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Colouring lines in projective space

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Abstract

Let V be a vector space of dimension v over a field of order q. The q-Kneser graph has the k-dimensional subspaces of V as its vertices, where two subspaces α and β are adjacent if and only if $\alpha \cap \beta$ is the zero subspace. This paper is motivated by the problem of determining the chromatic numbers of these graphs. This problem is trivial when $k = 1$ (and the graphs are complete) or when $v < 2k$ (and the graphs are empty). We establish some basic theory in the general case. Then specializing to the case $k = 2$, we show that the chromatic number is $q^2 + q$ when $v = 4$ and $(q^{v-1} - 1)/(q - 1)$ when $v > 4$. In both cases we characterise the minimal colourings.

Keywords: Kneser graph; Chromatic number; Projective space

1. Introduction

The *Kneser graph* $K_{v:k}$ has the subsets of size k from a fixed set of size v as its vertices, with two k-subsets adjacent if and only if they are disjoint as sets. The Kneser graphs play an important role in work on graph homomorphisms and graph colouring. In this paper we are concerned with a generalisation of these graphs, which we call *q-Kneser graphs*. We construct these as follows. Let $\mathbb{F}$ be a finite field of order q. The vertices of the q-Kneser graph $qK_{v:k}$ are the k-dimensional subspaces of a vector space of dimension v over $\mathbb{F}$; two k-subspaces are adjacent if their intersection is the zero subspace.

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Our work in this paper is concerned with determining the chromatic numbers of the q-Kneser graphs. The graphs $qK_{v:1}$ are complete graphs and if $v < 2k$ then $qK_{v:k}$ is an empty graph, and there is nothing we need say about these cases. We summarise our main results.

We show that if $v > 2k$, then

$$\chi(qK_{v:k}) \leqslant \frac{q^{v-k+1}-1}{q-1}$$

and if $v = 2k$, then

$$\chi(qK_{2k:k}) \leqslant q^k + q^{k-1}.$$

Naturally, these bounds are derived by giving explicit colourings. We prove that the stated bounds are tight when $k = 2$, where we can also characterise the minimal colourings. When $v \geqslant 5$ these are essentially unique, but when $v = 4$ there are a number of colourings.

We now explain why these colouring questions are interesting. We first recall what is known about the ordinary Kneser graphs $K_{v:k}$. It is easy to find a colouring of $K_{v:k}$ with $v - 2k + 2$ colours as follows: If α is a k-subset and the largest element of α is greater than $2k$, define this element to be the colour of α. This uses $v - 2k$ colours to colour all k-subsets not contained in $\{1, \dots, 2k\}$. The subsets not already coloured induce a copy of $K_{2k:k}$; since this graph is bipartite we can colour it with two colours. Thus we have coloured $K_{v:k}$ with $v - 2k + 2$ colours. Lovász proved in [6] that this upper bound is the correct value.

There are at least three reasons why Lovász's result is interesting. First if $v = 3k - 1$, then $K_{v:k}$ is triangle-free and has chromatic number $k+1$. Hence we have an explicit construction of triangle-free graphs with large chromatic number. By choosing v and k more carefully, we actually obtain graphs with large chromatic number and no short odd cycles. (See [5] for more details.) Second, the fractional chromatic number of $K_{v:k}$ is known to be v/k and so the Kneser graphs provide examples of graphs whose fractional chromatic number is much lower than their chromatic number. Third, Lovász's proof that the chromatic number of $K_{v:k}$ is $v - 2k + 2$ uses the Borsuk–Ulam theorem from topology in an essential way. Although other proofs are known now, they all are based on results that are at least morally equivalent to the Borsuk–Ulam theorem.

Next, we consider the connection between the ordinary Kneser graphs and the q-Kneser graphs. To clarify this, we need the so-called q-binomial coefficients. Choose a positive integer q and, for an integer n, define

$$[n] := \frac{q^n - 1}{q - 1}.$$

We define the *q-factorial function* $[n]!$ inductively by $[0]! = 1$ and

$$[n+1]! = [n+1][n]!.$$

We define the *q-binomial coefficient* $\begin{bmatrix} v \\ k \end{bmatrix}$ by

$$\begin{bmatrix} v \\ k \end{bmatrix} = \frac{[v]!}{[k]![v-k]!}.$$

This is also known as the Gaussian binomial coefficient, and is sometimes written $\begin{bmatrix} v \\ k \end{bmatrix}_q$. The q-binomial coefficients play the same role in the enumeration of subspaces that the usual binomial coefficients play in the enumeration of subsets. If $q = 1$, then we define $[n]$ to be n and it then follows that $[n]! = n!$ and

$$\begin{bmatrix} v \\ k \end{bmatrix} = \binom{v}{k}.$$

If q is a prime power, then $\left[\begin{smallmatrix} v \\ k \end{smallmatrix}\right]$ is equal to the number of subspaces of dimension k in a vector space of dimension v over the field of order q. We recall, from [3, p. 239] for example, that in a v-dimensional vector space over $GF(q)$, the number of ℓ-dimensional subspaces that intersect a given k-dimensional subspace in a given subspace of dimension j is

$$q^{(\ell-j)(k-j)} \begin{bmatrix} v-k \\ \ell-j \end{bmatrix}.$$

Lemma 1.1. *The q-Kneser graph $qK_{v:k}$ has $\left[\begin{smallmatrix} v \\ k \end{smallmatrix}\right]$ vertices and is regular with valency $q^{k^2}\left[\begin{smallmatrix} v-k \\ k \end{smallmatrix}\right]$.*

Proof. Only the valency is in question. Suppose $|\mathbb{F}| = q$ and α is a subspace of dimension k in $\mathbb{F}^v$. The subspaces of dimension $2k$ that contain α partition the set of k-dimensional subspaces that meet α in the zero subspace. This partition has $\left[\begin{smallmatrix} v-k \\ k \end{smallmatrix}\right]$ components. The number of k-subspaces in a space of dimension $2k$ that meet a given k-dimensional subspace in the zero subspace is q^{k^2}. $\square$

When $q = 1$, these expressions for the number of vertices and valency reduce to the corresponding values for the Kneser graph $K_{v:k}$, respectively $\binom{v}{k}$ and $\binom{v-k}{k}$. Many other parameters of the q-Kneser graphs, for example the eigenvalues of the adjacency matrix and their multiplicities, are given by expressions which involve q-binomial coefficients and which reduce to the corresponding value for the ordinary Kneser graphs when we set q equal to 1. In particular the fractional chromatic number of the ordinary Kneser graphs is v/k while the fractional chromatic number of the q-Kneser graphs is $[v]/[k]$. (See [4] for more details on fractional chromatic number.)

Because of the above connections, and because the colouring problem for the ordinary Kneser graphs is so interesting, it is reasonable to study colouring problems for the q-Kneser graphs. There is evidence that the relation between these two problems is complex. Lovász's result makes use of the fact that if $\alpha_1, \dots, \alpha_r$ are vertices in $K_{v:k}$, then their set of common neighbours consists of the k-sets in the complement of the union

$$\bigcup_{i=1}^{r} \alpha_i,$$

which is a Kneser graph on a smaller set. If $\alpha_1, \dots, \alpha_r$ is a set of vertices in $qK_{v:k}$, then their set of common neighbours does not depend only on the join of these vertices. Hence it is not easy to see how topological methods can be applied to colouring q-Kneser graphs. It could be argued that this adds interest to the q-colouring problem—it is not unreasonable to hope that real progress on colouring q-Kneser graphs will yield insights concerning the case $q = 1$. However, in this paper we show that $\chi(qK_{v:2}) = [v-1]$ for $v > 4$, and thus $qK_{5:2}$ has chromatic number [5]. But the ordinary Kneser graph $K_{5:2}$ is the Petersen graph, whose chromatic number is three. So putting $q = 1$ in this formula leads to the wrong answer.

2. Independent sets

We will require information about the independent sets of maximum size in the ordinary Kneser graphs and the q-Kneser graphs, and so we summarise this here.

If $v \geqslant 2k$, the k-sets of $\{1, \ldots, v\}$ that contain a given point i form an independent set in $K_{v:k}$ with size $\binom{v-1}{k-1}$ and this is the maximum possible size. If $v > 2k$ these are the only independent sets of this size but if $v = 2k$, there are many others—partition the k-sets into $\binom{v-1}{k-1}$ pairs, and choose one k-set from each pair.

The Erdős–Ko–Rado theorem asserts that if $v \geqslant (k-t+1)(t+1)$ and $\mathcal{F}$ is a collection of k-subsets of a v-set such that any two k-subsets in $\mathcal{F}$ have at least t elements in common, then

$$|\mathcal{F}| \leqslant \binom{v-t}{k-t}$$

and, if $v > (k-t+1)(t+1)$ and equality holds, then $\mathcal{F}$ consists of the k-subsets that contain a given t-subset. (This result was originally proved by Erdős, Ko and Rado under the assumption that v was large enough relative to k, and it was first proved in the form stated by Wilson [8].)

The optimal colourings of $K_{v:k}$ we described earlier consist of $v - 2k$ independent sets, each contained in an independent set of maximum size, together with a bipartition of $K_{2k:k}$. To prove that $\chi(K_{v:k}) = v - 2k + 2$ when $v > 2k$, it would suffice to show the following: in any colouring of $K_{v:k}$ with $v - 2k + 2$ colours, there is at least one colour class consisting of k-subsets with a common point. (The topological arguments offer no help here; they show that there is no colouring using fewer than $v - 2k + 2$ colours.)

We turn to the q-Kneser graphs. Frankl and Wilson [2] proved that if $\mathcal{F}$ is a collection of k-subspaces of $\mathbb{F}^v$ such that any two elements of $\mathcal{F}$ intersect in a subspace of dimension at least t, then

$$|\mathcal{F}| \leqslant \max\left\{\begin{bmatrix} v-t \\ k-t \end{bmatrix}, \begin{bmatrix} 2k-t \\ k \end{bmatrix}\right\}.$$

When $v \geqslant 2k$ and $t = 1$, this implies that an independent set in $qK_{v:k}$ has size at most $\begin{bmatrix} v-1 \\ k-1 \end{bmatrix}$.

What of the sets that meet this bound? Here we must pay careful attention to the wording in [2]. It is asserted that when $v > 2k$, if $\mathcal{F}$ has maximum size then it consists of the k-spaces that contain a specified t-space. No proof is offered, instead there is a claim that this result follows easily from the work in [2] and the results in a second paper. The difficulty is that the second paper is cited incorrectly; it is true that the characterisation follows readily from the work in [2] and Wilson's earlier paper [8]. For the case $t = 1$, a short proof is also offered in [4].

When $v = 2k$, the set of k-spaces contained in a given subspace of dimension $2k - 1$ form an intersecting family with the maximum possible size. Frankl and Wilson state that that they could not prove there are only two types of optimal families when $t \geqslant 2$. Thus they assert that they can prove this when $t = 1$, but it is not clear what proof they had in mind. (M. W. Newman and Godsil also have a proof of this now.)

In our later work, we require information about minimal sets of points in projective space that meet all subspaces of a given dimension. The basic result is the following.

Theorem 2.1. *Let S be a set of points in the projective space $PG(v-1, q)$ such that every subspace of projective dimension $k - 1$ contains a point of S. Then $|S| \geqslant [v - k + 1]$, and if equality holds then S consists of the points from a subspace.*

The result was first proved by Bose and Burton [1]. As we will need the case where we have a set of points S that meet all lines, we offer a proof for this case. If x is a point not in S then each line through x must contain a point from S, and so $|S|$ is bounded below by the number of lines on x. If equality holds then a line through x meets S in at most one point. Therefore if ℓ is a line that contains two points of S, then all points on ℓ must lie in S. Consequently S is a subspace.

3. Homomorphisms

If X and Y are two graphs, a homomorphism from X to Y is a map f from $V(X)$ to $V(Y)$ such that if u and v are adjacent vertices in X, then $f(u)$ and $f(v)$ are adjacent in Y. Since the graphs in this paper do not have loops, if $y \in V(Y)$, then the preimage $f^{-1}(y)$ is an independent set in X. Further, X can be coloured with r colours if and only if there is a homomorphism $f : X \to K_r$. It also follows that if there is a homomorphism $f : X \to Y$, then $\chi(X) \leqslant \chi(Y)$. Accordingly homomorphisms provide a useful tool for working on colouring problems.

The following homomorphisms between Kneser graphs are known. First, $K_{v:k}$ is an induced subgraph of $K_{v+1:k}$, we call this embedding the *extension map*. Next, if t is a positive integer, then $K_{v:k}$ is an induced subgraph of $K_{tv:tk}$. We call this the *multiplication map*. Finally, Stahl [7] discovered a homomorphism from $K_{v:k}$ to $K_{v-2:k-1}$. Given the existence of this map, we see that

$$\chi(K_{v:k}) \leqslant \chi(K_{v-2:k-1}),$$

which implies that $\chi(K_{v:k}) \leqslant v - 2k + 2$. Hence we call Stahl's map the *colouring map*. Stahl has conjectured that there is a homomorphism from $K_{v:k}$ to $K_{w,\ell}$ if and only if there is a homomorphism from $K_{v:k}$ to $K_{w,\ell}$ that is a composition of extension, multiplication and colouring maps.

We turn to the q-Kneser graphs. We have the following homomorphisms:

(a) The extension map, embedding $qK_{v:k}$ in $qK_{v+1:k}$.

(b) Since the field of order q is a subfield of the field of order q^r, we have a subfield map $qK_{v:k} \to q^r K_{v:k}$.
(c) A k-space in a v-dimensional vector space over $GF(q^r)$ can be viewed as a subspace of dimension rk in a space of dimension rv over $GF(q)$. This leads to a q-analog of the multiplication map, embedding $q^r K_{v:k}$ as an induced subgraph of $qK_{rv:rk}$.
(d) Each k-subspace is the row space of a unique $k \times v$ matrix in reduced row echelon form. The subspace spanned by the last $k-1$ rows of this matrix is a $(k-1)$-subspace of a $(v-1)$-dimensional space. Hence we have a homomorphism from $qK_{v:k}$ to $qK_{v-1:k-1}$.
(e) Finally $qK_{v:k}$ is an induced subgraph of $K_{[v]:[k]}$.

In Cases (c) and (e) above, the induced subgraph has the same fractional chromatic number as the target graph.

However there is no homomorphism from $qK_{5:2}$ to $qK_{3:1}$, because

$$\chi(qK_{3:1}) = q^2 + q + 1$$

while

$$\chi(qK_{5:2}) \geqslant \frac{\begin{bmatrix} 5 \\ 2 \end{bmatrix}}{\alpha(qK_{5:2})} = \frac{\begin{bmatrix} 5 \\ 2 \end{bmatrix}}{\begin{bmatrix} 4 \\ 1 \end{bmatrix}} = \frac{[5]}{[2]} = \frac{q^4 + q^3 + q^2 + q + 1}{q + 1} > q^3 + q.$$

Hence there is no q-analog of Stahl's colouring homomorphism from $qK_{v:k}$ to $qK_{v-1:k-2}$.

4. Chromatic number

Now we consider the chromatic number of the q-Kneser graphs. There are two obvious families of independent sets in $qK_{v:k}$, namely the set of all k-spaces containing a given one-dimensional subspace, and the set of all k spaces contained in a given subspace of dimension $2k-1$. There are $\begin{bmatrix} v-1 \\ k \end{bmatrix}$ k-spaces containing a given one-dimensional subspace, while a subspace of dimension $2k-1$ contains $\begin{bmatrix} 2k-1 \\ k \end{bmatrix}$ k-spaces.

Lemma 4.1. *If $v \geqslant 2k$ then*

$$\chi(qK_{v:k}) \leqslant [v - k + 1].$$

If $v = 2k$ then

$$\chi(qK_{v:k}) \leqslant q^k + q^{k-1}.$$

Proof. A subspace U of dimension $v-k+1$ has non-trivial intersection with each k-space, so we can colour each k-space S with any of the one-dimensional subspaces in $S \cap U$. As U contains $[v-k+1]$ such subspaces, this yields a colouring with $[v-k+1]$ colours.

If $v = 2k$ we can do even better than this by choosing a subspace U of dimension $k+1$, and a subspace T of dimension k in U. Now consider the one-dimensional subspaces in U

that do not lie in T, together with the subspaces of dimension $2k-1$ that contain T but not U. This gives a total of

$$[k+1]-[k]+[k]-[k-1]=[k+1]-[k-1]=q^k+q^{k-1}$$

points and subspaces.

For any k-space S, if $S \cap U \subseteq T$, then S lies in a $(2k-1)$-space that contains T but not U, and otherwise S contains a one-dimensional subspace of U that does not lie in T. Therefore we can use the points and subspaces as colours and obtain a colouring of $qK_{2k:k}$ with q^k+q^{k-1} colours. □

The bound $[v-k+1]$ on $qK_{v:k}$ also follows from the fourth homomorphism described in Section 3. This leads to a second description of the colouring: if we represent each k-space by a $k \times v$ matrix in reduced row-echelon form, we can colour each subspace with the first row.

It seems plausible to us that the upper bounds in Lemma 4.1 provide the correct value of the chromatic number in all cases.

5. Covering lines

For the remainder of this paper we specialise to the situation $k=2$, where it proves convenient to use the terminology of projective geometry. In this terminology, the one-dimensional, two-dimensional and three-dimensional subspaces of $\mathbb{F}^v$ are the points, lines and planes of $PG(v-1,q)$. Two subspaces are called *incident* if one contains the other. The q-Kneser graph $qK_{v:2}$ has the lines of $PG(v-1,q)$ as its vertices, with two lines being adjacent if they are skew (have no point in common).

An independent set of size three in $qK_{v:2}$ consists either of three concurrent lines, or three non-concurrent lines in the same plane. It follows that an independent set of maximum size consists either of the lines on a point or the lines in a plane, and further that any independent set of $qK_{v:2}$ is contained in a maximum independent set of one of these types. Any colouring of $qK_{v:2}$ thus defines a collection of points and planes of $PG(v-1,q)$ such that every line of $PG(v-1,q)$ is incident with one of the points or one of the planes. We will call such a set of points and planes a *cover* of $PG(v-1,q)$ and say that a point or plane *covers* the lines with which it is incident.

If the colour classes of a colouring each contain more than $q+1$ vertices, then the colouring determines a unique cover. The converse is not quite true, in that a cover does not determine a unique colouring of $qK_{v:2}$ because some lines may be incident with more than one element of a cover. However if a cover is minimal (under inclusion), all of the colourings it determines use the same number of colours, and so $\chi(qK_{v:2})$ is equal to the minimum size of a cover.

6. Projective 3-space

In this section we show that $\chi(qK_{4:2})=q^2+q$.

Lemma 6.1. *Suppose C is a cover of $PG(3,q)$ consisting of r points and s planes. If C contains $q+1$ collinear points, then $r+s \geqslant q^2+q+1$. (Dually, if C contains $q+1$ planes on one line, then $r+s \geqslant q^2+q+1$.)*

Proof. Let ℓ be a line all of whose points are in C. Then these points cover

$$(q+1)(q^2+q)+1 = q^3+2q^2+q+1$$

lines and therefore there are q^4 remaining lines to be covered.

Each point of C not in ℓ covers at most q^2 lines not already covered by the points of ℓ. Each plane of C meets ℓ in at least one point, and so covers at most q^2 lines not already covered by the points of ℓ. So we need in total at least q^2 points and planes to cover the q^4 uncovered lines. □

Lemma 6.2. *Suppose C is a cover of $PG(3,q)$ consisting of r points and s planes. If $r+s \leqslant q^2+q$, then $r, s \geqslant q$.*

Proof. Let x be a point not in C. There are q^2+q+1 lines on x, and each point of C covers at most one line on x. Since $r+s \leqslant q^2+q$, by assumption, we must have $r > 0$. Similarly, $s > 0$.

Suppose, for a contradiction that, $r \leqslant q-1$. Then at least q^2+2 of the q^2+q+1 lines on x are not covered by one of the r points. Hence they must be covered by one of the s planes. The first plane on x covers $q+1$ lines through x, each additional plane on x covers at most q further lines. Hence if there are t of our s planes on x, then

$$(q+1)+(t-1)q \geqslant q^2+2$$

and therefore

$$t-1 \geqslant q-1+\frac{1}{q}.$$

This implies that $t \geqslant q+1$.

Now count pairs (x, H) where x is a point not in our cover and H is a plane in the cover that contains x. We find that

$$s(q^2+q+1) \geqslant (q+1)[(q^3+q^2+q+1)-(q-1)]$$

and hence

$$s \geqslant \frac{(q^3+q^2+q-(q-2))(q+1)}{q^2+q+1} = q(q+1) - \frac{(q+1)(q-2)}{q^2+q+1}.$$

Since s is an integer, this implies that $s \geqslant q^2+q$. Consequently $r+s \geqslant q^2+q+1$, which contradicts our initial assumption. □

Theorem 6.3. *Suppose C is a cover of $PG(3,q)$ with $r+s \leqslant q^2+q$ points and planes. Then C contains exactly q^2+q points and planes, and, moreover, $q \mid r$ and dually $q \mid s$.*

Proof. Suppose that $r = kq + x$ where $0 \leqslant x < q$. By Lemma 6.2 we see that $s \geqslant q$, and therefore $r \leqslant q^2$ and so $k \leqslant q$.

Let P be a plane that is not in the cover. If P does not contain a point of the cover, then every line in P would have to be covered by one of the planes, and therefore $s \geqslant q^2 + q + 1$ which is not possible.

Therefore P contains points from the cover. Suppose it contains at most k. Since $k \leqslant q$, any k points on P cover at most $kq+1$ lines, so at least $(q^2+q+1)-(kq+1) = (q^2+q)-kq$ lines on P are not covered by one of these k points. Each of these lines must be covered by one of the s planes, whence $s \geqslant (q^2 + q) - kq$ and $r + s \geqslant q^2 + q + x$. Therefore we may assume that any plane not in the cover contains at least $k + 1$ of the points from the cover.

Each point of $PG(3, q)$ lies in exactly $q^2 + q + 1$ lines; since $r \leqslant q^2 + q$ it follows that any point not in the cover lies on a line that contains no point from the cover. Let ℓ be a line that does not contain points from the cover. Since $r = kq + x$ where $x < q$, at least two of the $q + 1$ planes on ℓ, P_1 and P_2, will each contain fewer than $k + 1$ points from the cover. By the preceding paragraph, P_1 and P_2 must be in the cover. To summarise, if ℓ is a line containing no points from the cover, then at least two planes on ℓ are in the cover.

The plane P_1 on ℓ contains at most k points from the cover, so at least $(q^2+q)-kq$ lines on P_1 do not contain a point from the cover. By the preceding paragraph each of these lines lies on a second plane from the cover, and we need at least $(q^2 + q) - kq$ additional planes from the cover on these lines. Consequently, $s \geqslant (q^2 + q) - kq$ and $r + s \geqslant q^2 + q + x$. As C satisfies $r + s \leqslant q^2 + q$, we must have $x = 0$. Therefore $r + s = q^2 + q$ and, moreover, $q \mid r$. □

Since $\chi(qK_{4:2})$ equal to the minimum size of a cover of $PG(3, q)$, we have $\chi(qK_{4:2}) = q^2 + q$.

7. Minimal covers

We have shown that $\chi(qK_{4:2}) = q^2 + q$, and given examples of colourings which meet this bound. In this section we completely characterise the minimal covers of $PG(3, q)$. Consider a cover constructed as follows: Choose a plane H, a point x on H, and s lines in H on x, where $1 \leqslant s \leqslant q$. The cover then consists of the $q(q + 1 - s)$ points of H not on these lines and the sq planes distinct from H that contain one of the s lines. We will call a cover of this type a *standard cover*.

We omit the proofs of Lemmas 7.1, 7.2, and 7.5 since they apply the same techniques used in Section 6. (The complete version of this paper is available online at the Math ArXiv.)

Lemma 7.1. *Let C be a cover of $PG(3, q)$ with $r = kq$ points and $s = q(q + 1 - k)$ planes.* If *P is a plane not in C, then it contains at least k points from C; if equality holds then the k points are collinear.*

Lemma 7.2. *Let C be a cover of $PG(3, q)$ with $r = kq$ points and $s = q(q + 1 - k)$ planes. Suppose P is a plane that contains at least $q + 1$ points from the cover. Then the points of C cover at most kq^3 lines not in P.*

Lemma 7.3. *If C is a cover of $PG(3,q)$ with $r = kq$ points and $s = q(q+1-k)$ planes, then each plane contains at least one line disjoint from C.*

Proof. Let P be a plane and assume by way of contradiction that each line on P is incident with a point from C. Since all planes from C meet P in a line, each plane in C contains at least one point from C. Therefore on each plane of C there are at least $q+1$ lines that are incident with a point from C, and consequently each plane in C covers at most q^2 of the lines that do not contain points from C. As there are

$$(q^2+1)(q^2+q+1) = q^4+q^3+2q^2+q+1$$

lines in total, the number of lines covered by the points in C is at least

$$\begin{aligned} q^4+q^3+2q^2+q+1-sq^2 &= q^4+q^3+2q^2+q+1-(q+1-k)q^3 \\ &= kq^3+2q^2+q+1. \end{aligned}$$

We know that q^2+q+1 of these lines lie in P, the remaining lines, of which there are at least kq^3+q^2, must intersect P in a point.

Since every line in P is incident with a point from C, there are at least $q+1$ points from C on P. By Lemma 7.2, the kq points of C cover at most kq^3 lines not on P, a contradiction. □

Lemma 7.4. *Let C be a cover of $PG(3,q)$ with q^2 points and q planes. Then C is standard.*

Proof. Assume C contains q^2 points and q planes. We will show first that there is a plane containing at least $q+1$ points from C.

Let ℓ_1 be a line not incident with a point in C. By Lemma 6.1 there is a plane H on ℓ_1 that is not in C and by Lemma 7.1, there are at least q points from C on H. If there are exactly q points then they lie on a line ℓ_2, and any plane containing ℓ_2 and a point in C not on ℓ_2 contains $q+1$ points from C. Otherwise H contains at least $q+1$ points from C.

We next show that no plane of C contains a point of C, and that the q planes of C lie on a common line.

Let P be a plane that contains at least $q+1$ points from C. By Lemma 7.2, the q^2 points in C cover at most q^4 lines not in P. By Lemma 7.3, there is a line on P that contains no point of C and so at most q^2+q lines on P are incident with points of C. Hence the number of lines incident with the q^2 points in C is at most q^4+q^2+q.

Since any two planes have a line in common, the q planes in C cover at most q^3+q^2+1 lines. The total number of lines is

$$q^4+q^3+2q^2+q+1 = (q^4+q^2+q)+(q^3+q^2+1),$$

whence the q^2 points in C must cover exactly q^4+q^2+q lines and the q planes must cover exactly q^3+q^2+1. We also see that the set of lines covered by the points of C is disjoint from the set of lines covered by the planes, and consequently no point of C can lie in a plane of C.

Further, since the q planes cover exactly q^3+q^2+1 lines, the q planes must lie on a line ℓ.

Let Q be the unique plane on ℓ not in the cover. Then the q^2 points of our cover must lie on Q, and hence the points of the cover are the points of $Q \setminus \ell$. □

Lemma 7.5. *Let C be a cover of $PG(3, q)$ with $r = kq$ points and $s = q(q + 1 - k)$ planes and let P be a plane not in C that contains at least $k + 1$ points from C. Then any line on P not incident with a point from C lies on at least two planes from C.*

Lemma 7.6. *Let C be a cover of $PG(3, q)$ with $r = kq$ points and $s = q(q + 1 - k)$ planes. Suppose there is a plane P that contains at least $q + 1$ points from the cover and a point y that lies on at least $q + 1$ planes. Then no plane in the cover contains a point from the cover.*

Proof. Let P be a plane that contains at least $q + 1$ points from C. By Lemma 7.2, our $r = kq$ points cover at most kq^3 lines not in P. Since there is a line in P that contains no points from C, our r points cover at most $kq^3 + q^2 + q$ lines.

Dually, the number of lines covered by the s planes in C is at most

$$sq^2 + q^2 + q = (q + 1 - k)q^3 + q^2 + q.$$

Suppose, for a contradiction, that some plane in C contains a point from C. Then there are $q + 1$ lines that are covered by both by a point in C and a plane from C. Hence the number of lines covered by the points and planes of C is at most

$$(kq^3 + q^2 + q) + ((q + 1 - k)q^3 + q^2 + q) - (q + 1) = q^4 + q^3 + 2q^2 + q - 1.$$

Since there are $q^4 + q^3 + 2q^2 + q + 1$ lines altogether, this provides our contradiction. □

Theorem 7.7. *A cover of $qK_{4:2}$ with $q^2 + q$ points and planes is standard.*

Proof. Let C be a cover of $PG(3, q)$ with $q^2 + q$ points and planes. We may assume that there are $r = kq$ points and $s = (q + 1 - k)q$ planes. By Lemma 7.4 and duality, we may assume that $2 \leqslant k \leqslant q - 1$.

As a first step, we show that there is a line that contains no point from C and lies on exactly one plane from C. Let m be a line that contains no point from C. There are $q + 1$ planes on m and kq points in the cover, so there is a plane H on m that contains fewer than k points. By Lemma 7.1 we see that H lies in the cover. At most $(k - 1)q + 1$ lines on H are incident with points of C and therefore there are at least $q(q + 2 - k)$ lines in H not incident with a point from C. Since there are only $q(q + 1 - k)$ planes in C, there is a line ℓ in H which contains no point from C and which is not contained in a second plane from C.

Next, we show that there is a plane that contains at least $q + 2$ points from C.

Let $H_1, \ldots, H_q$ denote the planes on ℓ other than H. These q planes do not belong to C and therefore by Lemma 7.1, there are at least k points from C on each of them. Since these planes partition the points of C into q classes, each plane contains exactly k points from C and, by Lemma 7.1, each set of k points lies on a line. Denote the line on H_i by m_i. The k points on H_i cover exactly $kq + 1$ lines on H_i; the remaining $(q + 1 - k)q$ lines on H_i are covered by planes of C. Since there are exactly $(q + 1 - k)q$ planes in C, each line of H_i

that is not covered by a point of C is contained in exactly one plane from C. Note that H contains no points of C.

The plane H_1 contains ℓ and therefore m_1 intersects ℓ in a point x. The lines other than m_1 on x in H_1 are covered by planes of C, and so there are q planes from C on x. For $i = 2, \dots, q$ these planes intersect H_i in q distinct lines through x, and these lines do not contain points of C. Therefore each of the lines $m_2, \dots, m_q$ intersects ℓ in x.

Let P be the plane determined by m_1 and m_2. The lines on P incident with x are m_1, m_2, the intersection of P with H and the intersection of P with $H_3, \dots, H_q$. The planes in C intersect H_1 in lines that contain no points of C, but $P \cap H_1 = m_1$ which does contain points from C. Therefore P is not in C.

Since P contains $2k$ points from C, by Lemma 7.5 any line on P not incident with a point from C lies on at least two planes from C. As there are q planes from C on x, at most $q/2$ lines on P incident with x do not contain points from C. Consequently at least $(q+2)/2$ lines on P incident with x contain points from C. Referring to our listing above of the lines on x in P, we see that m_1 and m_2 contain k points from C. As H contains no points of C, the line $P \cap H$ is disjoint from C. If $P \cap H_i$ contains a point from C then $P \cap H_i$ is m_i, because this is the only line on x in H_i that contains points from C. Then $P \cap H_i$ contains k points from C. Since $k \geqslant 2$, it follows that the number of points from C on P is at least

$$k\frac{q+2}{2} \geqslant q+2.$$

So we have shown that there is a plane that contains at least $q+2$ points from C; the dual of our argument shows that there is a point y on at least $q+2$ planes from C.

By Lemma 7.6, no plane in C contains a point from C. The $(q+1-k)q$ planes in C each meet P in a line, and so by Lemma 6.1 there are at least $q+1-k$ lines in P that contain no point of C. Thus there are at most q^2+k lines in P that do contain points of C. Each point of C in P covers q^2 lines not in P. Since there are at least $q+2$ points of C in P, each point of C not in P covers at most

$$(q^2+q+1)-(q+2) = q^2-1$$

lines not covered by points of C in P. So the number of lines covered by the points of C is at most

$$kq^3+q^2+k,$$

and if equality holds, all points of C lie in P and there are exactly $q+1-k$ lines in P disjoint from C, each of which lies in q planes from C.

Dually, the number of lines covered by the $(q+1-k)q$ planes of C is at most

$$(q+1-k)q^3+q^2+(q+1-k)$$

and, if equality holds, these planes have a common point y and there are exactly k lines incident with y that do not lie on a plane from C. Since the total number of lines is

$$q^4+q^3+2q^2+q+1 = (kq^3+q^2+k) + ((q+1-k)q^3+q^2+q+1-k),$$

our last two inequalities must be tight. Therefore all points of C lie in P and all the planes of C contain y. Hence C is a standard cover. $\square$

8. Higher dimensions

In this section we determine the chromatic number of $\chi(qK_{v:2})$ for $v \geqslant 5$ by determining the minimum number of points and planes needed to cover the lines of $PG(v-1,q)$.

Theorem 8.1. *Let C be a cover of $PG(v-1,q)$, where $v \geqslant 5$ with r points and s planes such that $r+s \leqslant [v-1]$. Then $r \geqslant [v-1]$.*

Proof. Suppose, for a contradiction, that $r < [v-1]$, and define $\delta := [v-1]-r$. We determine a lower bound on the number of lines that do not contain a point of C, by counting the flags (x, ℓ_x) where x is a point not in C and ℓ_x is a line on x containing no points of C. Each point x not in C lies on $[v-1]$ lines, and therefore there are at least $[v-1]-r=\delta$ lines through x that contain no points of C. As there are $[v]-r$ points not in C, the number of flags is at least

$$([v]-r)\delta = ([v]-[v-1]+[v-1]-r)\delta = (q^{v-1}+\delta)\delta.$$

Since each line containing no points of C lies in exactly $q+1$ flags, it follows that the number of such lines is at least

$$\frac{(q^{v-1}+\delta)\delta}{q+1}.$$

Each of these lines must be contained in one of the s planes, and a plane contains exactly q^2+q+1 lines. Therefore

$$s \geqslant \frac{(q^{v-1}+\delta)\delta}{(q+1)(q^2+q+1)}. \tag{1}$$

Since $r+s \leqslant [v-1]$, we have $s \leqslant \delta$ so

$$\frac{(q^{v-1}+\delta)\delta}{(q+1)(q^2+q+1)} \leqslant \delta,$$

from which we have

$$\delta \leqslant (q+1)(q^2+q+1) - q^{v-1}. \tag{2}$$

Observe that

$$\begin{aligned} q^4-(q+1)(q^2+q+1) &= q^4-q^3-2q^2-2q-1 \\ &= q(q(q^2-q-2)-2)-1 \end{aligned}$$

and therefore

$$q^{v-1}-(q+1)(q^2+q+1) = (q^{v-1}-q^4)+q(q(q^2-q-2)-2)-1.$$

If $q>2$ then $q^2-q-2>0$ and so the right side is positive. If $q=2$ then the right side is equal to

$$(2^{v-1}-16)-5,$$

which is positive if $v > 5$. Consequently, we conclude that the right-hand side of (2) is negative in these cases, which is a contradiction. Therefore, $r = [v-1]$ if $q > 2$ or if $v > 5$ and $q = 2$.

Finally we consider the case where $v = 5$ and $q = 2$. Let x be a point not in C. Since $r < [4] = 15$, by assumption, at least one of the 15 lines on x must be covered by one of the s planes in the cover. Consequently, x must lie on one of the s planes in the cover. Since, $r + s \leqslant [4] = 15$, we have $r \leqslant 15 - s$ so at least

$$31 - (15 - s) = 16 + s \tag{3}$$

points do not lie in C, and must lie on one of the s planes in the cover. Since planes contain 7 points, we must have $7s \geqslant 16 + s$ so $s \geqslant 3$.

Suppose $s > 3$. Then $r \leqslant 11$, so at least four of the 15 lines on x must lie on the s planes. Since a plane on x covers three lines on x, we must have that x lies on at least two planes in the cover. Consequently, $7s/2 \geqslant 16 + s$, which implies that $s \geqslant 7$. Since $s \leqslant \delta \leqslant 5$ by (2), this is a contradiction.

Therefore, $s = 3$ so at least 19 points must lie on the three planes in the cover by (3). However in $2K_{5:2}$ distinct planes intersect, so three planes cannot contain 19 distinct points. We have the desired contradiction, so $r = [4] = 15$ when $v = 5$ and $q = 2$. $\square$

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