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On a conjecture of the Randić index

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ABSTRACT

The Randić index of a graph G is defined as $R(G) = \sum_{u\sim v}(d(u)d(v))^{-\frac{1}{2}}$, where $d(u)$ is the degree of vertex u and the summation goes over all pairs of adjacent vertices u, v. A conjecture on $R(G)$ for connected graph G is as follows: $R(G) \geq r(G) - 1$, where $r(G)$ denotes the radius of G. We proved that the conjecture is true for biregular graphs, connected graphs with order $n \leq 10$ and tricyclic graphs.

1. Introduction

Let $G = (V, E)$ be a simple graph, where V is vertex set, E is edge set. If $|V| = n$, we call G a graph with order n. For $v \in V(G)$, $d(v)$ (or d_v) and $N(v)$ denote the degree and the neighbor set of vertex v, respectively. The distance $d_G(u, v)$ is defined to be the number of edges in a shortest path from u to v in G. The eccentricity of a vertex x, denoted by ρ, is equal to $\max_{y\in V(G)} d_G(x, y)$. And the radius of G is defined as $r(G) = \min_{x\in V(G)} \max_{y\in V(G)} d_G(x, y)$ and a center u is a vertex for which $\max_{y\in V(G)} d_G(u, y) = r(G)$. The minimum and maximum degrees of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A bipartite graph G is called (a,b)-biregular if all vertices in one part of G have degree a and all vertices in the other part have degree b. For terminology and notation not defined here, we refer the readers to [2].

The Randić index is a graph invariant defined as

$$R = R(G) = \sum_{u\sim v} \frac{1}{\sqrt{d(u)d(v)}},$$

where $u \sim v$ denotes adjacent vertices u, v.

Recently many researches on extremal aspects of the theory of Randić index have been reported (see [6]). But some problems are still open. In [4], S. Fajtlowicz proposed the following conjecture.

Conjecture ([4]). *For all connected graphs G,*

$$R(G) \geq r(G) - 1,$$

where $r(G)$ denotes the radius of G.

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In [3], Caporossi and Hansen proved the following:

Theorem A (*[3]*).

(1) *For all trees* T, $R(T) \geq r(T) + \sqrt{2} - \frac{3}{2} \geq r(T) - 0.086$;
(2) *For all trees* T *except even paths*, $R(T) \geq r(T)$.

In [7], Bolian Liu and I. Gutman proved the following:

Theorem B (*[7]*).

(1) *Let* G *be an* (n, m) *unicyclic (bicyclic) graph* $(n \geq 3)$. *Then* $R(G) \geq r(G) - 1$.
(2) *Let* G *be a graph of order* n *with* $\delta(G) \geq 2$. *If* $n \leq 9$, *then* $R(G) \geq r(G) - 1$.

Recently in [6], X. Li and I. Gutman pointed out that " It does not seem to be easy to extend these results to general graphs."

In [7], the conjecture is proved for unicyclic and bicyclic graphs and connected graphs of order $n \leq 9$ with $\delta(G) = 2$. In this paper, we prove that the conjecture is true for biregular graphs, connected graphs with order $n \leq 10$ and tricyclic graphs.

2. Biregular graphs

Lemma 2.1. *Let* G *be a connected graph and* u *be a center of* G. *Denote* $S_i = \{v \in V(G) : d_G(u, v) = i\}$ *for* $0 \leq i \leq r$ *(*r *is the radius). Then* $|S_i| \geq 2$ *for* $1 \leq i \leq r - 1$.

Proof. If $r \leq 1$, then there is nothing to prove.

Now we consider $r \geq 2$.

Let u be a vertex with the eccentricity $\rho \geq 2$ and $S_i = \{v \in V(G) : d_G(u, v) = i\}$ for $0 \leq i \leq \rho$. Suppose that some S_i with $1 \leq i \leq \rho - 1$ has only one vertex w. There is a vertex y such that $d(u, y) = 1$ and $d(y, w) = i - 1$. Then the distances from y to the vertices in S_j $(j < i)$ are at most $j + 1$, and $j + 1 \leq i < \rho$. And the distances from y to the vertices in S_j $(j \geq i)$ are $j - 1$, owing to a path through w, and $j - 1 < \rho$. Thus the eccentricity of y is less than ρ. Then u is not a center.

Hence if u is a center, then all sets S_i $(1 \leq i \leq r - 1)$ have at least 2 vertices. ■

Theorem 2.2. *Let* G *be a connected biregular graph, then*

$$R(G) \geq r(G).$$

Proof. Let (X, Y) be the two parts of $V(G)$, $|X| = x$, $|Y| = y$. Each vertex of X has degree a and each vertex of Y has degree b.

If a or $b = 1$, then G is a star graph. By direct calculation, $r(G) = 1 \leq R(G)$.

In the following, let $\min\{a, b\} \geq 2$.

Since G is a biregular graph, then

$$\begin{cases} x + y = n \\ xa = yb. \end{cases}$$

Thus

$$\frac{a}{b} = \frac{y}{x}$$

and

$$R(G) = \sum_{u\sim v} \frac{1}{\sqrt{ab}} = \frac{1}{\sqrt{ab}} xa = \sqrt{\frac{a}{b}}x = \sqrt{\frac{y}{x}}x = \sqrt{xy}.$$

Let u be a center of G, i.e., a point for which $\max_{y\in V(G)} d_G(u, y) = r(G) = r$, and $S_i = \{v \in V(G) : d_G(u, v) = i\}$ for $0 \leq i \leq r$. By Lemma 2.1, $|S_i| \geq 2$ $(1 \leq i \leq r - 1)$.

Note that $S_i \cap S_j = \emptyset$ for $i \neq j$ and $n = \sum_{i=0}^{r} |S_i|$.

If r is even, then $S_0, S_2, \ldots, S_r \subset X$ and $S_1, S_3, \ldots, S_{r-1} \subset Y$.

Using $|S_i| \geq 2$ $(i = 1, 2, \ldots, r-1)$, we get $x = |S_0| + |S_2| + \cdots + |S_r| \geq 1 + \frac{r-2}{2} \times 2 + 1 = r$, $y = |S_1| + |S_3| + \cdots + |S_{r-1}| \geq \frac{r-1+1}{2} \times 2 = r$.

Hence $xy \geq r^2$, $R(G) = \sqrt{xy} \geq r$.

If r is odd, then $r - 1$ is even, $S_0, S_2, \ldots, S_{r-1} \subset X$ and $S_1, S_3, \ldots, S_r \subset Y$.

Thus $x = |S_0| + |S_2| + \cdots + |S_{r-1}| \geq 1 + \frac{r-1}{2} \times 2 = r$.

$y = |S_1| + |S_3| + \cdots + |S_r| \geq \frac{r-2+1}{2} \times 2 + 1 = r$.

Hence $xy \geq r^2$, $R(G) = \sqrt{xy} \geq r$. ■

Corollary 2.3. *Let* G *be a complete bipartite graph, then*

$$R(G) \geq r(G).$$

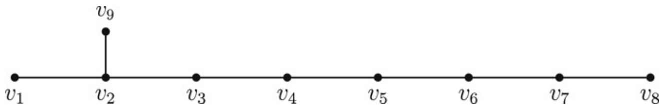

Fig. 1. A spanning tree T has 9 vertices, radius 4, and T is not P_9.

3. Graphs with order ≤ 10

Lemma 3.1 ([1]). *Let G be a graph of order n, containing no isolated vertices. Then*

$$R(G) \geq \sqrt{n-1},$$

where equality holds if and only if G is a star.

Theorem 3.2. *Let G be a connected graph of order at most 7, then*

$$R(G) \geq r(G) - 1.$$

Proof. By the proof of Lemma 2.1, we know that $r(G) - 1 \leq \lfloor \frac{n}{2} \rfloor - 1 \leq \sqrt{n-1}$ for $n \leq 7$. By Lemma 3.1, $r(G) - 1 \leq \sqrt{n-1} \leq R(G)$. ■

Theorem 3.3. *Let G be a connected graph of order $n = 8$, then*

$$R(G) \geq r(G) - 1.$$

Proof. We consider the following two cases:

Case 1. There exists a spanning tree T that is not isomorphic to P_8.
Then $r(T) \leq 3$. Hence $r(G) - 1 \leq r(T) - 1 \leq 3 - 1 = 2 \leq \sqrt{8-1} = \sqrt{7} \leq R(G)$.
Case 2. $G \cong P_8$ or C_8.
It is easy to check that $R(G) \geq r(G) - 1 = 3$ by counting.
Hence the conjecture is true for $n = 8$. ■

Theorem 3.4. *Let G be a connected graph of order $n = 9$, then*

$$R(G) \geq r(G) - 1.$$

Proof. By Theorem B (2), it is sufficient to prove that the conjecture is true for $\delta(G) = 1$.
We consider the following two cases.
Case 1. G has the only spanning tree P_9 and $\delta(G) = 1$, i.e., $G = P_9$.

$$R(P_9) = \frac{1}{2} \times 6 + \frac{2}{\sqrt{2}} = 3 + \sqrt{2} \geq 4 - 1 = 3 = r(G) - 1.$$

Case 2. There is a spanning tree T which is not P_9.
(1) If $r(T) \leq 3$, by Lemma 3.1, then $r(G) - 1 \leq r(T) - 1 \leq 3 - 1 = 2 \leq \sqrt{9-1} \leq R(G)$.
(2) If $r(T) = 4$, then the spanning tree of G is $P_8 = v_1 \cdots v_8$ with one edge added.
For example, a spanning tree is shown as in Fig. 1.
Since the vertex v_9 can be connected to at most 3 vertices of the path $v_1 \cdots v_8$ (otherwise a spanning tree with shorter radius appears), by the theorems on trees, unicyclic and bicyclic graphs ([3], [7]), the conjecture is true. ■

Theorem 3.5. *Let G be a connected graph of order $n = 10$, then*

$$R(G) \geq r(G) - 1.$$

Proof. There are two cases:
Case 1. If G has a spanning tree T which is not path P_{10}.
Then $r(G) - 1 \leq 4 - 1 = \sqrt{10-1} \leq R(G)$.
Case 2. If G has the only spanning tree P_{10}, then $G \cong P_{10}$ or C_{10}.
For C_{10}, $R(G) = \frac{1}{2} \times 10 = 5 \geq 5 - 1 \geq r(G) - 1$.
For P_{10}, $R(G) = \frac{1}{2} \times 7 + \frac{2}{\sqrt{2}} \geq 3.5 + 1.41 = 4.91 \geq 5 - 1 \geq r(G) - 1$. ■

Up to now, we have shown that the conjecture $R(G) \geq r(G) - 1$, is true for all connected graph with order $n \leq 10$.

4. Tricyclic graphs

Since in Section 3, the conjecture is true for order $n \le 10$, we may suppose that the order of graphs is at least 11 in this section.

Lemma 4.1 ([7]). *Let G be a unicyclic graph and v_1v_2 be an edge in a cycle of G with $d(v_1) = d_1$, $d(v_2) = d_2$. Then the minimum value for the difference $R(G) - R(G - v_1v_2)$ is reached when $d_1 = d_2 = \frac{n+1}{2}$.*

$$\text{Thus } R(G) - R(G - v_1v_2) \ge 2\left(\frac{n+1}{2} - 2 + \frac{1}{\sqrt{2}}\right)\left(\sqrt{\frac{2}{n+1}} - \sqrt{\frac{2}{n-1}}\right) + \frac{2}{n+1}$$
$$\ge \sqrt{2}\cdot n\cdot\left(\sqrt{\frac{1}{n+1}} - \sqrt{\frac{1}{n-1}}\right) + \frac{2}{n+1}. \quad (1)$$

The cyclomatic number of a connected graph G is defined as $c(G) = m - n + 1$. A graph G with $c(G) = k$ is called k-cycles graph. Now we investigate the conjecture for tricyclic graphs.

Lemma 4.2 ([7]). *Let x be a positive integer with $x \ge 3$. Denote $k(x) = \sqrt{2}\cdot x\cdot\left(\sqrt{\frac{1}{x+1}} - \sqrt{\frac{1}{x-1}}\right) + \frac{2}{x+1}$. Then $k(x)$ is monotonously increasing in x.*

Lemma 4.3 ([5]). *Let G be a connected k-cycles graph. There are k edges $e_1, \dots, e_k$ of cycles of G such that $G - e_1 - \cdots - e_k$ is a spanning tree of G.*

Lemma 4.4. *Let x be a positive integer with $x \ge 11$. Denote $f(x) = \sqrt{2}\cdot(x+1)\cdot\left(\sqrt{\frac{1}{x+3}} - \sqrt{\frac{1}{x+1}}\right) + \frac{2}{x+3}$. Then $f(x)$ is monotonously increasing in x.*

Proof. Let

$$f(x) = \sqrt{2}g(x) = \sqrt{2}\left[(x+1)\cdot\left(\sqrt{\frac{1}{x+3}} - \sqrt{\frac{1}{x+1}}\right) + \frac{\sqrt{2}}{x+3}\right]. \quad (2)$$

We consider the first derivative of $g(x)$.

$$g'(x) = \frac{1}{\sqrt{x+3}} - \frac{1}{\sqrt{x+1}} + (x+1)\left[\frac{1}{-2\sqrt{x+3}(x+3)} + \frac{1}{2\sqrt{x+1}(x+1)}\right] - \frac{\sqrt{2}}{(x+3)^2}$$
$$= \frac{x+5}{2\sqrt{x+1}(x+3)} - \frac{1}{2\sqrt{x+1}} - \frac{\sqrt{2}}{(x+3)^2}.$$

Thus

$$2\sqrt{x+1}(x+3)^2g'(x) = (x+5)\sqrt{x+1}\sqrt{x+3} - (x+3)^2 - 2\sqrt{2}\sqrt{x+1}. \quad (3)$$

Let $t(x) = 2\sqrt{x+1}(x+3)^2g'(x) = (x+5)\sqrt{x+1}\sqrt{x+3} - (x+3)^2 - 2\sqrt{2}\sqrt{x+1}$.

We will prove that $t(x)$ monotonically increases in x ($x \ge 11$).

Note

$$t'(x) = \left[(x+5)\sqrt{x+1}\sqrt{x+3} - (x+3)^2 - 2\sqrt{2}\sqrt{x+1}\right]'$$
$$= \sqrt{x^2+4x+3} + (x+5)\frac{x+2}{\sqrt{x^2+4x+3}} - 2(x+3) - \frac{\sqrt{2}}{\sqrt{x+1}}.$$

Thus

$$\sqrt{x^2+4x+3}\cdot t'(x) = x^2+4x+3+(x+5)(x+2) - 2(x+3)\sqrt{x^2+4x+3} - \sqrt{2}\sqrt{x+3}$$
$$= 2x^2+11x+13 - 2(x+3)\sqrt{x^2+4x+3} - \sqrt{2}\sqrt{x+3}$$
$$> 2x^2+11x+13 - 2(x+3)(x+2) - \sqrt{2}\sqrt{x+3}$$
$$= x+1-\sqrt{2}\sqrt{x+3}.$$

It is easy to verify that $x+1-\sqrt{2}\sqrt{x+3} > 0$ for $x \ge 11$.

Thus $t(x) = (x+5)\sqrt{x+1}\sqrt{x+3} - (x+3)^2 - 2\sqrt{2}\sqrt{x+1}$ is a monotonically increasing function for $x \ge 11$.

Hence $t(x) \ge t(11) > 1.5857 > 0$ for $x \ge 11$.

By (3), we have $g'(x) > 0$ for $x \ge 11$ and $g(x)$ monotonically increases in x for $x \ge 11$.

From (2), $f(n) \ge \sqrt{2}\cdot g(11) \ge -0.2206$ for $n \ge 11$. ■

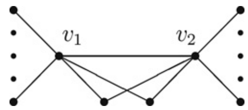

Fig. 2. The bicyclic graph has the minimum difference $R(G) - R(G - v_1v_2)$ when n is even.

Lemma 4.5. *Let G be a bicyclic graph and v_1v_2 be an edge in a cycle of G. Then $R(G) - R(G - v_1v_2) \geq -0.2683$.*

Proof. Let v_1v_2 be an edge in a cycle of G with $d(v_1) = d_1$, $d(v_2) = d_2$. By definition of Randić index, it is not difficult to obtain the following result

$$R(G) - R(G - v_1v_2) = \sum_{v_1 \sim v_x \neq v_2} \frac{1}{\sqrt{d_x}} \left(\frac{1}{\sqrt{d_1}} - \frac{1}{\sqrt{d_1 - 1}} \right) + \sum_{v_2 \sim v_y \neq v_1} \frac{1}{\sqrt{d_y}} \left(\frac{1}{\sqrt{d_2}} - \frac{1}{\sqrt{d_2 - 1}} \right) + \frac{1}{\sqrt{d_1 d_2}}. \quad (4)$$

Since v_1v_2 is an edge in a cycle, there exist at least two vertices $v_x \neq v_2$, $v_y \neq v_1$ for which $d_x, d_y \geq 2$, because v_x and v_y have to be connected by a path, different from $\{v_1, v_2\}$ (or $d_x = d_y$).

Hence from expression (4)

$$R(G) - R(G - v_1v_2) \geq \left(d_1 - 3 + \frac{2}{\sqrt{2}} \right) \left(\frac{1}{\sqrt{d_1}} - \frac{1}{\sqrt{d_1 - 1}} \right) + \left(d_2 - 3 + \frac{2}{\sqrt{2}} \right) \left(\frac{1}{\sqrt{d_2}} - \frac{1}{\sqrt{d_2 - 1}} \right) + \frac{1}{\sqrt{d_1 d_2}}. \quad (5)$$

Note that G is a bicyclic graph ($n \geq 4$).

Case 1. n is even.

The minimum value for the difference $R(G) - R(G - v_1v_2)$ is reached when $d_1 = d_2 = \frac{n+2}{2}$. And the graph is depicted as Fig. 2.

From inequality (5),

$$\begin{aligned} R(G) - R(G - v_1v_2) &\geq 2 \left(\frac{n+2}{2} - 3 + \frac{2}{\sqrt{2}} \right) \left(\sqrt{\frac{2}{n+2}} - \sqrt{\frac{2}{n}} \right) + \frac{2}{n+2} \\ &\geq \sqrt{2}(n-1) \left(\sqrt{\frac{1}{n+2}} - \sqrt{\frac{1}{n}} \right) + \frac{2}{n+2} \\ &\geq \sqrt{2}(n+1) \left(\sqrt{\frac{1}{n+2}} - \sqrt{\frac{1}{n}} \right) + \frac{2}{n+2} := h(n). \end{aligned}$$

Then $h(x) = k(x+1)$, where $k(x)$ is the function in Lemma 4.2. By Lemma 4.2, $h(x)$ is also monotonously increasing in x.

Hence $R(G) - R(G - v_1v_2) \geq h(n) \geq h(11) \geq -0.2683$ for $n \geq 11$.

Case 2. n is odd.

The minimum value for the difference $R(G) - R(G - v_1v_2)$ is reached when $d_1 = \frac{n+3}{2}$, $d_2 = \frac{n+1}{2}$. And

$$\begin{aligned} R(G) - R(G - v_1v_2) &\geq \left(\frac{n+3}{2} - 3 + \frac{2}{\sqrt{2}} \right) \left(\sqrt{\frac{2}{n+3}} - \sqrt{\frac{2}{n+1}} \right) \\ &\quad + \left(\frac{n+1}{2} - 3 + \frac{2}{\sqrt{2}} \right) \left(\sqrt{\frac{2}{n+1}} - \sqrt{\frac{2}{n-1}} \right) + \frac{2}{\sqrt{(n+1)(n+3)}} \\ &\geq \left(\frac{n+3}{2} - 3 + \frac{2}{\sqrt{2}} \right) \left(\sqrt{\frac{2}{n+3}} - \sqrt{\frac{2}{n-1}} \right) + \frac{2}{n+3} \\ &\geq \frac{n}{\sqrt{2}} \left(\sqrt{\frac{1}{n+3}} - \sqrt{\frac{1}{n-1}} \right) + \frac{2}{n+3} := m(n). \end{aligned}$$

Analog of the proof in Lemma 4.4, it is no difficult to verify that $m(n)$ is monotonously increasing in x. Then $R(G) - R(G - v_1v_2) \geq m(n) \geq m(11) \geq -0.2381$ for $n \geq 11$.

By the above discussions, the lemma follows. ■

Lemma 4.6. *Let G be a tricyclic graph and v_1v_2 be an edge in a cycle of G. Then $R(G) - R(G - v_1v_2) \geq -0.2673$.*

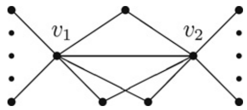

Fig. 3. The first extremal tricyclic graph with the minimal difference $R(G) - R(G - v_1v_2)$.

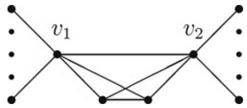

Fig. 4. The second extremal tricyclic graph with the minimal difference $R(G) - R(G - v_1v_2)$.

Proof. Let v_1v_2 be an edge in a cycle of G with $d(v_1) = d_1, d(v_2) = d_2$.

Completely similar to the proof of Lemma 4.5, from expression (4), and note that G is a tricyclic graph ($n \geq 4$), there are two extremal graphs (Figs. 3 and 4) when $R(G) - R(G - v_1v_2)$ attains the minimal value.

Case 1. The extremal graph is Fig. 3.

Subcase 1.1. n is odd.

The minimum value for the difference $R(G) - R(G - v_1v_2)$ is reached when $d_1 = d_2 = \frac{n+3}{2}$. Then

$$\begin{aligned}
R(G) - R(G - v_1v_2) &\geq 2\left(\frac{n+3}{2} - 4 + \frac{3}{\sqrt{2}}\right)\left(\sqrt{\frac{2}{n+2}} - \sqrt{\frac{2}{n+1}}\right) + \frac{2}{n+3} \\
&= \sqrt{2}\left(n - 5 + \frac{3}{\sqrt{2}}\right)\left(\sqrt{\frac{1}{n+3}} - \sqrt{\frac{1}{n+1}}\right) + \frac{2}{n+3} \\
&\geq \sqrt{2}(n+1)\left(\sqrt{\frac{1}{n+3}} - \sqrt{\frac{1}{n+1}}\right) + \frac{2}{n+3}.
\end{aligned}$$

By Lemma 4.4, $R(G) - R(G - v_1v_2) \geq f(11) \geq -0.2206$.

Subcase 1.2. n is even. The minimal value for the difference $R(G) - R(G - v_1v_2)$ is attained when $d_1 = \frac{n+4}{2}, d_2 = \frac{n+2}{2}$. Then

$$\begin{aligned}
R(G) - R(G - v_1v_2) &\geq \left(\frac{n+4}{2} - 4 + \frac{3}{\sqrt{2}}\right)\left(\sqrt{\frac{2}{n+4}} - \sqrt{\frac{2}{n+2}}\right) \\
&\quad + \left(\frac{n+2}{2} - 4 + \frac{3}{\sqrt{2}}\right)\left(\sqrt{\frac{2}{n+2}} - \sqrt{\frac{2}{n}}\right) + \frac{2}{\sqrt{(n+2)(n+4)}} \\
&\geq \sqrt{2}\left(\frac{n+4}{2} - 4 + \frac{3}{\sqrt{2}}\right)\left(\sqrt{\frac{1}{n+4}} - \sqrt{\frac{1}{n}}\right) + \frac{2}{n+4} \\
&\geq \frac{n+1}{\sqrt{2}}\left(\sqrt{\frac{1}{n+4}} - \sqrt{\frac{1}{n}}\right) + \frac{2}{n+4} := l(n).
\end{aligned}$$

Similarly, $l(x)$ is monotonously increasing in x for $x \geq 11$.

Thus $R(G) - R(G - v_1v_2) \geq l(11) \geq -0.2673$.

Case 2. The extremal graph is Fig. 4.

Subcase 2.1. n is even.

The minimal value for the difference $R(G) - R(G - v_1v_2)$ is attained when $d_1 = d_2 = \frac{n+2}{2}$. Then

$$\begin{aligned}
R(G) - R(G - v_1v_2) &\geq 2\left(\frac{n+2}{2} - 3 + \frac{2}{\sqrt{3}}\right)\left(\sqrt{\frac{2}{n+2}} - \sqrt{\frac{2}{n}}\right) + \frac{2}{n+2} \\
&\geq \sqrt{2}(n+1)\left(\sqrt{\frac{1}{n+2}} - \sqrt{\frac{1}{n}}\right) + \frac{2}{n+2}.
\end{aligned}$$

By Lemma 4.2, $R(G) - R(G - v_1v_2) \geq -0.2683$ for $n \geq 11$.

Subcase 2.2. n is odd.

The minimal value for the difference $R(G) - R(G - v_1v_2)$ is attained when $d_1 = \frac{n+3}{2}$, $d_2 = \frac{n+1}{2}$. Then

$$\begin{aligned} R(G) - R(G - v_1v_2) &\geq \left(\frac{n+3}{2} - 3 + \frac{2}{\sqrt{3}}\right)\left(\sqrt{\frac{2}{n+3}} - \sqrt{\frac{2}{n+1}}\right) \\ &\quad + \left(\frac{n+1}{2} - 3 + \frac{2}{\sqrt{3}}\right)\left(\sqrt{\frac{2}{n+1}} - \sqrt{\frac{2}{n-1}}\right) + \frac{2}{\sqrt{(n+1)(n+3)}} \\ &\geq \frac{n}{\sqrt{2}}\left(\sqrt{\frac{1}{n+3}} - \sqrt{\frac{1}{n-1}}\right) + \frac{2}{n+2}. \end{aligned}$$

Similarly, $R(G) - R(G - v_1v_2) \geq -0.2271$ for $n \geq 11$.

By the above discussions, if G is a tricyclic graph and v_1v_2 is an edge in a cycle of G, the $R(G) - R(G - v_1v_2) \geq -0.2673$ for $n \geq 11$. ■

Theorem 4.7. *Let G be a tricyclic graph with order n ($n \geq 5$, $m = n + 2$), then*

$$R(G) \geq r(G) - 1.$$

Proof. Let e_1 be an edge in a cycle of G. By Lemma 4.3, $G - e_1$ is a bicyclic graph. And by Lemma 4.6, $R(G) - R(G - e_1) \geq -0.2673$.

Let e_2 be an edge in a cycle of $G - e_1$. Similarly, by Lemmas 4.3 and 4.5, $R(G - e_1) - R(G - e_1 - e_2) \geq -0.2683$.

Let e_3 be an edge in the cycle of $G - e_1 - e_2$. Denote $T = G - e_1 - e_2 - e_3$ by a spanning tree of G. By the inequality (1) in Lemmas 4.1 and 4.3, $R(G - e_1 - e_2) - R(T) \geq \sqrt{2} \cdot 11 \cdot \left(\sqrt{\frac{1}{11+1}} - \sqrt{\frac{1}{11-1}}\right) + \frac{2}{11+1} \geq -0.2620$ for $n \geq 11$.

Then

$$\begin{aligned} R(G) &= R(G) - R(G - e_1) + R(G - e_1) - R(G - e_1 - e_2) + R(G - e_1 - e_2) - R(T) + R(T) \\ &\geq R(T) - 0.2673 - 0.2683 - 0.2620 \\ &= R(T) - 0.7976 \\ &\geq r(T) - 0.086 - 0.7976 \quad \text{(Theorem A)} \\ &= r(T) - 0.8836 \geq r(G) - 0.8836 \geq r(G) - 1. \quad ■ \end{aligned}$$

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