2}$.

Since the smaller the value of x is the smaller can φ be for the first 4 inequalities and the larger the value of x is the smaller can φ be for the 5th inequality, the last item is critical for determining an optimal x. Hence we set $\varphi^x = (\varphi - \varphi^{-1})^{-1}$ according to the 5th inequality and use this value to find the minimum value for φ according to the other inequalities. φ must satisfy:

1. $1 \geqslant (\varphi - \varphi^{-1})^{-2} \cdot (\varphi^{-1} + \varphi^{-4})$.
2. $1 \geqslant 2 \cdot (\varphi - \varphi^{-1})^{-3} \cdot \varphi^{-2}$.
3. $1 \geqslant (\varphi - \varphi^{-1})^{-4} \cdot (2 \cdot \varphi^{-3} + \varphi^{-4})$.
4. $1 \geqslant (\varphi - \varphi^{-1})^{-1} \cdot \varphi^{-1} + (\varphi - \varphi^{-1})^{-3} \cdot \varphi^{-3}$.

Let $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ be the largest root of, respectively, the following equations.

$$1 = (\varphi - \varphi^{-1})^{-2} \cdot (\varphi^{-1} + \varphi^{-4}).$$
$$1 = 2 \cdot (\varphi - \varphi^{-1})^{-3} \cdot \varphi^{-2}.$$
$$1 = (\varphi - \varphi^{-1})^{-4} \cdot (2 \cdot \varphi^{-3} + \varphi^{-4}).$$
$$1 = (\varphi - \varphi^{-1})^{-1} \cdot \varphi^{-1} + (\varphi - \varphi^{-1})^{-3} \cdot \varphi^{-3}.$$

The minimum value of φ satisfying the 4 inequalities is the maximum of the values of $\varphi_1, \varphi_2, \varphi_3$ and φ_4 (which are approximately 1.549907, 1.569804, 1.556978 and 1.570214). Since φ_4 is the largest of them and $1 = (\varphi - \varphi^{-1})^{-1} \cdot \varphi^{-1} + (\varphi - \varphi^{-1})^{-3} \cdot \varphi^{-3}$ is equivalent to $(\varphi^2 - 1)^2 \cdot (\varphi^2 - 2) = 1$, we obtain the following lemma.

Lemma 4.1. *If a set of unit clauses, 2-literal clauses and 3-literal clauses with n variables contains at least one unit clause or one 2-literal clause, satisfiability of this set of clauses can be determined within* $O(\varphi_0^n)$ *branches of subproofs where* φ_0 *is the largest root of the equation* $(\varphi^2 - 1)^2 \cdot (\varphi^2 - 2) = 1$.

Since $1.5 < \varphi_0 < 2$ and $x = \log(\varphi_0 - \varphi_0^{-1})^{-1}/\log(\varphi_0) < 1$, this lemma follows from the above case analysis. The above analysis incorporates many proof strategies. By using these strategies, we can cut away many branches of the proofs.

Theorem 4.1. *Satisfiability of any set of unit clauses, 2-literal clauses and 3-literal clauses with n variables can be determined within* $O(\varphi_0^n)$ *branches of subproofs.*

Note that the time used in the process of dividing a proof to several subproofs is a polynomial function of the size of the set of clauses. A little increase of φ_0 (which is approximately 1.570214) is enough to get rid of the polynomial factor. Hence we obtain the following corollary.

Corollary 4.1. *Satisfiability of any set of unit clauses, 2-literal clauses and 3-literal clauses with n variables can be determined within* $O(1.571^n)$ *time units, if the size of the set of clauses is bounded by a polynomial function of n.*

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