Mixed interior and boundary bubbling solutions for Neumann problem in $\mathbb{R}^2$

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**Abstract**

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with smooth boundary, we consider the following problem:

$$
-\Delta u + u = \lambda u^{p-1} e^u, \quad u > 0, \quad \text{in } \Omega,
$$

$$
\frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial \Omega,
$$

where $\Omega$ is a bounded domain in $\mathbb{R}^2$ with smooth boundary, $\lambda > 0$ is a small parameter, $0 < p < 2$, and $\nu$ denotes the outer normal vector to $\partial \Omega$. We construct solutions of this problem with $k$ interior bubbling points and $l$ boundary bubbling points, for any $k \geq 1$ and $l \geq 1$.

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1. Introduction

In this paper, we consider the following boundary value problem

$$
\begin{align*}
-\Delta u + u &= \lambda u^{p-1} e^u, \quad u > 0, \quad \text{in } \Omega; \\
\frac{\partial u}{\partial \nu} &= 0, \quad \text{on } \partial \Omega,
\end{align*}
$$

where $\Omega$ is a bounded domain in $\mathbb{R}^2$ with smooth boundary, $\lambda > 0$ is a small parameter, $0 < p < 2$, and $\nu$ denotes the outer normal vector to $\partial \Omega$. This problem is the Euler–Lagrange equation for the functional

$$
J^p_\lambda(u) = \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + u^2) - \frac{\lambda}{p} \int_{\Omega} e^{up}, \quad u \in H^1(\Omega).
$$
If $p = 1$, Senba and Suzuki [21,22] have analyzed the asymptotic behavior of solutions to problem (1.1). If $u_\lambda$ is a family of solutions to problem (1.1) when $p = 1$, then there exist non-negative integers $k, l \geq 1$, such that

$$\lim_{\lambda \to 0} \lambda \int_\Omega e^{u_\lambda} = 4\pi (2k + l).$$

(1.3)

Let $m = k + l$. Up to subsequences, there exist points $\xi_j, j = 1, \ldots, m$ with $\xi_j \in \Omega$ for $j \leq k$ and $\xi_j \in \partial \Omega$ for $k < j \leq m$, for which

$$u_\lambda(x) \to \sum_{j=1}^{k} 8\pi G(x, \xi_j) + \sum_{j=k+1}^{m} 4\pi G(x, \xi_j), \quad \text{as } \lambda \to 0,$$

(1.4)

uniformly on compact subset of $\Omega \setminus \{\xi_1, \ldots, \xi_m\}$. Moreover, the $m$-tuple $(\xi_1, \ldots, \xi_m)$ can be characterized as critical point of a functional defined on $\Omega^k \times (\partial \Omega)^l$, given by

$$\varphi_m(\xi) = \varphi_m(\xi_1, \ldots, \xi_m) = \sum_{j=1}^{m} c_j^2 H(\xi_j, \xi_j) + \sum_{l \neq j} c_l c_j G(\xi_l, \xi_j),$$

(1.5)

where

$$c_j = 8\pi \quad \text{for } j = 1, \ldots, k, \quad \text{and} \quad c_j = 4\pi \quad \text{for } j = k + 1, \ldots, m,$$

and $G(x, y)$ is the Green’s function of the problem

$$\begin{cases} -\Delta_x G(x, y) + G(x, y) = \delta_y(x), & \text{in } \Omega; \\ \frac{\partial G(x, y)}{\partial v_x} = 0, & \text{on } \partial \Omega, \end{cases}$$

(1.6)

and $H(\cdot, \cdot)$ its regular part, namely,

$$H(x, y) = \begin{cases} G(x, y) + \frac{1}{2\pi} \log |x - y|, & \text{if } y \in \Omega; \\ G(x, y) + \frac{1}{\pi} \log |x - y|, & \text{if } y \in \partial \Omega. \end{cases}$$

(1.7)

Conversely, del Pino and Wei in [11], constructed bubbling solutions $u_\lambda$ to problem (1.1) when $p = 1$ with the above properties (1.3) and (1.4). Moreover, the location of the bubbling points corresponds to critical points of the function $\varphi_m$ defined by (1.5). Furthermore, they obtained the following expansion of the energy functional

$$J_1^1(u_\lambda) = -4\pi (2k + l)(2 - \log 8) - 8\pi (2k + l) \log \varepsilon - \frac{1}{2} \varphi_m(\xi) + o(1),$$

where $o(1) \to 0$ as $\lambda \to 0$.

This paper is devoted to construct solutions to problem (1.1) with bubbling profiles at points inside $\Omega$ and on the boundary of $\Omega$ when $p$ is between 0 and 2. In particular, we recover the result in [11] when $p = 1$. 
Let $\varepsilon$ be a parameter, which depends on $\lambda$, defined as

$$
p\lambda \left(-\frac{4}{p} \log \varepsilon\right)^{\frac{2(p-1)}{p}} \varepsilon^{\frac{2(p-2)}{p}} = 1. \tag{1.8}\n$$

Observe that, as $\lambda \to 0$, then $\varepsilon \to 0$, and $\lambda = \varepsilon^2$ if $p = 1$. Our result states as follows.

**Theorem 1.1.** Let $0 < p < 2$, and $k, l, m \geq 1$ be integers with $m = k + l$. There exists $\lambda_0 > 0$ so that, for any $0 < \lambda < \lambda_0$, problem (1.1) has a solution $u_\lambda$, with the following properties:

1. $u_\lambda$ has $m$ local maximum points $\xi_j^*$, $j = 1, \ldots, m$ such that $\xi_j^* \in \Omega$ for $1 \leq j \leq k$, and $\xi_j^* \in \partial \Omega$ for $k + 1 \leq j \leq m$. Furthermore

$$
\lim_{\lambda \to 0} \varphi_m (\xi_1^*, \ldots, \xi_m^*) = \min_{\Omega^k \times (\partial \Omega)^l} \varphi_m,
$$

where $\varphi_m$ is defined by (1.5).

2. One has

$$
u_\lambda (x) = p^{-\frac{1}{2}} \sqrt{\lambda \varepsilon} \frac{p-2}{p} \left[ \sum_{j=1}^{k} 8\pi G(x, \xi_j^*) + \sum_{j=k+1}^{m} 4\pi G(x, \xi_j^*) + o(1) \right]\n$$

where $\varepsilon$ satisfies (1.8), and $o(1) \to 0$, as $\lambda \to 0$, on each compact subset of $\Omega^2 \setminus \{\xi_1^*, \ldots, \xi_m^*\}$, and $G(\cdot, \cdot)$ is the Green’s function given in (1.6).

3. Moreover

$$
\lim_{\lambda \to 0} \varepsilon^{\frac{2(2-p)}{p}} \int_{\Omega} e^{\nu_\lambda} = 4\pi (2k + l). \tag{1.10}\n$$

Furthermore

$$
J^p_\lambda (u_\lambda) = \lambda \varepsilon^{\frac{2(2-p)}{p}} \left[ -4\pi (2k + l) - \frac{2 - p \log 8}{(2 - p) p} - \frac{8\pi p}{(2k + l) \log \varepsilon} - \frac{1}{2(2 - p)} \varphi_m (\xi^*) \right] + O \left( |\log \varepsilon|^{-1} \right) \tag{1.11}\n$$

where $O(1)$ uniformly bounded as $\lambda \to 0$.

The proof of our result relies on a very well-known Lyapunov–Schmidt reduction procedure, introduced in [1,17]. We use Lyapunov–Schmidt reduction method to reduce the problem to a finite dimensional one, with some reduced energy. Then, the solutions in Theorem 1.1 turn out to be generated by critical points of the reduced energy functionals. Such an idea has been used in many papers. See for instance [7–16,19] and references therein. A key step in this procedure is to find a good first approximation for the solution. Usually, this ansatz is built as a sum of terms, each one of which turns out to solve an associate limit problem, properly scaled and translated. For our problem, let us introduce the following limit problem

$$
\Delta w + e^w = 0 \quad \text{in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^w \, dx < +\infty. \tag{1.12}\n$$
It is well known that the solutions to (1.12) can be all written in the following form
\[ w_\mu(z) = \log \left( \frac{8\mu^2}{\mu^2 + |z|^2} \right) \text{ and } w_\mu,\xi(z) := w_\mu(z - \xi), \]
(1.13)
where \( \mu \) is any positive number and \( \xi \) any point in \( \mathbb{R}^2 \) (see [6]).

If we use the above solution, properly scaled and translated, as our approximate solution, we get a very good approximation of a solution in a region far away from the points. In other words, the error is relatively small far away from these points, which unfortunately turns out to be not good enough close to these points, thus we need to improve the approximation. We do this by adding two other terms in the expansion of the solution: this can be done in a very natural way, which has first been used, for instance, in [14] for studying the following problem
\[
\begin{cases}
\Delta u + u^q = 0, & u > 0, \text{ in } \Omega; \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\]
(1.14)
where \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^2 \), and \( q \) is a large exponent. Later on, which has been applied in papers [4,15,16,19].

It is important to remark about the analogy existing between our result and the Dirichlet problem:
\[
\begin{cases}
\Delta u + \lambda u^{p-1}e^{u^p} = 0, & u > 0, \text{ in } \Omega; \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\]
(1.15)
For \( p = 1 \), asymptotic behavior of solutions to (1.15) for which \( \lambda \int_\Omega e^u \) remains uniformly bounded, as \( \lambda \to 0 \), is well understood after the works [3,18,20]; indeed, \( \lambda e^u \) approaches a superposition of Dirac deltas centered in points in the interior of \( \Omega \). On the other hand, construction of solutions with this behavior has been achieved in [2,8,13]. In the previous paper [12], the author and M. Musso constructed bubbling solutions to (1.15) for \( 0 < p < 2 \): in this case it is proven that if \( \Omega \) is not simply connected, then there exists a family of solutions to (1.15). Construction of bubbling solutions for problem (1.15) for \( p = 2 \) is somehow different from the case \( 0 < p < 2 \). This has been treated in [10].

In order to state this result, let us introduce the following function of \( k \) distinct points \( \xi_1, \ldots, \xi_k \in \Omega \) and \( k \) positive numbers \( m_1, \ldots, m_k \),
\[
\varphi_{k,2}(\xi, m) = a \sum_{j=1}^{k} m_j^2 + 2 \sum_{j=1}^{k} m_j \log m_j + \sum_{j=1}^{k} m_j^2 H(\xi_j, \xi_j) + \sum_{i \neq j} m_i m_j G(\xi_i, \xi_j),
\]
(1.16)
where \( a > 0 \) is an absolute constant, and \( G(x, y) \) is the Green's function and \( H(\cdot, \cdot) \) its regular part. The authors in [10] established that, if \( \varphi_{k,2} \) has a topologically nontrivial critical value, with corresponding critical point \( (\xi_1, \ldots, \xi_k, m_1, \ldots, m_k) \in \Omega^k \times \mathbb{R}^k_+ \), then there exists a solution \( u_\lambda \) of (1.15) when \( p = 2 \) with the shape
\[
u \lambda(\lambda) = \sqrt{\lambda} \left[ \sum_{j=1}^{k} m_j G(x, \xi_j) + o(1) \right], \quad \text{as } \lambda \to 0,
\]
(1.17)
where \( o(1) \to 0 \) as \( \lambda \to 0 \) uniformly on compact sets of \( \Omega \setminus \{\xi_1, \ldots, \xi_k\} \). Furthermore,
\[
\int_{\Omega}^2(\nu_\lambda) = 2k\pi + \alpha \lambda + 4\pi \lambda \varphi_{k,2}(\xi, m) + \lambda o(1)
\]
where $\alpha$ is an absolute constant, $\varphi_{k, 2}$ is defined in (1.16) and $o(1) \to 0$ as $\lambda \to 0$. In particular, in the case $\Omega$ is not simply connected they constructed the solution $u_\lambda$ of (1.15) for $p = 2$, with two bubbling points, namely satisfying

$$u_\lambda(x) = \sqrt{\lambda} \left[ \sum_{j=1}^{2} m_j G(x, \xi_j) + o(1) \right], \quad \text{as } \lambda \to 0,$$

where $(m_1, m_2, \xi_1, \xi_2)$ is a critical point of $\varphi_{2, 2}$ defined in (1.16) and $o(1) \to 0$ as $\lambda \to 0$ uniformly on compact sets of $\Omega$.

This paper is organized as follows. In Section 2, describing a first approximation solution to problem (1.1) and estimating the error. We describe the proof of the main result in Section 3. Section 4 is devoted to perform the finite dimensional reduction. Section 5 contains the asymptotic expansion of the reduced energy.

In this paper, the symbol $C$ denotes a generic positive constant independent of $\lambda$, it could be changed from one line to another. The symbols $O(\varepsilon)$ (respectively $o(\varepsilon)$) will denote quantities for which $O(\varepsilon)$ $|\varepsilon|$ stays bounded (respectively, $o(\varepsilon)$ $|\varepsilon|$ tends to zero) as parameter $\lambda$ goes to zero. In particular, we will often use the notation $o(1)$ stands for a quantity which tends to zero as $t \to 0$.

2. Preliminaries and ansatz for the solution

In this section we describe the approximate solution for problem (1.1) and then we estimate the error of such approximation in appropriate norms.

We choose a sufficiently small but fixed number $\delta > 0$ and define

$$\mathcal{M}_\delta := \{ \xi := (\xi_1, \ldots, \xi_m) \in \Omega^k \times (\partial \Omega)^l \mid \min_{i=1,\ldots,k} \text{dist}(\xi_i, \partial \Omega) \geq \delta, \min_{i \neq j} |\xi_i - \xi_j| \geq \delta \}.$$  (2.1)

Let us consider $m$ distinct points $(\xi_1, \ldots, \xi_m) \in \mathcal{M}_\delta$, with $\xi_1, \ldots, \xi_k$ in $\Omega$ and $\xi_{k+1}, \ldots, \xi_m$ on $\partial \Omega$. Moreover we consider $m$ positive numbers $\mu_j$ such that

$$\delta < \mu_j < \delta^{-1}, \quad \text{for all } j = 1, \ldots, m.$$  (2.2)

We define the function

$$u_j(x) = \log \frac{8\mu_j^2}{(\mu_j^2 \varepsilon^2 + |x - \xi_j|^2)^2},$$

and a correction term defined as the solution of

$$\begin{cases} 
-\Delta H_j + H_j = -u_j, & \text{in } \Omega; \\
\frac{\partial H_j}{\partial v} = -\frac{\partial u_j}{\partial v}, & \text{on } \partial \Omega. 
\end{cases}$$  (2.3)

**Lemma 2.1.** For any $0 < \alpha < 1$, $\xi = (\xi_1, \ldots, \xi_m) \in \mathcal{M}_\delta$, then we have

$$H_j(x) = c_j H(x, \xi_j) - \log(8\mu_j^2) + o(\varepsilon^\alpha),$$  (2.4)

uniformly in $\bar{\Omega}$ as $\varepsilon \to 0$, where $H(\cdot, \cdot)$ is the regular part of Green's function defined in (1.7).
Proof. First, on the boundary, we have
\[
\frac{\partial H_j}{\partial \nu} = - \frac{\partial u_j}{\partial \nu} = 4 \frac{(x - \xi_j) \cdot \nu(x)}{\mu_j^2 \varepsilon^2 + |x - \xi_j|^2}.
\]
Thus,
\[
\lim_{\varepsilon \to 0} \frac{\partial H_j}{\partial \nu} = 4 \frac{(x - \xi_j) \cdot \nu(x)}{|x - \xi_j|^2}, \quad \forall x \in \partial \Omega \setminus \{\xi_j\}.
\]
On the other hand, the regular part of Green's function \(H(x, y)\) satisfies
\[
\begin{cases}
-\Delta H(x, y) + H(x, y) = -\frac{4}{c_j} \log \frac{1}{|x - y|}, & \text{in } \Omega; \\
\frac{\partial H(x, y)}{\partial \nu} = \frac{4 (x - y) \cdot \nu(x)}{|x - y|^2}, & \text{on } \partial \Omega.
\end{cases}
\tag{2.5}
\]
Set \(z(x) = H_j(x) - c_j H(x, \xi_j) + \log(8 \mu_j^2)\), then we get
\[
\begin{cases}
-\Delta z(x) + z(x) = \log \frac{1}{|x - \xi_j|^4} - \log \frac{1}{(\mu_j^2 \varepsilon^2 + |x - \xi_j|^2)^2}, & \text{in } \Omega; \\
\frac{\partial z(x)}{\partial \nu} = \frac{\partial H_j(x)}{\partial \nu} - 4 \frac{(x - \xi_j) \cdot \nu(x)}{|x - \xi_j|^2}, & \text{on } \partial \Omega.
\end{cases}
\]
A direct computation shows that, there is a positive constant \(C\) such that
\[
\left\| \frac{\partial H_j(x)}{\partial \nu} - 4 \frac{(x - \xi_j) \cdot \nu(x)}{|x - \xi_j|^2} \right\|_{L^q(\partial \Omega)} \leq C \varepsilon^{1/q}, \quad \forall q > 1,
\tag{2.6}
\]
and
\[
\left\| \log \frac{1}{|x - \xi_j|^4} - \log \frac{1}{(\mu_j^2 \varepsilon^2 + |x - \xi_j|^2)^2} \right\|_{L^q(\Omega)} \leq C \varepsilon, \quad \text{for any } 1 < q < 2.
\]
Then by elliptic regularity theory, we obtain
\[
\|z_\varepsilon\|_{W^{1+s, q}(\Omega)} \leq \left( \left\| \frac{\partial z_\varepsilon}{\partial \nu} \right\|_{L^q(\partial \Omega)} + \|\Delta z_\varepsilon\|_{L^q(\Omega)} \right) \leq C \varepsilon^{1/q}
\tag{2.7}
\]
for any \(0 < s < \frac{1}{q}\). By the Morrey embedding we obtain
\[
\|z_\varepsilon\|_{C^\beta(\widehat{\Omega})} \leq C \varepsilon^{1/q}
\]
for any \(0 < \beta < \frac{1}{2} + \frac{1}{q}\). Then we obtain that (2.4) holds with \(\alpha = \frac{1}{q}\). \(\square\)
We now define the first ansatz is given by

\[ U(x) = \frac{1}{p \gamma^{p-1}} \sum_{j=1}^{n} [u_j(x) + H_j(x)], \]

with some number \( \gamma \), to be fixed later on. We want to show that \( U(x) \) is a good approximation for a solution to (1.1) far from the points \( \xi_j \), but unfortunately it is not good enough for our construction close to the points \( \xi_j \). Thus we need to further adjust this ansatz. In order to do this, we set

\[ w_{\mu_j}(y) = w_{\mu_j}(y - \xi_j') = \log \frac{8 \mu_j^2}{\mu_j^2 + |y - \xi_j'|^2}. \]

Define the function \( w_{ij} \) to be the radial solution of

\[ \Delta w_{ij} + e^{w_{\mu_j}} w_{ij} = e^{w_{\mu_j}} f^i \quad \text{in} \quad \mathbb{R}^2, \]

for \( i = 0, 1 \), (2.8)

where

\[ f^0 = -\left( w_{\mu_j} + \frac{1}{2} (w_{\mu_j})^2 \right), \]

\[ f^1 = -\left( 2 w_{\mu_j} w_{0j} + \frac{1}{2} \left[ w_{0j} + \frac{(w_{\mu_j})^2}{2} \right] + w_{0j} + \frac{p - 2}{2(p - 1)} (w_{\mu_j})^2 + \frac{(w_{\mu_j})^3}{2} \right). \]

In fact, as shown in [14] (see also [5]), there exist radially symmetric solutions with the properties that

\[ w_{ij}(y) = C_{ij} \log \frac{|y - \xi_j'|}{\mu_j} + O \left( \frac{1}{|y - \xi_j'|} \right) \quad \text{as} \quad |y - \xi_j| \to \infty, \]

(2.9)

for some explicit constants \( C_{ij} \), which can be explicitly computed. In particular, when \( i = 0 \), the constant \( C_{0j} \) is given by

\[ C_{0j} = -8 \int_{0}^{+\infty} \frac{t^2 - 1}{(t^2 + 1)^3} \left[ \log \frac{8 \mu_j^{-2}}{(1 + t^2)^2} + \frac{1}{2} \left( \log \frac{8 \mu_j^{-2}}{(1 + t^2)^2} \right)^2 \right] dt \]

\[ = -4 \int_{0}^{+\infty} \frac{t^2 - 1}{(t^2 + 1)^3} \left[ \log \frac{8 \mu_j^{-2}}{(1 + t^2)^2} + \frac{1}{2} \left( \log \frac{8 \mu_j^{-2}}{(1 + t^2)^2} \right)^2 \right] d(t^2) \]

\[ = -4 \int_{1}^{+\infty} \frac{r - 2}{r^3} \left[ \log(8 \mu_j^{-2}) - 2 \log r + \frac{1}{2} \left( \log(8 \mu_j^{-2}) \right)^2 - 2 \log(8 \mu_j^{-2}) \log r + 2(\log r)^2 \right] dr. \]

Since

\[ \int_{1}^{+\infty} \frac{r - 2}{r^3} dr = 0, \quad \int_{1}^{+\infty} \frac{r - 2}{r^3} \log r dr = \frac{1}{2}, \quad \text{and} \quad \int_{1}^{+\infty} \frac{r - 2}{r^3} (\log r)^2 dr = \frac{3}{2}. \]
Hence
\[ C_{0j} = 4 \log 8 - 8 - 8 \log \mu_j. \] \hspace{1cm} (2.10)

Let \( H_{ij} \), for \( i = 0, 1 \), be a new correction defined as the solution of
\[
\begin{align*}
-\Delta H_{ij} + H_{ij} &= -w_{ij}(x/\varepsilon), \text{ in } \Omega; \\
\frac{\partial H_{ij}}{\partial \nu} &= -\frac{\partial w_{ij}}{\partial \nu}, \text{ on } \partial \Omega.
\end{align*}
\] \hspace{1cm} (2.11)

**Lemma 2.2.** For any \( 0 < \alpha < 1 \), for \( i = 0, 1 \), one has
\[ H_{ij}(x) = -\frac{C_{ij} \xi_j}{4} \log \frac{|x - \xi_j|}{\varepsilon} - C_{ij} \log \varepsilon + O(\varepsilon^{q}) \] \hspace{1cm} (2.12)
uniformly in \( \bar{\Omega} \) as \( \varepsilon \to 0 \), where \( H \) is the regular part of Green's function defined in (1.7).

**Proof.** The proof is the same as Lemma 2.1. First we note that, on the boundary, we have
\[
\lim_{\varepsilon \to 0} \frac{\partial H_{ij}}{\partial \nu} = -C_{ij} \frac{(x - \xi_j) \cdot \nu(x)}{|x - \xi_j|^2}, \quad \forall x \in \partial \Omega \setminus \{\xi_j\}.
\]
Define \( \tilde{z}(x) = H_{ij}(x) + \frac{C_{ij} \xi_j}{4} \log \frac{|x - \xi_j|}{\varepsilon} - C_{ij} \log \varepsilon \), by using (2.5), then we can get
\[
\begin{align*}
-\Delta \tilde{z}(x) + \tilde{z}(x) &= -C_{ij} \log \frac{1}{|x - \xi_j|} - w_{ij}, \text{ in } \Omega; \\
\frac{\partial \tilde{z}(x)}{\partial \nu} &= \frac{\partial H_{ij}(x)}{\partial \nu} + 4C_{ij} \frac{(x - \xi_j) \cdot \nu(x)}{|x - \xi_j|^2}, \text{ on } \partial \Omega.
\end{align*}
\]
From (2.9), we can get
\[
\left\| C_{ij} \log \frac{1}{|x - \xi_j|} - w_{ij} \right\|_{L^q(\Omega)} \leq C \varepsilon, \quad \text{for any } 1 < q < 2,
\]
for some constant \( C > 0 \), and
\[
\left\| \frac{\partial H_{ij}(x)}{\partial \nu} + 4C_{ij} \frac{(x - \xi_j) \cdot \nu(x)}{|x - \xi_j|^2} \right\|_{L^q(\partial \Omega)} \leq C \varepsilon^{1/q}, \quad \forall q > 1.
\]
Then by the same procedure as proof of Lemma 2.1, we obtain that (2.12) holds. \( \square \)

Now we define the first approximation solution to (1.1) as
\[
U_{\lambda}(x) = \frac{1}{p \gamma^{p-1}} \sum_{j=1}^{m} \left[ u_j(x) + H_j(x) + \frac{p-1}{p} \frac{1}{\gamma^p} (w_{0j}(x) + H_{0j}(x)) \right. \\
+ \left. \left( \frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} (w_{1j}(x) + H_{1j}(x)) \right]. \hspace{1cm} (2.13)
\]
From Lemma 2.1 and Lemma 2.2, one has, away from the points $\xi_j$,

$$U_\lambda(x) = \frac{1}{p^\gamma p - 1} \sum_{j=1}^m c_j G(x, \xi_j) \left[ 1 - \frac{p - 1}{p} \frac{C_0 j}{4} - \left( \frac{p - 1}{p} \right)^2 \frac{C_1 j}{4} + O(\varepsilon^\alpha) \right].$$

(2.14)

Consider now the change of variables

$$v(y) = p^\gamma p - 1 u(\varepsilon y) - p^\gamma p,$$

with $\gamma p = -\frac{4}{p} \log \varepsilon$.

By the choice of $\varepsilon$ in (1.8), then problem (1.1) reduces to

$$\begin{cases}
-\Delta v + \varepsilon^2 v = f(v) - p^\gamma p \varepsilon^2, & v > 0, \quad \text{in } \Omega_\varepsilon; \\
\frac{\partial v}{\partial \nu} = 0, & \text{on } \partial \Omega_\varepsilon,
\end{cases}$$

(2.15)

where $\Omega_\varepsilon = \varepsilon^{-1} \Omega$, and

$$f(v) = \left(1 + \frac{v}{p^\gamma p}\right)^{p-1} e^{p^\gamma p \left[1 + \frac{v}{p^\gamma p} \right]^{p-1}}.$$  

(2.16)

Let us define the first approximation solution to (2.15) as

$$V_\lambda(y) = p^\gamma p - 1 U_\lambda(\varepsilon y) - p^\gamma p,$$

(2.17)

with $U_\lambda$ defined by (2.13).

We write $y = \varepsilon^{-1} x$, $\xi_j = \varepsilon^{-1} \xi_j$. For $|x - \xi_j| < \delta$ with $\delta$ sufficiently small but fixed, by using Lemma 2.1, Lemma 2.2 and (2.14), and the fact that $u_j(\varepsilon y) - p^\gamma p = w_{\mu_j}(y - \xi_j)$, we have

$$V_\lambda(y) = u_j(\varepsilon y) + H_j(\varepsilon y) + \frac{p - 1}{p} \frac{1}{p^\gamma p} \left( w_{0j}(\varepsilon y) + H_{0j}(\varepsilon y) \right)$$

$$+ \left( \frac{p - 1}{p} \right)^2 \frac{1}{p^\gamma p} \left( w_{1j}(\varepsilon y) + H_{1j}(\varepsilon y) \right) - p^\gamma p$$

$$+ \sum_{l \neq j} \left[ u_l(\varepsilon y) + H_l(\varepsilon y) + \frac{p - 1}{p} \frac{1}{p^\gamma p} \left( w_{0l}(\varepsilon y) + H_{0l}(\varepsilon y) \right) \\
+ \left( \frac{p - 1}{p} \right)^2 \frac{1}{p^\gamma p} \left( w_{1l}(\varepsilon y) + H_{1l}(\varepsilon y) \right) \right]$$

$$= w_{\mu_j}(y - \xi_j) + c_j H(\varepsilon y, \xi_j) - \log(8 \mu_j^2) + O(\varepsilon^\alpha)$$

$$+ \frac{p - 1}{p} \frac{1}{p^\gamma p} \left[ w_{0j}(\varepsilon y) - \frac{C_0 j c_j}{4} H(x, \xi_j) + C_0 j \log(\mu_j) + C_0 j \log \varepsilon + O(\varepsilon^\alpha) \right]$$

$$+ \left( \frac{p - 1}{p} \right)^2 \frac{1}{p^\gamma p} \left[ w_{1j}(\varepsilon y) - \frac{C_1 j c_j}{4} H(x, \xi_j) + C_1 j \log(\mu_j) + C_1 j \log \varepsilon + O(\varepsilon^\alpha) \right]$$
Thus, \( \mu \) solution of equation

\[
\begin{align*}
&+ \sum_{l \neq j} c_l G(\xi_l, \xi_j) \left[ 1 - \frac{C_{0l}}{4} \frac{p-1}{p} \frac{1}{\gamma^p} - \frac{C_{1l}}{4} \left( \frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} \right] \\
&= \omega_j(y) + \frac{p-1}{p} \frac{1}{\gamma^p} \omega_0j(y) + \left( \frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} w_{1j}(y) + O(\varepsilon \gamma) - \log(8\mu_j^2) + c_j H(\xi_j, \xi_j) + \sum_{l \neq j} c_l G(\xi_l, \xi_j) - \frac{p-1}{4} C_{0j} \\
&- \frac{p-1}{p} \frac{1}{\gamma^p} \left[ \frac{C_{0j}}{4} \left( c_j H(\xi_j, \xi_j) + \sum_{l \neq j} c_l G(\xi_l, \xi_j) - 4 \log \mu_j \right) + \frac{(p-1)C_{1j}}{4} \right] \\
&- \left( \frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} \left[ \frac{C_{1j}}{4} \left( c_j H(\xi_j, \xi_j) + \sum_{l \neq j} c_l G(\xi_l, \xi_j) - 4 \log \mu_j \right) \right].
\end{align*}
\] (2.18)

where

\[
\begin{align*}
\omega_j(y) &= \omega_{\mu,j}(y - \xi_j), \\
\omega_0j(y) &= \omega_{0j}(y - \xi_j), \\
w_{1j}(y) &= \omega_{1j}(y - \xi_j).
\end{align*}
\]

We now choose the parameters \( \mu_j \): we assume they are defined by the relation

\[
\log(8\mu_j^2) = c_j H(\xi_j, \xi_j) + \sum_{l \neq j} c_l G(\xi_l, \xi_j) - \frac{p-1}{4} C_{0j} \\
- \frac{p-1}{p} \frac{1}{\gamma^p} \left[ \frac{C_{0j}}{4} \left( c_j H(\xi_j, \xi_j) + \sum_{l \neq j} c_l G(\xi_l, \xi_j) - 4 \log \mu_j \right) + \frac{(p-1)C_{1j}}{4} \right] \\
- \left( \frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} \left[ \frac{C_{1j}}{4} \left( c_j H(\xi_j, \xi_j) + \sum_{l \neq j} c_l G(\xi_l, \xi_j) - 4 \log \mu_j \right) \right].
\] (2.19)

Taking into account the explicit expression (2.10) of the constant \( C_{0j} \), we observe that \( \mu_j \) bifurcates, as \( \lambda \) goes to zero, from the value

\[
\bar{\mu}_j = 8 - \frac{p}{2 - p} e^{\frac{p-1}{2 - p}} \left[ c_j H(\xi_j, \xi_j) + \sum_{l \neq j} c_l G(\xi_l, \xi_j) \right]
\] (2.20)

solution of equation

\[
\log(8\mu_j^2) = c_j H(\xi_j, \xi_j) + \sum_{l \neq j} c_l G(\xi_l, \xi_j) - \frac{p-1}{4} C_{0j}.
\] (2.21)

Thus, \( \mu_j \) is a perturbation of order \( \frac{1}{\gamma^p} \) of the value \( \bar{\mu}_j \), namely

\[
\log(8\mu_j^2) = \left[ \frac{2(p-1)}{2 - p} (1 - \log 8) + \frac{1}{2 - p} \left( c_j H(\xi_j, \xi_j) + \sum_{l \neq j} c_l G(\xi_l, \xi_j) \right) \right] \left( 1 + O\left( \frac{1}{\gamma^p} \right) \right).
\] (2.22)
Then, by this choice of the parameters \( \mu_j \), we deduce that, if \( |y - \xi'_j| < \delta/\varepsilon \) with \( \delta \) sufficiently small but fixed, we can rewrite

\[
V_\lambda(y) = w_j(y) + \frac{p-1}{p} \frac{1}{\gamma^p} w_{0j}(y) + \left( \frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} w_{1j}(y) + \theta(y),
\]

\( (2.23) \)

with

\[
\theta(y) = O(\varepsilon |y - \xi'_j|) + O(\varepsilon^\alpha).
\]

In the rest of this paper, we will look for solutions for problem (2.15) in the form \( v = V_\lambda + \phi \), where \( \phi \) will represent a lower order correction. For small \( \phi \), we can rewrite problem (2.15) as a nonlinear perturbation of its linearization, namely,

\[
\begin{cases}
L(\phi) = E_\lambda + N(\phi), & x \in \Omega_\varepsilon; \\
\frac{\partial \phi}{\partial \nu} = 0, & x \in \partial \Omega_\varepsilon,
\end{cases}
\]

\( (2.24) \)

where

\[
L(\phi) := -\Delta \phi + \varepsilon^2 \phi - W \phi, \quad \text{with } W = f'(V_\lambda),
\]

\( (2.25) \)

\[
E_\lambda = \Delta V_\lambda + f(V_\lambda) - \varepsilon^2 V_\lambda + 4\varepsilon^2 \log \varepsilon,
\]

\( (2.26) \)

and

\[
N(\phi) = f(V_\lambda + \phi) - f(V_\lambda) - f'(V_\lambda) \phi.
\]

\( (2.27) \)

For any \( h \in L^\infty(\Omega_\varepsilon) \), let us define a weighted \( L^\infty \)-norm defined as

\[
\|h\|_* := \sup_{y \in \Omega_\varepsilon} \left( \sum_{j=1}^m \left( 1 + |y - \xi'_j| \right)^{2-\sigma} \varepsilon^2 \right)^{-1} |h(y)|
\]

\( (2.28) \)

where we fix \( 0 < \sigma < 1 \). With respect to this norm, the error term \( E_\lambda \) given in (2.26) can be estimated in the following way.

**Lemma 2.3.** Let \( \delta > 0 \) be a small but fixed number and assume that the points \( \xi = (\xi_1, \ldots, \xi_m) \in M_\delta \). There exists \( C > 0 \), such that we have

\[
\|E_\lambda\|_* \leq C \frac{\gamma^{3p}}{|\log \varepsilon|^3}
\]

\( (2.29) \)

for all \( \lambda \) small enough.

**Proof.** First we observe that

\[
-\varepsilon^2 V_\lambda + 4\varepsilon^2 \log \varepsilon = O(\varepsilon^2).
\]

\( (2.30) \)
Far away from the points $\xi_j$, namely for $|x - \xi_j| > \delta$, i.e. $|y - \xi_j| > \frac{\delta}{\varepsilon}$, for all $j = 1, \ldots, m$, from (2.14) we have that

$$\Delta V_\lambda(y) = p \gamma^{p-1} e^2 \Delta U(y) = O(\gamma^{p-1} \varepsilon^4).$$

On the other hand, in this region we have

$$1 + \frac{V_\lambda(y)}{p \gamma p} = 1 + \frac{4 \log \varepsilon + O(1)}{p \gamma p} = \frac{O(1)}{|\log \varepsilon|}$$

(2.31)

where $O(1)$ denotes a smooth function, uniformly bounded, as $\varepsilon \to 0$, in the considered region. Hence

$$f(V_\lambda) = \left(1 + \frac{V_\lambda}{p \gamma p}\right)^{p-1} e^{\gamma p [(1 + \frac{V_\lambda}{p \gamma p})^{p-1} - 1]} = \frac{\varepsilon^\frac{4}{p}}{|\log \varepsilon|^{p-1}} O(1).$$

Thus if we are far away from the points $\xi_j$, or equivalently for $|y - \xi_j| > \frac{\delta}{\varepsilon}$, the size of the error, measured with respect to the $\| \cdot \|_\infty$-norm, is relatively small. In other words, if we denote by $1_{\text{outer}}$ the characteristic function of the set $\{y : |y - \xi_j| > \frac{\delta}{\varepsilon}, j = 1, \ldots, m\}$, then in this region we have

$$\| E_{1_{\text{outer}}} \|_\infty \leq C \frac{\varepsilon^{\frac{2(2-p)}{p}}}{|\log \varepsilon|^{p-1}}.$$ (2.32)

Let us now fix the index $j$ in $\{1, \ldots, m\}$, for $|y - \xi_j| < \frac{\delta}{\varepsilon}$, we have

$$\Delta V_\lambda(y) = -e^{w_j(y)} + \frac{p - 1}{p} \frac{1}{\gamma p} \Delta w_{0j}(y) + \left(\frac{p - 1}{p}\right)^2 \frac{1}{\gamma^{2p}} \Delta w_{1j}(y) + O(\varepsilon^2).$$ (2.33)

On the other hand, for any $R > 0$ large but fixed, in the ball $|y - \xi_j| < R \varepsilon := R |\log \varepsilon|^\alpha$, with $\alpha \geq 3$, we can use Taylor expansion to first get

$$\left(1 + \frac{V_\lambda}{p \gamma p}\right)^{p-1} = 1 + \frac{p - 1}{p} \frac{1}{\gamma p} w_j + \left(\frac{p - 1}{p}\right)^2 \frac{1}{\gamma^{2p}} \left[w_{0j} + \frac{p - 2}{2(p - 1)} (w_j)^2\right]$$

$$+ \left(\frac{p - 1}{p}\right)^3 \frac{1}{\gamma^{3p}} (\log |y - \xi_j|),$$

$$\gamma^p \left[\left(1 + \frac{V_\lambda}{p \gamma p}\right)^p - 1\right] = w_j + \left(\frac{p - 1}{p}\right) \frac{1}{\gamma p} \left[w_{0j} + \frac{(w_j)^2}{2}\right]$$

$$+ \left(\frac{p - 1}{p}\right)^2 \frac{1}{\gamma^{2p}} \left[w_{1j} + w_j w_{0j} + \frac{1}{3} (w_{0j} + (w_j)^2)\right]$$

and

$$e^{\gamma p [(1 + \frac{V_\lambda}{p \gamma p})^{p-1} - 1]} = e^{w_j} \left[1 + \left(\frac{p - 1}{p}\right) \frac{1}{\gamma p} \left[w_{0j} + \frac{(w_j)^2}{2}\right]$$

$$+ \left(\frac{p - 1}{p}\right)^2 \frac{1}{\gamma^{2p}} \left[w_{1j} + w_j w_{0j} + \frac{1}{3} (w_{0j} + (w_j)^2)\right] + \frac{1}{3} (\log |y - \xi_j|)\right].$$
Thus we obtain

\[
    f(V_\lambda) = \left(1 + \frac{V_\lambda}{pY^p}\right)^{p-1} e^{\gamma p (1 + \frac{\nu_j}{pY^p})^{p-1}}
    \]

\[
    = e^{w_j} \left\{ 1 + \left(\frac{p-1}{p}\right) \frac{1}{\gamma^p} \left[ w_{0j} + \frac{(w_j)^2}{2} + w_j \right] + \left(\frac{p-1}{p}\right)^2 \frac{1}{\gamma^{2p}} \left[ w_{1j} + 2w_j w_{0j} + \frac{1}{2} \left( w_{0j} + \frac{(w_j)^2}{2} \right)^2 + w_{0j} \right] + \frac{p-2}{2(p-1)} (w_j)^2 + \frac{(w_j)^3}{2} \right\} + O \left( \frac{\log |y - \xi'_j|}{\gamma^3p} \right).\]

Thus, thanks to the fact that we have improved our original approximation with the terms \(w_{0j}\) and \(w_{1j}\), and the definition of \(\ast\)-norm, we get that

\[
    \| E_{1B(\xi'_j, R_\epsilon)} \|_\ast \leq C \frac{\gamma^3}{\epsilon^3}, \quad \text{for any } j = 1, \ldots, m. \tag{2.34}
\]

Here \(1_{B(\xi'_j, R_\epsilon)}\) denotes the characteristic function of \(B(\xi_j, R_\epsilon)\). Finally, in the remaining region, namely where \(R_\epsilon < |y - \xi_j| < \frac{\delta}{\epsilon}\), for any \(j = 1, \ldots, m\), we have from one hand that \(|\Delta V_\lambda(y)| \leq Ce^{w_j(y)}\), and also \(|f(V_\lambda(y))| \leq Ce^{w_j(y)}\) as consequence of (2.18). This fact, together with (2.34) and (2.32), (2.30) we obtain estimate (2.29). \(\square\)

By the above computation, we find that very close to the point \(\xi_j\) in \(\Omega\), we have

\[
    \| f'(V_\lambda) - e^{w_j} \|_\ast \to 0 \quad \text{as } \lambda \to 0, \tag{2.35}
\]

and there exists some positive constant \(D_0\) such that

\[
    f'(V_\lambda) \leq D_0 \sum_{j=1}^m e^{w_j}. \tag{2.36}
\]

Moreover, we can get

\[
    \| f''(V_\lambda) \|_\ast \leq C. \tag{2.37}
\]

**Proof of (2.35) and (2.36).** We have

\[
    f'(V_\lambda) = \frac{p-1}{p} \frac{1}{Y^p} \left(1 + \frac{V_\lambda}{pY^p}\right)^{p-2} e^{\gamma p [1 + \frac{\nu_j}{pY^p}]^{p-1}} + \left(1 + \frac{V_\lambda}{pY^p}\right)^{2(p-1)} e^{\gamma p [1 + \frac{\nu_j}{pY^p}]^{p-1}} := I_a + I_b.
\]

Far away from the points \(\xi_j\), namely for \(|x - \xi_j| > \delta, i.e. |y - \xi'_j| > \frac{\delta}{\epsilon}\), for all \(j = 1, \ldots, m\), a consequence of (2.31) is that
large but fixed, we use Taylor expansion to get

\[ I_a = \frac{\varepsilon^{\frac{4}{p}}}{|\log \varepsilon|^{p-1}} O(1) \quad \text{and} \quad I_b = \frac{\varepsilon^{\frac{4}{p}}}{|\log \varepsilon|^{2(p-1)}} O(1). \]

Then we have

\[ f'(V_\lambda) 1_{\text{outer}} = \frac{\varepsilon^{\frac{4}{p}}}{|\log \varepsilon|^{p-1}} O(1). \tag{2.38} \]

On the other hand, fix the index \( j \) in \( \{1, \ldots, m\} \), for \( |y - \xi_j'| < R_\varepsilon \) with \( R_\varepsilon = R|\log \varepsilon| \), for any \( R > 0 \) large but fixed, we use Taylor expansion to get

\[
I_a = \frac{p-1}{p} \frac{1}{\gamma p} \left[ 1 + \frac{1}{p \gamma^p} \left( w_j(y) + \frac{p-1}{p} \gamma^p W_0(y) + \left( \frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} W_1(y) + \theta(y) \right) \right]^{p-2}
\times e^{\gamma \left[ 1 + \frac{1}{p \gamma^p} \left( w_j(y) + \frac{p-1}{p} \gamma^p W_0(y) + \left( \frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} W_1(y) + \theta(y) \right) \right]}^p - 1
= \frac{p-2}{p} \frac{1}{\gamma^p} \left[ \frac{p-1}{p} \frac{1}{\gamma^p} w_j(y) + \left( \frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} W_0(y) + \left( \frac{p-1}{p} \right)^3 \frac{1}{\gamma^{3p}} W_1(y) + \theta(y) \right]
\times e^{w_j(y)} e^{\frac{p-1}{p} \frac{1}{\gamma^p} W_0(y)} e^{\theta(y)} e^{\frac{p-1}{p} \frac{1}{\gamma^{2p}} W_1(y) + \theta(y)^2},
\]

and

\[
I_b = \left[ 1 + \frac{1}{p \gamma^p} \left( w_j(y) + \frac{p-1}{p} \gamma^p W_0(y) + \left( \frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} W_1(y) + \theta(y) \right) \right]^{2(p-1)}
\times e^{\gamma \left[ 1 + \frac{1}{p \gamma^p} \left( w_j(y) + \frac{p-1}{p} \gamma^p W_0(y) + \left( \frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} W_1(y) + \theta(y) \right) \right]}^p - 1
= \left[ 1 + \frac{2(p-1)}{p} \frac{1}{\gamma^p} w_j(y) + 2 \left( \frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} W_0(y) + 2 \left( \frac{p-1}{p} \right)^3 \frac{1}{\gamma^{3p}} W_1(y) + 2 \frac{2(p-1)}{p} \frac{1}{\gamma^p} \theta(y) \right]
\times e^{w_j(y)} e^{\frac{p-1}{p} \frac{1}{\gamma^p} W_0(y)} e^{\theta(y)} e^{\frac{p-1}{p} \frac{1}{\gamma^{2p}} W_1(y) + \theta(y)^2}.
\]

By the definition of \( W_0 \) and \( W_1 \), we get that

\[
I_a 1_{B(\xi'_j, R_\varepsilon)} = \frac{O(1)}{|\log \varepsilon|}, \quad I_b 1_{B(\xi'_j, R_\varepsilon)} - e^{w_j(y)} = \frac{O(1)}{|\log \varepsilon|}. \tag{2.39}
\]

Finally, in the remaining region, namely where for any \( j = 1, \ldots, m \), we have \( R_\varepsilon < |y - \xi_j'| < \frac{\delta}{\varepsilon} \), we have

\[
|I_a| \leq C e^{w_j(y)}, \quad |I_b| \leq C e^{w_j(y)}. \tag{2.40}
\]
Let us now fix the index $j$, but fixed, by Taylor expansion, we have
\[
\| f'(V_\lambda) - e^{w_j} \|_* = O(1) / |\log \epsilon |
\]
which implies (2.35). Combing (2.38), (2.39) with (2.40) we obtain estimate (2.36). \qed

Proof of (2.37). We have
\[
f''(V_\lambda) = \frac{(p-1)(p-2)}{p^2} \frac{1}{\gamma^{2p}} \left[ 1 + \frac{V_\lambda}{p\gamma^{\frac{p}{p-1}}} \right]^{p-3} e^{\gamma^{\frac{p}{p-1}}(1+\frac{V_\lambda}{p\gamma^{\frac{p}{p-1}}})^{p-1}}
\]
\[
+ \frac{3(p-1)}{p} \frac{1}{\gamma^{p}} \left[ 1 + \frac{V_\lambda}{p\gamma^{\frac{p}{p-1}}} \right]^{2p-3} e^{\gamma^{\frac{p}{p-1}}(1+\frac{V_\lambda}{p\gamma^{\frac{p}{p-1}}})^{p-1}}
\]
\[
+ \left[ 1 + \frac{V_\lambda}{p\gamma^{\frac{p}{p-1}}} \right]^{3(p-1)} e^{\gamma^{\frac{p}{p-1}}(1+\frac{V_\lambda}{p\gamma^{\frac{p}{p-1}}})^{p-1}} := I_c + I_d + I_e.
\]

By a similar computation as above: Far away from the points $\xi_j$, namely for $|x - \xi_j| > \delta$, i.e. $|y - \xi_j| > \frac{\delta}{\epsilon}$, for all $j = 1, \ldots, m$, we have
\[
I_c = \frac{\epsilon^4}{|\log \epsilon|^{p-1}} O(1), \quad I_d = \frac{\epsilon^4}{|\log \epsilon|^{2(p-1)}} O(1), \quad \text{and} \quad I_e = \frac{\epsilon^4}{|\log \epsilon|^{3(p-1)}} O(1).
\]
Then
\[
f''(V_\lambda) 1_{\text{outer}} = \frac{\epsilon^4}{|\log \epsilon|^{p-1}} O(1) \tag{2.41}
\]
where again $O(1)$ denotes a function which is uniformly bounded, as $\epsilon \to 0$, in the considered region.

Let us now fix the index $j$ in $[1, \ldots, m]$, for $|y - \xi_j| < R_{\epsilon}$ with any $R_{\epsilon} := R|\log \epsilon|$ for some $R > 0$ large but fixed, by Taylor expansion, we have
\[
I_c = \frac{(p-1)(p-2)}{p^2} \frac{1}{\gamma^{2p}}
\]
\[
\times \left[ 1 + \frac{1}{p\gamma^{\frac{p}{p-1}}} \left( w_j(y) + \frac{p-1}{p} \frac{1}{\gamma^{p}} w_{0j}(y) + \left( \frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} w_{1j}(y) + \theta(y) \right) \right]^{p-3} e^{\gamma^{\frac{p}{p-1}}(1+\frac{1}{p\gamma^{\frac{p}{p-1}}}(w_j(y) + \frac{p-1}{p} \frac{1}{\gamma^{p}} w_{0j}(y) + (\frac{p-1}{p})^2 \frac{1}{\gamma^{2p}} w_{1j}(y) + \theta(y)))^{p-1}}
\]
\[
= \frac{(p-2)(p-3)}{p^2} \frac{1}{\gamma^{2p}} \left( \frac{p-1}{p} + \frac{p-1}{p} \frac{1}{\gamma^{p}} w_j(y) + \left( \frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} w_{0j}(y) \right.
\]
\[
\times e^{w_j(y)e^{\frac{1}{p\gamma^{\frac{p}{p-1}}}(w_j(y) + \frac{p-1}{p} \frac{1}{\gamma^{p}} w_{0j}(y) + (\frac{p-1}{p})^2 \frac{1}{\gamma^{2p}} w_{1j}(y) + \theta(y)))}
\]
\[
\left. + \frac{p-1}{p} \frac{1}{\gamma^{3p}} w_{1j}(y) + \frac{p-1}{p} \frac{1}{\gamma^{p}} \theta(y) \right)
\]
\[
\times e^{w_j(y)e^{\frac{1}{p\gamma^{\frac{p}{p-1}}}(w_j(y) + \frac{p-1}{p} \frac{1}{\gamma^{p}} w_{0j}(y) + (\frac{p-1}{p})^2 \frac{1}{\gamma^{2p}} w_{1j}(y) + \theta(y)))^2}
\].
\[ I_d = \frac{3(p-1)}{p} \frac{1}{\gamma^p} \left[ 1 + \frac{1}{p \gamma^p} \left( w_j(y) + \frac{p-1}{p} \frac{1}{\gamma^p} w_0(y) + \left( \frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} w_1(y) + \theta(y) \right) \right]^{2p-3} \times e^{\gamma^p \left[ (1 + \frac{1}{p \gamma^p}) (w_j(y) + \frac{p-1}{p} \frac{1}{\gamma^p} w_0(y) + \left( \frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} w_1(y) + \theta(y)) \right]^{p-1}} = \frac{3(2p-3)}{p} \frac{1}{\gamma^p} \left[ \frac{p-1}{2p-3} + \frac{p-1}{p} \frac{1}{\gamma^p} w_j(y) + \left( \frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} w_0(y) \right]^{2p-3} \times e^{\gamma^p \left[ (1 + \frac{1}{p \gamma^p}) (w_j(y) + \frac{p-1}{p} \frac{1}{\gamma^p} w_0(y) + \left( \frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} w_1(y) + \theta(y)) \right]^{p-1}} \times e^{\gamma\frac{1}{p} \frac{1}{\gamma^p} w_0(y) \left( \frac{p-1}{p} \frac{1}{\gamma^p} \right)^2 \frac{1}{\gamma^{2p}} w_1(y) + \theta(y)} \times e^{\gamma\frac{1}{p} \frac{1}{\gamma^p} \left( w_j(y) + \frac{p-1}{p} \frac{1}{\gamma^p} w_0(y) + \left( \frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} w_1(y) + \theta(y) \right)^2} \times e^{\gamma\frac{1}{p} \frac{1}{\gamma^p} \left( w_j(y) + \frac{p-1}{p} \frac{1}{\gamma^p} w_0(y) + \left( \frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} w_1(y) + \theta(y) \right)^2}.

and

\[ I_e = \left[ 1 + \frac{1}{\gamma^p} \left( w_j(y) + \frac{p-1}{p} \frac{1}{\gamma^p} w_0(y) + \left( \frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} w_1(y) + \theta(y) \right) \right]^{3(p-1)} \times e^{\gamma^p \left[ (1 + \frac{1}{p \gamma^p}) (w_j(y) + \frac{p-1}{p} \frac{1}{\gamma^p} w_0(y) + \left( \frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} w_1(y) + \theta(y)) \right]^{p-1}} = \left[ 1 + \frac{3(p-1)}{p} \frac{1}{\gamma^p} w_j(y) + \frac{3}{p} \frac{1}{\gamma^{2p}} w_0(y) + \left( \frac{3(p-1)}{p} \right)^2 \frac{1}{\gamma^{3p}} w_1(y) \right]^{3(p-1)} \times e^{\gamma\frac{1}{p} \frac{1}{\gamma^p} \left( w_j(y) + \frac{p-1}{p} \frac{1}{\gamma^p} w_0(y) + \left( \frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} w_1(y) + \theta(y) \right)^2} \times e^{\gamma\frac{1}{p} \frac{1}{\gamma^p} \left( w_j(y) + \frac{p-1}{p} \frac{1}{\gamma^p} w_0(y) + \left( \frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} w_1(y) + \theta(y) \right)^2}.

Therefore, we get

\[ I_c^1 B_{(\xi_j, R_c)} = O\left( \frac{1}{\log \varepsilon} \right), \quad I_d^1 B_{(\xi_j, R_c)} = O\left( \frac{1}{\log \varepsilon^2} \right), \quad I_e^1 B_{(\xi_j, R_c)} = O(1). \quad (2.42) \]

Finally, for \( R_c < |y - \xi_j| < \frac{\varepsilon}{2} \), for any \( j \), we have

\[ |I_c| \leq \frac{C}{|\log \varepsilon|}, \quad |I_d| \leq \frac{C}{|\log \varepsilon|^2}, \quad |I_e| = O(1) + C e^{w_j(y)}. \quad (2.43) \]

From (2.41), (2.42) with (2.43), by the definition of *-norm, we obtain that (2.37) holds. \( \square \)

3. The existence result

The operator \( L \) defined in (2.25) can be seen as a superposition of linear operators,

\[ \mathcal{L}_\varepsilon(\phi) = -\Delta \phi - \frac{8}{(1 + |z|^2)^2} \phi, \]

namely, equation \(-\Delta w - e^w = 0\) linearized around the radial solution \( w(y) = \log \frac{8}{(1 + |y|^2)^2} \). The key face to develop a satisfactory solvability theory for the operator \( L \) is the nondegeneracy of \( w \) up to the natural invariances of the equation under translations and dilations. In fact, the functions
Proposition 3.1. Let \( \delta > 0 \) be fixed. There exist positive numbers \( \lambda_0 \) and \( C \), such that for any points \( \xi_j, j = 1, \ldots, m \), in \( \mathcal{M}_\delta \), \( \mu_j \) is given by (2.22), then problem (3.5) has a unique solution \( \phi \) which satisfies
∥φ∥_∞ ≤ \frac{C}{|\log \varepsilon|^2},
for all \lambda < \lambda_0. Moreover, if we consider the map \xi' \mapsto \phi into the space C(\overline{\Omega}_\varepsilon), the derivative D_{\xi'}\phi exists and defines a continuous function of \xi'. Besides, there is a constant C > 0, such that

∥D_{\xi'}\phi∥_∞ ≤ \frac{C}{|\log \varepsilon|}.

(3.6)

In order to find a solution to the original problem we need to find \xi' such that

\sum_{i,j} c_{ij}(\xi') = 0 \quad \text{for all } i = 1, J_j, j = 1, \ldots, m.

(3.7)

This problem is indeed variational: it is equivalent to finding critical points of a function of \xi = \varepsilon \xi'.

Associated to (1.1), let us consider the energy functional

J_{\lambda}(u) = \frac{1}{2} \int_{\Omega} (|\nabla u|^2 + u^2) \, dx - \frac{\lambda}{p} \int_{\Omega} e^{up} \, dx,

and the finite dimensional restriction

F_{\lambda}(\xi) = J_{\lambda}((U_\lambda + \tilde{\phi})(x, \xi)),

where

(U_\lambda + \tilde{\phi})(x, \xi) = \gamma + \frac{1}{p \gamma^{p-1}} \left( (V_\lambda + \phi) \left( \frac{x}{\varepsilon}, \frac{\xi}{\varepsilon} \right) \right)

with V_j defined in (2.17), \phi is the unique solution to problem (3.5) given by Proposition 3.1.

The next result, whose proof is postponed until Section 5.

Proposition 3.2. (i) The functional F_{\lambda}(\xi) is of class C^1. Moreover, for all \lambda > 0 sufficiently small, if D_{\xi} F_{\lambda}(\xi) = 0, then \xi satisfies (3.7).

(ii) Let \delta > 0 be fixed. There exist positive numbers \lambda_0 and C, such that for any points \xi_j, j = 1, \ldots, m in \mathcal{M}_\delta, \mu_j is given by (2.22), the following expansion holds

\lambda^{-1} \varepsilon^{-\frac{2(p-2)}{p}} F_{\lambda}(\xi) = -4\pi (2k + l) \frac{2 - p \log 8}{(2 - p)p} \log \varepsilon - \frac{8\pi}{p} (2k + l) \log \varepsilon - \frac{1}{2(2 - p)} \varphi_m(\xi)

+ O(|\log \varepsilon|^{-1}),

(3.10)

where

\varphi_m(\xi) = \varphi_m(\xi_1, \ldots, \xi_m) = \sum_{j=1}^m c_j^2 H(\xi_j, \xi_j) + \sum_{i \neq j} c_i c_j G(\xi_i, \xi_j).

(3.11)

Proof of Theorem 1.1. First, from the same argument as Lemma 6.1 in [11], we have that

\min_{\partial \mathcal{M}_\delta} \varphi_m(\xi) \to +\infty, \quad \text{as } \delta \to 0.

(3.12)
We state it here for completeness. Let \( \xi = (\xi_1, \ldots, \xi_m) \in \partial M_\delta \). There are two possibilities: either there exists \( j_0 \leq k \) such that \( d(\xi_{j_0}, \partial \Omega) = \delta \), or exists \( i_0 \neq j_0, |\xi_{i_0} - \xi_{j_0}| = \delta \).

In the first case, a consequence of the properties of the Green's function is that for all \( \xi \in \Omega \),

\[
H(\xi, \xi) \geq C \frac{1}{d(\xi, \partial \Omega)}.
\]

In the second case, we may assume that there exists a fixed constant \( C \) such that \( d(\xi_i, \partial \Omega) \geq C \), \( i = 1, \ldots, k \), as otherwise it follows into the first case. But then it is easy to see that

\[
G(\xi_i, \xi_j) \geq C \frac{1}{|\xi_i - \xi_j|}.
\]

Then by (3.13) and (3.14) we obtain (3.12).

From (i) of Proposition 3.2, the function

\[
(U_\lambda + \bar{\phi})(x, \xi) = \gamma + \frac{1}{p^* - 1} \left( (V_\lambda + \phi)(x, \xi) \right)
\]

where \( V_\lambda \) defined by (2.17) and \( \phi(\xi) \) is the unique solution of problem (3.5), is a solution of problem (1.1) if we adjust \( \xi \) so that it is a critical point of \( F_\lambda(\xi) \) defined by (3.9). This is equivalent to finding a critical point of

\[
\tilde{F}_\lambda(\xi) := a \lambda^{-1} e^{\frac{2d - m}{p}} F_\lambda(\xi) + b + c \log \varepsilon,
\]

for suitable constants \( a, b \) and \( c \). On the other hand, from (ii) of Proposition 3.2, for \( \xi \in M_\delta \), we have that

\[
\tilde{F}_\lambda(\xi) = \varphi_m(\xi) + O(|\log \varepsilon|^{-1}) \Theta_\lambda(\xi),
\]

where \( \varphi_m \) is given by (3.11), and \( \Theta_\lambda(\xi) \) is uniformly bounded in consider region as \( \lambda \to 0 \).

From (3.12), the function \( \varphi_m \) is \( C^1 \), bounded from below in \( M_\delta \), we have that, for \( \delta \) is arbitrarily small, \( \varphi_m \) has an absolute minimum in \( M_\delta \). This implies that \( \tilde{F}_\lambda \) also has an absolute minimum \( (\xi_1^*, \ldots, \xi_m^*) \in M_\delta \) such that

\[
\lim_{\lambda \to 0} \varphi_m(\xi_1^*, \ldots, \xi_m^*) = \min_{M_\delta} \varphi_m.
\]

Moreover, while (1.9) holds as a direct consequence of the construction of \( U_\lambda \), and Theorem 1.1(3) holds from (ii) of Proposition 3.2. \( \square \)

**Remark 3.1.** Using Ljusternik–Schnirelmann theory, one can get a second, distinct solution satisfying Theorem 1.1. The proof is similar to [7].

### 4. The finite dimensional reduction

This section is devoted to the proof of Proposition 3.1. Given \( h \in L^\infty(\Omega_\varepsilon) \), we first consider the problem of finding a function \( \phi \) such that for certain scalars \( c_{ij} \), it satisfies
First we show the following result.

**Proposition 4.1.** Let $\delta > 0$ be fixed. There exist positive numbers $\lambda_0$ and $C$, such that for any points $\xi_j$, $j = 1, \ldots, m$, in $\mathcal{M}_\delta$, $\mu_j$ is given by (2.22), and $h \in L^\infty(\Omega_\varepsilon)$, there is a unique solution $\phi := T_\lambda(h)$ to problem (4.1) for all $\lambda \leq \lambda_0$. Moreover,

$$
\|\phi\|_\infty \leq C \left( \log \frac{1}{\varepsilon} \right) \|h\|_*.
$$

(4.2)

The proof will be split into a series of lemmas which we state and prove next.

**Lemma 4.1.** There exist constants $R_1 > 0$, $C > 0$ such that for $\lambda > 0$ small enough and for any points $\xi_j \in \tilde{\Omega}$, $j = 1, \ldots, m$, in $\mathcal{M}_\delta$, set $\tilde{\Omega}_\varepsilon = \Omega_\varepsilon \setminus \bigcup_{j=1}^m B(\xi'_j, R_1)$, we have

$$
\psi : \tilde{\Omega}_\varepsilon \to [1, \infty)
$$

smooth and positive verifying

$$
L(\psi) := -\Delta \psi + \varepsilon^2 \psi - W \psi \geq \sum_{j=1}^m \frac{1}{|y - \xi'_j|^{2+\sigma}} + \varepsilon^2 \quad \text{in} \ \tilde{\Omega}_\varepsilon,
$$

with

$$
\frac{\partial \psi}{\partial \nu} \geq 0 \quad \text{on} \ \partial \tilde{\Omega}_\varepsilon, \quad \psi > 0 \quad \text{in} \ \tilde{\Omega}_\varepsilon.
$$

Moreover $\psi$ is bounded uniformly,

$$
1 \leq \psi \leq C \quad \text{in} \ \tilde{\Omega}_\varepsilon.
$$

**Proof.** We take

$$
\psi_{1j}(r) = 1 - \frac{1}{r^{\sigma}}, \quad \text{where} \ r = |y - \xi'_j|.
$$

(4.3)

A direct computation shows that, we have

$$
-\Delta \psi_{1j} = \sigma^2 \frac{1}{r^{2+\sigma}}.
$$
If $\xi'_j \in \Omega_\varepsilon$, then we have
\[
\frac{\partial \psi_{1j}}{\partial \nu} = O(\varepsilon^{1+\sigma}).
\]

If $\xi'_j \in \Omega_\varepsilon$ and $|y - \xi'_j| > R$, we have
\[
\frac{\partial \psi_{1j}}{\partial \nu} = \sigma \frac{(y - \xi'_j) \cdot \nu}{r^{2+\sigma}}.
\]

We write the boundary $\partial \Omega_\varepsilon$ near point $\xi'_j$ as the graph $\{(y_1, y_2): y_2 = G_\varepsilon(y_1)\}$ with $G_\varepsilon(y_1) = \frac{1}{\varepsilon} G(\varepsilon y_1)$ and $G$ a smooth function such that $G(0) = 0$ and $G'(0) = 0$. Fix $\delta > 0$ small. Then for $R_1 < r < \delta/\varepsilon$ we have that $r$ is comparable with $y_1$, $G'_\varepsilon(y_1) = O(\varepsilon r)$ and $G_\varepsilon(y_1) = O(\varepsilon r^2)$. Then
\[
\frac{\partial \psi_{1j}}{\partial \nu} = \frac{\sigma}{r^{2+\sigma}} \frac{1}{\sqrt{G'_\varepsilon(y_1)^2 + 1}} (-y_1 G'_\varepsilon(y_1) + G_\varepsilon(y_1))
= \frac{\sigma}{r^{2+\sigma}} \frac{1}{\sqrt{O(\delta^2) + 1}} O(\varepsilon r^2)
= O\left(\frac{\varepsilon}{r^\sigma}\right), \quad \forall R_1 < r < \frac{\delta}{\varepsilon}.
\]

Hence, we obtain that
\[
\frac{\partial \psi_{1j}}{\partial \nu} = o(\varepsilon), \quad \text{on } \Omega_\varepsilon.
\]

Next, let us define
\[
\psi = \sum_{j=1}^m \psi_{1j} + C \psi_0,
\]
where $\psi_0$ is the solution of the following problem:
\[
-\Delta \psi_0 + \varepsilon^2 \psi_0 = \varepsilon^2 \quad \text{in } \Omega_\varepsilon; \quad \frac{\partial \psi_0}{\partial \nu} = \varepsilon \quad \text{on } \partial \Omega_\varepsilon.
\]

It is directly checked that $\frac{\partial}{\sigma^2} \psi$ satisfies the required condition. □

**Lemma 4.2.** The operator $L$ satisfies the maximum principle in $\tilde{\Omega}_\varepsilon$ for $R \geq R_1$ large but independent of $\lambda$, with $R_1$ in Lemma 4.1. Namely, if $L(\phi) \geq 0$ in $\Omega_\varepsilon$ and $\phi \geq 0$ on $\partial \Omega_\varepsilon$, then $\phi \geq 0$ in $\Omega_\varepsilon$.

**Proof.** Given $a > 0$, we consider the function
\[
Z(y) = \sum_{j=1}^m z_0(a|y - \xi'_j|), \quad y \in \Omega_\varepsilon,
\]
where $z_0(r) = \frac{r^2 - 1}{r^2 + 1}$ is the radial solution in $\mathbb{R}^2$ of
\[
\Delta z_0 + \frac{8}{(1 + r^2)^2}z_0 = 0.
\]

First, we observe that, if $|y - \xi^j| \geq R$ for $R > \frac{1}{a}$, then $Z(y) > 0$. By the definition of $z_0$ we have
\[
-\Delta Z(y) + \epsilon^2 Z(y) = \sum_{j=1}^{m} \frac{(8a^2 + \epsilon^2)(a^2|y - \xi^j|^2 - 1)}{(1 + a^2|y - \xi^j|^2)^3} \geq \sum_{j=1}^{m} \frac{8a^2 + \epsilon^2}{3(1 + a^2|y - \xi^j|^2)^2} \geq \sum_{j=1}^{m} \frac{4}{27a^2|y - \xi^j|^4}
\]
provided $R > \frac{\sqrt{2}}{a}$. On the other hand, from (2.35), in the same region, we have
\[
f'(V_\lambda)Z(y) \leq D_0 \sum_{j=1}^{m} e^{w_j}Z(y) \leq \sum_{j=1}^{m} \frac{C}{|y - \xi^j|^4}.
\]

Hence if $a$ is taken small and fixed, and $R > 0$ is chosen sufficiently large depending on this $a$, then we have $L(Z) > 0$ in $\tilde{\Omega}_\epsilon$. Thus the function $Z(y)$ is what we are looking for.  

Let us fix such a number $R > 0$ which we may take large whenever it is needed. Define the following inner norm of $\phi$ in the following way
\[
\| \phi \|_i = \sup_{y \in \bigcup_{j=1}^{m} B(\xi^j, R)} |\phi(y)|.
\]

**Lemma 4.3.** There exists a uniform constant $C > 0$ such that if $L(\phi) = h$ in $\Omega_\epsilon$, $\phi = 0$ on $\partial \Omega_\epsilon$, then
\[
\| \phi \|_\infty \leq C \left[ \| \phi \|_i + \| h \|_* \right],
\]
for any $h \in L^\infty(\Omega_\epsilon)$.

**Proof.** Define now the function
\[
\tilde{\phi}(y) = 2\| \phi \|_i Z(y) + \| h \|_* \psi(y),
\]
where $Z$ is the function defined in (4.4), and the function $\psi$ satisfying the properties of Lemma 4.1. First, observe that by the definition of $Z$, choosing $R$ large if necessary,
\[
\tilde{\phi}(y) \geq 2\| \phi \|_i Z(y) \geq \| \phi \|_i \geq |\phi(y)| \quad \text{for} \quad |y - \xi^j| = R,
\]
and, by the positivity of $Z(y)$ and $\psi(y)$,
\[
\tilde{\phi}(y) \geq 0 = \phi(y) \quad \text{for} \quad y \in \partial \Omega_\epsilon.
\]
Finally, by the definition of $\| \cdot \|_*$ we have that

$$\left| h(y) \right| \leq \left( \sum_{j=1}^{m} (1 + |y - \xi'_j|)^{-2-\sigma} + \varepsilon^2 \right) \| h \|_*,$$

we then have

$$L(\tilde{\phi}) = 2 \| \phi \|_{L(Z)} + \| h \|_* L(\psi)$$

$$\geq \| h \|_* \left( \sum_{j=1}^{m} (1 + |y - \xi'_j|)^{-2-\sigma} + \varepsilon^2 \right)$$

$$\geq \left| h(y) \right| \geq L(\phi)(y),$$

provided $R$ large enough. Hence, from Lemma 4.2, we obtain that

$$\left| \phi(y) \right| \leq \tilde{\phi}(y) \quad \text{for} \quad y \in \tilde{\Omega}_\varepsilon,$$

and, since $Z(y) \leq 1$ and from Lemma 4.1 we get

$$\| \phi \|_\infty \leq C \left[ \| \phi \|_{i} + \| h \|_* \right].$$

Next we prove uniform a priori estimates for the problem (4.1) when $\phi$ satisfies additionally orthogonality under dilations. Specifically, we consider the problem

$$\begin{cases}
L(\phi) = h, & \text{in } \Omega_{\varepsilon}; \\
\frac{\partial \phi}{\partial \nu} = 0, & \text{on } \partial \Omega_{\varepsilon}; \\
\int_{\Omega_{\varepsilon}} \phi Z_{ij} \eta_j = 0, & \text{for } i = 0, \ldots, J_j, \ j = 1, \ldots, m.
\end{cases} \quad (4.6)$$

and prove the following estimate.

**Lemma 4.4.** Let $\delta > 0$ be fixed. There exist positive numbers $\lambda_0$ and $C$, such that for any points $\xi_j, \ j = 1, \ldots, m$, in $\mathcal{M}_\delta$, $u_j$ is given by (2.22), and $h \in L^\infty(\Omega_{\varepsilon})$, and any solution $\phi$ to problem (4.6), one has

$$\| \phi \|_\infty \leq C \| h \|_*.$$  \hspace{1cm} (4.7)

**Proof.** We carry out the proof of lemma by a contradiction. If the result was false, then there exist a sequence $\lambda_n \to 0$, points $\xi^n_j, \ j = 1, \ldots, m$ in $\mathcal{M}_\delta$, function $h_n$ with $\| h_n \|_* \to 0$ and $\phi_n$ with $\| \phi_n \|_\infty = 1$,

$$\begin{cases}
L(\phi_n) = h_n, & \text{in } \Omega_{\varepsilon^n}; \\
\frac{\partial \phi_n}{\partial \nu} = 0, & \text{on } \partial \Omega_{\varepsilon^n}; \\
\int_{\Omega_{\varepsilon^n}} \phi_n Z_{ij} \eta_j = 0, & \text{for all } i = 0, \ldots, J_j, \ j = 1, \ldots, m.
\end{cases} \quad (4.8)$$

Let $\delta > 0$ be fixed. There exist positive numbers $\lambda_0$ and $C$, such that for any points $\xi_j, j = 1, \ldots, m$, in $\mathcal{M}_\delta$, $u_j$ is given by (2.22), and $h \in L^\infty(\Omega_{\varepsilon})$, and any solution $\phi$ to problem (4.6), one has

$$\| \phi \|_\infty \leq C \| h \|_*.$$  \hspace{1cm} (4.7)

**Proof.** We carry out the proof of lemma by a contradiction. If the result was false, then there exist a sequence $\lambda_n \to 0$, points $\xi^n_j, j = 1, \ldots, m$ in $\mathcal{M}_\delta$, function $h_n$ with $\| h_n \|_* \to 0$ and $\phi_n$ with $\| \phi_n \|_\infty = 1$,
Then from Lemma 4.3, we see that \( \| \phi_n \|_i \) stays away from zero. Up to a subsequence, for one of the indices, say \( j \), we can assume that there exists \( R > 0 \) such that,

\[
\sup_{|y - \xi^n_j| < R} |\phi_n(y)| \geq \kappa > 0 \quad \text{for all } n.
\]

Let us set \( \hat{\phi}_n(z) = \phi_n((\xi^n_j)' + z) \). Elliptic estimate allow us to assume that \( \hat{\phi}_n \) converges uniformly over compact subsets of \( \mathbb{R}^2 \) to a bounded, nonzero solution \( \hat{\phi} \) of

\[
\Delta \hat{\phi} + \frac{8 \mu^2_j}{(\mu^2_j + |z|^2)^2} \hat{\phi} = 0.
\]

This implies that \( \hat{\phi} \) is a linear combination of the functions \( z_{ij} \), \( i = 0, \ldots, J_j \). But orthogonality conditions over \( \hat{\phi}_n \) pass to the limit thanks to \( \| \hat{\phi}_n \|_\infty \leq 1 \). By the dominated convergence theorem then yields that

\[
\int_{\Omega_\varepsilon} \eta(|z|) z_{ij} \hat{\phi} = 0 \quad \text{for } i = 0, \ldots, J_j, \text{ thus a contradiction with } \liminf_{n \to \infty} \| \phi_n \|_i > 0. \]

Now we establish a priori estimates for the problem (4.6) with the orthogonality condition \( \int_{\Omega_\varepsilon} \eta_j Z_{0j} \phi = 0 \) dropped. We consider the problem

\[
\begin{aligned}
L(\phi) &= h, \quad &\text{in } \Omega_\varepsilon; \\
\partial \phi \over \partial \nu &= 0, \quad &\text{on } \partial \Omega_\varepsilon; \\
\int_{\Omega_\varepsilon} \eta_j Z_{ij} \phi &= 0, \quad &\text{for } i = 1, J_j, \quad j = 1, \ldots, m,
\end{aligned}
\]

(4.9)

Lemma 4.5. Let \( \delta > 0 \) be fixed. There exist positive numbers \( \lambda_0 \) and \( C \), such that for any points \( \xi_j, j = 1, \ldots, m, \) in \( \mathcal{M}_\delta \), \( \mu_j \) is given by (2.22), and \( h \in L^\infty(\Omega_\varepsilon) \), and any solution \( \phi \) to problem (4.9), one has

\[
\| \phi \|_\infty \leq C \left( \log \frac{1}{\varepsilon} \right) \| h \|_* , \quad (4.10)
\]

for all \( \lambda < \lambda_0 \).

Proof. Let \( \phi \) satisfies (4.9). We modify \( \phi \) to \( \tilde{\phi} \), such that \( \tilde{\phi} \) satisfies all orthogonality condition. For this, let us set \( R > R_0 + 1 \) large and fixed. Set

\[
a_{0j} = \frac{1}{\mu_j \left( \frac{4}{c_j} \log \frac{1}{R \varepsilon} + H(\xi, \xi_j) \right)}.
\]

Define

\[
\hat{Z}_{0j}(y) = Z_{0j}(y) - \frac{1}{\mu_j} + a_{0j} G(\xi_j, \varepsilon y).
\]

We note that the function \( \hat{Z}_{0j} \) satisfies the Neumann boundary condition. Let \( \chi \) be a radial smooth cut-off function on \( \mathbb{R}^2 \) so that \( 0 \leq \chi \leq 1 \), \( |\nabla \chi| \leq C \) in \( \mathbb{R}^2 \), \( \chi \equiv 1 \) in \( B_R(0) \) and \( \chi \equiv 0 \) in \( \mathbb{R}^2 \setminus B_{R+1}(0) \). Set

\[
\chi_j(y) = \chi \left( |y - \xi_j| \right) \quad \text{for } j = 1, \ldots, k; \quad \chi_j(y) = \chi \left( F^*_j(y) \right) \quad \text{for } j = k + 1, \ldots, m. \quad (4.11)
\]
Now, we define
\[ \tilde{Z}_0 = \chi_j Z_0 + (1 - \chi_j) \hat{Z}_0. \]

Given \( \phi \) satisfying (4.9), we set
\[ \tilde{\phi} = \phi + \sum_{j=1}^{m} d_j \tilde{Z}_0, \]
where \( d_j = -\int_{\Omega_\epsilon} \nabla_j Z_0 \phi \int_{\Omega_\epsilon} Z_0^2 \nabla_j \).

Therefore, our result is a direct consequence of the following Claim.

Claim.
\[ |d_j| \leq C |\log \epsilon||h||_* \quad \forall j = 1, \ldots, m. \quad (4.12) \]

First, using the notation \( L = -\Delta + \epsilon^2 I - W \), we observe that \( \tilde{\phi} \) satisfies
\[
\begin{align*}
L(\tilde{\phi}) &= h + \sum_{j=1}^{m} d_j L(\tilde{Z}_0), \quad \text{in } \Omega_\epsilon; \\
\frac{\partial \tilde{\phi}}{\partial \nu} &= 0, \quad \text{on } \partial \Omega_\epsilon. 
\end{align*}
\]

Thus by Lemma 4.4, we have
\[ \| \tilde{\phi} \|_{L^\infty(\Omega_\epsilon)} \leq C \sum_{j=1}^{m} |d_j| \| L(\tilde{Z}_0) \|_* + C \| h \|_* \quad (4.14) \]

Multiplying the first equation in (4.13) by \( \tilde{Z}_0 \) for \( s = 1, \ldots, m \), integrating by parts and using the second equation in (4.13), we find
\[ \sum_{j=1}^{m} d_j \int_{\Omega_\epsilon} L(\tilde{Z}_0) \tilde{Z}_0 \leq C \| h \|_* \left( 1 + \sum_{j=1}^{m} \| L(\tilde{Z}_0) \|_* \right) + C \sum_{j=1}^{m} |d_j| \| L(\tilde{Z}_0) \|^2_* \quad (4.15) \]

Next we estimate the size of \( \| L(\tilde{Z}_0) \|_* \). From (3.4), we have
\[ L(\tilde{Z}_0) = e^{\omega_j} Z_0 - W \tilde{Z}_0 + O(\epsilon (1 + |y - \xi_j|)^3) \]
\[ = e^{\omega_j} \left( \frac{1}{\mu_j} - a_{0j} G(\xi_j, \epsilon y) \right) + O(\epsilon (1 + |y - \xi_j|)^3). \]

Thus, we have
\[ \| (1 - \chi_j) L(\tilde{Z}_0) \|_* \leq \frac{C}{\log(1/\epsilon)}, \]
where \( C \) is a constant, which depends on the chosen large constant \( R \). Hence
\( \mathcal{L}((\tilde{Z}_0)_j) = \chi_j \mathcal{L}(Z_0)_j + (1 - \chi_j) \mathcal{L}(\tilde{Z}_0)_j + 2 \nabla \chi_j \nabla (Z_0)_j - \Delta \chi_j (Z_0)_j - \tilde{Z}_0)_j \)

\[
= O(\varepsilon^{2+\alpha}) + (1 - \chi_j) e^{w_j} \left( \frac{1}{\mu_j} - a_{0j} G(\xi_j, \varepsilon y) \right) + 2 \nabla \chi_j \nabla (Z_0)_j - \Delta \chi_j (Z_0)_j - \tilde{Z}_0)_j.
\] (4.16)

Since, for \( r = |y - \xi'_j| \in (R, R + 1) \), we have

\[
\tilde{Z}_0)_j - Z_0)_j = a_{0j} G(\xi_j, \varepsilon y) - \frac{1}{\mu_j} = a_{0j} \left( 4 \log \frac{1}{\varepsilon} \right) \frac{1}{r} + H(\xi_j, \varepsilon y) - \frac{1}{\mu_j}.
\] (4.17)

Therefore, for \( r = |y - \xi'_j| \in (R, R + 1) \), we have

\[
\nabla (\tilde{Z}_0)_j - Z_0)_j = -\frac{C}{\log(1/\varepsilon)} \frac{1}{r} + O \left( \frac{\varepsilon^\alpha}{\log(1/\varepsilon)} \right).
\] (4.18)

From (4.16), (4.17) and (4.18) we obtain

\[
\| \mathcal{L}((\tilde{Z}_0)_j) \|_\ast \leq \frac{C}{\log(1/\varepsilon)}.
\] (4.19)

Now we estimate the left term of (4.15). From (4.16), we see that for \( j \neq s \),

\[
\int_{\Omega_\varepsilon} \mathcal{L}((\tilde{Z}_0)_j) \tilde{Z}_{0s} = O(\varepsilon^\alpha) + \int_{\Omega_\varepsilon} O \left( \frac{1}{\log(1/\varepsilon)} \left( |\nabla \chi_j| + |\Delta \chi_j| \right) \right) \tilde{Z}_{0s} = O \left( \left( \frac{1}{\log(1/\varepsilon)} \right)^2 \right).
\]

Moreover, for \( j = s \), we have

\[
\int_{\Omega_\varepsilon} \mathcal{L}((\tilde{Z}_{0s}) \tilde{Z}_{0s} = I_1 + I_2 + O(\varepsilon),
\]

where

\[
I_2 = \int_{\Omega_\varepsilon} O(\varepsilon^{2+\alpha}) + (1 - \chi_s) e^{w_j} \left( \frac{1}{\mu_s} - a_{0s} G(\xi_j, \varepsilon y) \right)
\]

\[
= O(\varepsilon^\alpha) + O \left( \frac{1}{\log(1/\varepsilon)} \right)
\]

and
\[ I_1 = \int_{\Omega_{\varepsilon}} \left[ 2 \nabla \chi_s \nabla (Z_{0s} - \hat{Z}_{0s}) + \Delta \chi_s (Z_{0s} - \hat{Z}_{0s}) \right] \hat{Z}_{0s} \]

\[ = \int_{\Omega_{\varepsilon}} \nabla \chi_s (Z_{0s} - \hat{Z}_{0s}) \hat{Z}_{0s} - \int_{\Omega_{\varepsilon}} \nabla \chi_s (Z_{0s} - \hat{Z}_{0s}) \nabla \hat{Z}_{0s} + O(\varepsilon). \]

We observe that in the consider region, \( r \in (R, R + 1) \) with \( r = |y - \xi|^j \), \( |Z_{0s} - \hat{Z}_{0s}| \leq \frac{C}{\log(1/\varepsilon)} \) while \( |
abla Z'_0| \leq \frac{1}{R^3} + \frac{C}{\log(1/\varepsilon)} \). Thus

\[ \left| \int_{\Omega_{\varepsilon}} \nabla \chi_s (Z_{0s} - \hat{Z}_{0s}) \hat{Z}_{0s} \right| \leq \frac{D}{R^3} \frac{1}{\log(1/\varepsilon)}, \]

where \( D \) may be chosen independent of \( R \). Now we have

\[ \int_{\Omega_{\varepsilon}} \nabla \chi_s (Z_{0s} - \hat{Z}_{0s}) \nabla \hat{Z}_{0s} = -\frac{E}{\log(1/\varepsilon)} \left[ 1 + O\left( \frac{1}{R} \right) \right] \]

where \( E \) may be chosen independent of \( \varepsilon \). Thus we choose \( R \) large enough, we then have \( I_1 \sim -\frac{E}{\log(1/\varepsilon)} \). Therefore, we have

\[ \int_{\Omega_{\varepsilon}} L(\hat{Z}_{0s}) \hat{Z}_{0s} = -\frac{E}{\log(1/\varepsilon)} \left[ 1 + O\left( \frac{1}{R} \right) \right], \]

and

\[ \int_{\Omega_{\varepsilon}} L(\hat{Z}_{0j}) \hat{Z}_{0s} = O\left( \frac{1}{R \log(1/\varepsilon)} \right) \text{ for } j \neq s. \]

Thus, we obtain that (4.12) holds. This finishes the proof of Lemma 4.5. \( \square \)

**Proof of Proposition 4.1.** We first establish the validity of the a priori estimate (4.2). The previous lemma yields

\[ ||\phi||_{\infty} \leq C \left( \log \frac{1}{\varepsilon} \right) \left[ \|h\|_* + \sum_{i=1}^J \sum_{j=1}^m |c_{ij}| \right]. \]  \( \text{ (4.20) } \)

Let \( \chi_j \) be a smooth cut-off function defined as (4.11). We multiply the first equation of (4.1) by \( Z_{ij} \chi_j \), we find

\[ \{ L(\phi), Z_{ij} \chi_j \} = \{ h, Z_{ij} \chi_j \} + c_{ij} \int_{\Omega_{\varepsilon}} \eta_j \chi_j |Z_{ij}|^2. \]  \( \text{ (4.21) } \)

We have

\[ -L(Z_{ij} \chi_j) = \Delta \chi_j Z_{ij} + 2 \nabla Z_{ij} \nabla \chi_j + \varepsilon O((1 + r)^{-3}). \]
with \( r = |y - \xi_j'|. \) Since \( \Delta \chi_j = O(\varepsilon^2), \nabla \chi_j = O(\varepsilon), \) and \( Z_{ij} = O(r^{-1}), \nabla Z_{ij} = O(r^{-2}), \) we get

\[-L(Z_{ij} \chi_j) = O(\varepsilon^3) \varepsilon O((1 + r)^{-3}).\]

Then we have

\[
|\langle L(\phi), Z_{ij} \chi_j \rangle| = |\langle \phi, L(Z_{ij} \chi_j) \rangle| \leq C \| \phi \|_{\infty}.
\]

Combining this with (4.20) and (4.21) we find

\[
|c_{ij}| \leq C \left[ \| h \|_{\ast} + \varepsilon \left( \log \frac{1}{\varepsilon} \right) \sum_{a,b} |c_{ab}| \right].
\]

Then, \( |c_{ij}| \leq C \| h \|_{\ast}. \) Combining this with (4.20) we obtain the estimate (4.2) holds.

Next prove the solvability of problem (4.1). To this purpose we consider the space

\[
\mathbb{H} = \{ \phi \in H^1(\Omega_{\varepsilon}) : \int_{\Omega_{\varepsilon}} \phi Z_{ij} \eta_j = 0 \text{ for } i = 1, J_j, j = 1, 2, \ldots, m \},
\]

endowed with the usual inner product \( \langle \phi, \psi \rangle = \int_{\Omega_{\varepsilon}} (\nabla \phi \nabla \psi + \varepsilon^2 \phi \psi). \) Problem (4.1), expressed in a weak form, is equivalent to find \( \phi \in \mathbb{H} \) such that

\[
\langle \phi, \psi \rangle = \int_{\Omega_{\varepsilon}} (W \phi + h) \psi \, dx, \quad \text{for all } \psi \in \mathbb{H}.
\]

With the aid of Riesz's representation theorem, this equation gets rewritten in \( \mathbb{H} \) in the operator form

\[
(Id - K)\phi = \tilde{h},
\]

for certain \( \tilde{h} \in \mathbb{H}, \) where \( K \) is a compact operator in \( \mathbb{H}. \) The homogeneous equation \( \phi = K\phi \) in \( \mathbb{H}, \) which is equivalent to (4.1) with \( h \equiv 0, \) has only the trivial solution in view of the a priori estimate (4.2). Now, Fredholm's alternative guarantees unique solvability of (4.22) for any \( \tilde{h} \in \mathbb{H}. \) This finishes the proof. □

The result of Proposition 4.1 implies that the unique solution \( \phi = T_{\lambda}(h) \) of (4.1) defines a continuous linear map from the Banach space \( C_\ast \) of all functions \( h \) in \( L^\infty \) for which \( \| h \|_{\ast} < \infty \) into \( L^\infty, \) with norm bounded uniformly in \( \lambda. \)

**Lemma 4.6.** *The operator \( T_{\lambda} \) is differentiable with respect to the variable \( \xi_a \) in \( \tilde{\Omega} \) with \( \xi \in M_{\delta}, \) one has the estimate

\[
\left\| \partial_{(\xi_a')}^b T_{\lambda}(h) \right\|_{\infty} \leq C \left( \log \frac{1}{\varepsilon} \right)^2 \| h \|_{\ast} \quad \text{for } b = 1, J_j, a = 1, 2, \ldots, m.
\]

**Proof.** Differentiating Eq. (4.1), formally \( Z := \partial_{(\xi_a')}^b \phi \) should satisfy

\[
L(Z) = -\partial_{(\xi_a')}^b W \phi + \sum_{i=1}^{J_j} c_{ia} \partial_{(\xi_a')}^b (\eta_a Z_{ia}) + \sum_{i=1}^{J_j} \sum_{j=1}^{m} d_{ij} Z_{ij} \eta_j
\]
with \( d_{ij} = d_{ij}^a c_{ij} \), and the orthogonality conditions now become
\[
\int_{\Omega_\varepsilon} Z_{ij} \eta_j Z = -\int_{\Omega_\varepsilon} d_{ij}^a (Z_{ij} \eta_j) \phi.
\]

We consider the constants \( b_{ia} \) defined as
\[
b_{ia} \int_{\Omega_\varepsilon} \eta_a Z_{ia}^2 = \int_{\Omega_\varepsilon} d_{ia}^a (Z_{ia} \eta_a) \phi.
\]

Define
\[
\tilde{Z} = Z + \sum_{i=1}^{J_j} b_{ia} \eta_a Z_{ia},
\]
and
\[
g = -d_{i(\xi_a b)} W \phi + \sum_{i=1}^{J_j} c_{ia} d_{i(\xi_a b)} (Z_{ia} \eta_2 a) + \sum_{i=1}^{J_j} b_{ia} L (\eta_2 a Z_{ia}).
\]

We then have
\[
\begin{cases}
L(\tilde{Z}) = g + \sum_{i=1}^{J_j} \sum_{j=1}^{m} b_{ia} \eta_a Z_{ia}, & \text{in } \Omega_\varepsilon; \\
\tilde{Z} = 0, & \text{on } \partial \Omega_\varepsilon; \\
\int_{\Omega_\varepsilon} Z_{ia} \eta_a \tilde{Z} = 0, & \text{for } i = 0, \ldots, J_j, a = 1, \ldots, m.
\end{cases}
\]

Furthermore, \( \tilde{Z} = T_\lambda(g) \). Using the result of Proposition 4.1 we find that
\[
\|g\|_* \leq C \left( \log \frac{1}{\varepsilon} \right) \|h\|_*.
\]
hence,
\[
\|\partial_{(\xi_a b)} T_\lambda(h)\|_\infty \leq C \left( \log \frac{1}{\varepsilon} \right)^2 \|h\|_* \text{ for } b = 1, J_j, a = 1, 2, \ldots, m. \quad \square
\]

Next, we will prove Proposition 3.1.

**Proof of Proposition 3.1.** In terms of the operator \( T_\lambda \) defined in Proposition 3.1, problem (3.5) becomes
\[
\phi = T_\lambda (N(\phi) + E_\lambda) := A(\phi).
\]

(4.24)
For a given number $M > 0$ let us consider the region

$$
\mathcal{F}_M := \left\{ \phi : \|\phi\|_\infty \leq \frac{M}{|\log \varepsilon|^2} \right\}.
$$

From Proposition 4.1, we get

$$
\|A(\phi)\|_\infty \leq C|\log \varepsilon| \left[ \|N(\phi)\|_* + \|E_\lambda\|_* \right].
$$

From Lemma 2.3, we have $\|E_\lambda\|_* \leq \frac{C}{|\log \varepsilon|^2}$. And, by the definition of $N(\phi)$ in (2.27), and from (2.37) then we have

$$
\|N(\phi)\|_* \leq C\|\phi\|_\infty^2.
$$

Thus

$$
\|A(\phi)\|_\infty \leq C|\log \varepsilon| \left( C\|\phi\|_\infty^2 + \frac{1}{|\log \varepsilon|^3} \right).
$$

We then get that $A(\mathcal{F}_M) \subset \mathcal{F}_M$ for a sufficiently large but fixed $M$ and all small $\lambda$. Moreover, for any $\phi_1, \phi_2 \in \mathcal{F}_M$, one has

$$
\|N(\phi_1) - N(\phi_2)\|_* \leq C \left( \max_{i=1,2} \|\phi_i\|_\infty \right) \|\phi_1 - \phi_2\|_\infty.
$$

In fact,

$$
N(\phi_1) - N(\phi_2) = f(V_\lambda + \phi_1) - f(V_\lambda + \phi_2) - f'(V_\lambda)(\phi_1 - \phi_2)
$$

$$
= \int_0^1 \left( \frac{d}{dt} f(V_\lambda + \phi_2 + t(\phi_1 - \phi_2)) \right) dt - f'(V_\lambda)(\phi_1 - \phi_2)
$$

$$
= \int_0^1 (f'(V_\lambda + \phi_2 + t(\phi_1 - \phi_2)) - f'(V_\lambda)) dt (\phi_1 - \phi_2).
$$

Thus, for a certain $t^* \in (0,1)$, and $s \in (0,1),

$$
|N(\phi_1) - N(\phi_2)| \leq C \left| f'(V_\lambda + \phi_2 + t^*(\phi_1 - \phi_2)) - f'(V_\lambda) \right| \|\phi_1 - \phi_2\|_\infty
$$

$$
\leq C \left| f''(V_\lambda + s\phi_2 + t^*(\phi_1 - \phi_2)) \right| (\|\phi_1\|_\infty + \|\phi_2\|_\infty) \|\phi_1 - \phi_2\|_\infty.
$$

Thanks to (2.37) and the fact that $\|\phi_1\|_\infty, \|\phi_2\|_\infty \to 0$ as $\lambda \to 0$, we conclude that

$$
\|N(\phi_1) - N(\phi_2)\|_* \leq C \left( \|f''(V_\lambda)\|_* + (\|\phi_1\|_\infty + \|\phi_2\|_\infty) \|\phi_1 - \phi_2\|_\infty \right)
$$

$$
\leq C (\|\phi_1\|_\infty + \|\phi_2\|_\infty) \|\phi_1 - \phi_2\|_\infty.
$$
Then we have
\[
\| A(\phi_1) - A(\phi_2) \|_\infty \leq C |\log \varepsilon| \| N(\phi_1) - N(\phi_2) \|_\infty \leq C |\log \varepsilon| \left( \max_{i=1,2} \| \phi_i \|_\infty \right) \| \phi_1 - \phi_2 \|_\infty.
\]
Thus the operator \( A \) has a small Lipschitz constant in \( \mathcal{F}_M \) for all small \( \lambda \), and therefore a unique fixed point of \( A \) exists in this region.

We shall next analyze the differentiability of the map \( \xi' = (\xi'_1, \ldots, \xi'_m) \mapsto \phi \). Assume for instance that the partial derivative \( \partial(\xi'_j) \phi \) exists for \( i = 1, J j \).

Since \( \phi = T_\lambda(N(\phi) + E_\lambda) \), formally that
\[
\partial(\xi'_j) \phi = (\partial(\xi'_j) T_\lambda)(N(\phi) + E_\lambda) + T_\lambda(\partial(\xi'_j) N(\phi) + \partial(\xi'_j) E_\lambda).
\]
From Lemma 4.6, we have
\[
\| \partial(\xi'_j) T_\lambda(N(\phi) + E_\lambda) \|_\infty \leq C |\log \varepsilon| |N(\phi) + E_\lambda|_\infty \leq C \frac{1}{|\log \varepsilon|}.
\]
On the other hand,
\[
\partial(\xi'_j) N(\phi) = \left( f'(V_\lambda + \phi) - f'(V_\lambda) - f''(V_\lambda) \phi \right) \partial(\xi'_j) V_\lambda + \partial(\xi'_j) \left[ f'(V_\lambda) - e^{w_j} \right] \phi.
\]
Then,
\[
\| \partial(\xi'_j) N(\phi) \|_\infty \leq C \left\{ \phi \|_\infty^2 + \frac{1}{|\log \varepsilon|} \phi \|_\infty + \| \partial(\xi'_j) \phi \|_\infty + \| \partial(\xi'_j) \phi \|_\infty \right\}.
\]
Since \( \| \partial(\xi'_j) E_\lambda \|_\infty \leq C |\log \varepsilon| \), and by Proposition 4.1 we then have
\[
\| \partial(\xi'_j) \phi \|_\infty \leq C |\log \varepsilon|,
\]
for all \( i = 1, J j, j = 1, \ldots, m \). Then, the regularity of the map \( \xi' \mapsto \phi \) can be proved by standard arguments involving the implicit function theorem and the fixed point representation (4.24). This concludes proof of Proposition 3.1.

5. Variational reduction: Proof of Proposition 3.2

In this section, we prove Proposition 3.2.

Proof of (i) of Proposition 3.2. A direct consequence of the results obtained in Proposition 3.1 and the definition of function \( U_\lambda \) is the fact the map \( \xi \mapsto F_\lambda(\xi) \) is of class \( C^1 \).

Define
\[
I_\lambda(v) = \frac{1}{2} \int_{\Omega_\varepsilon} \left( |\nabla v|^2 + \varepsilon^2 v^2 \right) dy - \int_{\Omega_\varepsilon} e^{\gamma p \left[ (1 + \frac{v}{\varepsilon}) \varepsilon^{p-1} \right]} dy. \tag{5.1}
\]
Let us differentiate the function \( F_\lambda(\xi) \) with respect to \( \xi \). Since

\[
J_\lambda^p \left( (U_\lambda + \bar{\phi})(x, \xi) \right) = \frac{1}{p^2 \gamma^{2(p-1)}} I_\lambda \left( (V_\lambda + \phi) \left( \frac{x}{\varepsilon}, \frac{\xi}{\varepsilon} \right) \right),
\]

(5.2)

we can differentiate directly \( I_\lambda(V_\lambda(\xi) + \phi(\xi)) \) under the integral sign, for \( a \in \{1, \ldots, m\} \) and \( b \in \{1, J_j\} \), so that

\[
\partial(\xi^a)^b F_\lambda(\xi) = \frac{1}{p^2 \gamma^{2(p-1)}} J^p \left( \sum_{i=1}^j \sum_{j=1}^m c_{ij} \eta_j Z_{ij} \left[ \partial(\xi^a)^b V_\lambda(\xi) + \partial(\xi^a)^b \eta_j Z_{ij} \phi(\xi) \right] \right),
\]

since \( \int_{\Omega_\varepsilon} \eta_j Z_{ij} \phi(\xi) = 0 \). By the expansion of \( V_\lambda \), we have

\[
\partial(\xi^a)^b V_\lambda = \partial(\xi^a)^b w_a(y) + \frac{p-1}{p} \frac{1}{\gamma^p} \partial(\xi^a)^b w_{0a}(y) + \left( \frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} \partial(\xi^a)^b w_{1m}(y) + \partial(\xi^a)^b \theta(y)
\]

\[
= -Z_{ba} + \frac{p-1}{p} \frac{1}{\gamma^p} \partial(\xi^a)^b w_{0a}(y) + \left( \frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} \partial(\xi^a)^b w_{1m}(y) + \partial(\xi^a)^b \theta(y).
\]

Moreover,

\[
\int_{\Omega_\varepsilon} c_{ij} \partial(\xi^a)^b \eta_j Z_{ij} \phi(\xi) = o(1) \int_{\Omega_\varepsilon} c_{ij} \eta_j Z_{ij} \partial(\xi^a)^b (V_\lambda)
\]

Then, if \( D_\xi F_\lambda(\xi) = 0 \), for \( i, b = 1, J_j; \ j, a = 1, \ldots, m \), we then have

\[
\sum_{i=1}^j \sum_{j=1}^m c_{ij} \int_{\Omega_\varepsilon} \eta_j Z_{ij} (Z_{ba} + o(1)) = 0.
\]

(5.3)

This is a strictly diagonal dominant system. It implies that \( c_{ij} = 0 \) for \( i = 1, J_j; \ j = 1, \ldots, m \). This concludes the proof of (i) of Proposition 3.2. □

**Proof of (ii) of Proposition 3.2.** We have

\[
F_\lambda(\xi) = J_\lambda^p ((U_\lambda(\xi) + \bar{\phi}(\xi))
\]

\[
= \frac{1}{2} \int_{\Omega} \left[ \left| \nabla (U_\lambda + \bar{\phi}) \right|^2 + (U_\lambda + \bar{\phi})^2 \right] + \frac{\lambda}{p} \int_{\Omega} e^{(U_\lambda + \bar{\phi})^p}.
\]
From (5.2) we have that

\[ J^p_{\lambda}(U_\lambda(\xi) + \tilde{\phi}(\xi)) - J^p_{\lambda}(U_\lambda(\xi)) = \frac{1}{p^2 \gamma^{2(p-1)}} \left[ I_{\lambda}(V_\lambda + \phi) - I_{\lambda}(V_\lambda) \right]. \]

Since by construction \( I'_\lambda(V_\lambda + \phi)[\phi] = 0 \), we have

\[ J^p_{\lambda}(U_\lambda(\xi) + \tilde{\phi}(\xi)) - J^p_{\lambda}(U_\lambda(\xi)) = \frac{1}{p^2 \gamma^{2(p-1)}} \int_0^1 D^2 I_{\lambda}(V_\lambda + t\phi) \phi^2 (1-t) \ dt. \]

Since \( \|E_\lambda\|_* \leq \frac{C}{|\log \epsilon|^3} \), \( \|\phi\|_{\infty} \leq \frac{C}{|\log \epsilon|^2} \), \( \|N(\phi)\|_* \leq \frac{C}{|\log \epsilon|^4} \) and (2.37), we get that

\[ \left| J^p_{\lambda}(U_\lambda(\xi) + \tilde{\phi}(\xi)) - J^p_{\lambda}(U_\lambda(\xi)) \right| \leq \frac{C}{\gamma^{2(p-1)}|\log \epsilon|^3}. \] (5.4)

Next we expand

\[ J^p_{\lambda}(U_\lambda(\xi)) = \frac{1}{2} \int \left[ \left| \nabla (U_\lambda(\xi)) \right|^2 + U_\lambda(\xi)^2 \right] - \frac{\lambda}{p} \int e^{(U_\lambda(\xi))^p}. \] (5.5)

Now we write

\[ U_j(x) := u_j(x) + H_j(x), \quad U_{0j} := w_{0j}(x) + H_{0j}(x), \quad U_{1j} := w_{1j}(x) + H_{1j}(x). \]

By (2.13),

\[ U_\lambda(x) = \frac{1}{p \gamma^{p-1}} \sum_{j=1}^m \left( U_j(x) + \frac{p-1}{p} \frac{1}{\gamma^p} U_{0j}(x) + \left( \frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} U_{1j}(x) \right). \]

We have

\[ \frac{1}{2} \int \left[ \left| \nabla (U_\lambda(\xi)) \right|^2 + U_\lambda(\xi)^2 \right] \]

\[ = \frac{1}{p^2 \gamma^{2(p-1)}} \left\{ \frac{1}{2} \sum_{j=1}^m \left( |\nabla U_j|^2 + U_j^2 \right) + \sum_{i \neq j} \int \nabla U_i \nabla U_j + U_i U_j \right\} \]

\[ + \frac{p-1}{p} \frac{1}{\gamma^p} \sum_{j=1}^m \left( \nabla U_j \nabla U_{0j} + U_j U_{0j} \right) + \left( \frac{p-1}{p} \right)^2 \frac{1}{\gamma^{2p}} \sum_{j=1}^m \left( \nabla U_j \nabla U_{1j} + U_j U_{1j} \right). \]
Let us estimate the first two terms. We observe that the remaining terms are $O\left(\frac{1}{\gamma^{2\beta-1}}\right)$. First, we note that $U_j$ satisfies

$$-\Delta U_j + U_j = \varepsilon^2 e^{u_j}, \quad \text{in } \Omega, \quad \frac{\partial U_j}{\partial 
u} = 0 \quad \text{on } \partial \Omega.$$ 

Then we have

$$\int_{\Omega} \left(|\nabla U_j(x)|^2 + U_j(x)^2\right) dx$$

$$= \varepsilon^2 \int_{\Omega} e^{u_j} U_j(x) = \varepsilon^2 \int_{\Omega} e^{u_j} (u_j(x) + H_j(x))$$

$$= \varepsilon^2 \int_{\Omega} \frac{8\mu_j^2}{(\varepsilon^2 \mu_j^2 + |x - \xi_j|^2)^2} \left(\log \frac{8\mu_j^2}{(\varepsilon^2 \mu_j^2 + |x - \xi_j|^2)^2} + c_j H(x, \xi_j) - \log(8\mu_j^2) + O(\varepsilon^\alpha)\right)$$

$$= \varepsilon^2 \int_{\Omega} \frac{8\mu_j^2}{(\varepsilon^2 \mu_j^2 + |x - \xi_j|^2)^2} \left(\log \frac{1}{(\varepsilon^2 \mu_j^2 + |x - \xi_j|^2)^2} + c_j H(x, \xi_j) + O(\varepsilon^\alpha)\right)$$

$$= \int_{\Omega_{\varepsilon \mu_j}} \frac{8}{(1 + |y|^2)^2} \left(\log \frac{1}{(1 + |y|^2)^2} + c_j H(\xi_j + \varepsilon \mu_j y, \xi_j) - 4 \log(\varepsilon \mu_j)\right) + O(\varepsilon^\alpha)$$

$$= \int_{\Omega_{\varepsilon \mu_j}} \frac{8}{(1 + |y|^2)^2} \log \frac{1}{(1 + |y|^2)^2} + c_j \int_{\Omega_{\varepsilon \mu_j}} \frac{8}{(1 + |y|^2)^2} (H(\xi_j + \varepsilon \mu_j y, \xi_j) - H(\xi_j, \xi_j))$$

$$+ c_j \int_{\Omega_{\varepsilon \mu_j}} \frac{8}{(1 + |y|^2)^2} H(\xi_j, \xi_j) - 4 \log(\varepsilon \mu_j) \int_{\Omega_{\varepsilon \mu_j}} \frac{8}{(1 + |y|^2)^2} + O(\varepsilon^\alpha). \quad (5.7)$$

But

$$\int_{\Omega_{\varepsilon \mu_j}} \frac{8}{(1 + |y|^2)^2} = c_j + O(\varepsilon), \quad \int_{\Omega_{\varepsilon \mu_j}} \frac{8}{(1 + |y|^2)^2} \log \frac{1}{(1 + |y|^2)^2} = -2c_j + O(\varepsilon^\alpha). \quad (5.8)$$
Moreover, for \(0 < \alpha < 1\),

\[
\int_{\Omega_{\epsilon, \mu_j}} \frac{8}{(1 + |y|^2)^2} (H(\xi_j + \epsilon \mu_j y, \xi_j) - H(\xi_j, \xi_j)) = \int_{\Omega_{\epsilon, \mu_j}} \frac{1}{(1 + |y|^2)^2} O(\epsilon^\alpha |y|^\alpha) = O(\epsilon^\alpha). \tag{5.9}
\]

Therefore from (5.7)–(5.9), we have

\[
\int_{\Omega} \left( \| \nabla U_j(x) \|^2 + U_j(x)^2 \right) \, dx = -2c_j + c_j^2 H(\xi_j, \xi_j) - 4c_j \log \epsilon - 4c_j \log \mu_j + O(\epsilon^\alpha)
\]

\[
= -2c_j + c_j^2 H(\xi_j, \xi_j) - 4c_j \log \epsilon - 2c_j \log(8\mu_j^2) + 2c_j \log(8) + O(\epsilon^\alpha). \tag{5.10}
\]

Now, we calculate that

\[
\sum_{l \neq j} \int_{\Omega} (\nabla U_l \nabla U_j + U_l U_j) \, dx
\]

\[
= \epsilon^2 \sum_{l \neq j} \int_{\Omega} e^{u_l} U_j = \epsilon^2 \sum_{l \neq j} \int_{\Omega} e^{u_l} (U_j + H_j) \, dx
\]

\[
= \epsilon^2 \sum_{l \neq j} \int_{\Omega} \frac{8\mu_l^2}{(\epsilon^2 \mu_j^2 + |x - \xi_l|^2)^2} \left( \frac{8\mu_j^2}{(\epsilon^2 \mu_j^2 + |x - \xi_j|^2)^2} + c_j H(x, \xi_j) - \log(8\mu_j^2) + O(\epsilon^\alpha) \right)
\]

\[
= \epsilon^2 \sum_{l \neq j} \int_{\Omega} \frac{8\mu_l^2}{(\epsilon^2 \mu_j^2 + |x - \xi_l|^2)^2} \left( \log \frac{1}{(\epsilon^2 \mu_j^2 + |x - \xi_j|^2)^2} + c_j H(x, \xi_j) + O(\epsilon^\alpha) \right)
\]

\[
= \sum_{l \neq j} \int_{\Omega_{\epsilon, \mu_j}} \frac{8}{(1 + |y|^2)^2} \left( \log \frac{1}{(\epsilon^2 \mu_j^2 + |x - \xi_l - \xi_j|^2)^2} + c_j H(\xi_l + \epsilon \mu_l y, \xi_j) \right) + O(\epsilon^\alpha)
\]

\[
= \sum_{l \neq j} c_l c_j G(\xi_l, \xi_j) + O(\epsilon^\alpha). \tag{5.11}
\]

Thus, from (5.6), (5.10) and (5.11) we have

\[
\frac{1}{2} \int_{\Omega} \left( \| \nabla U_\lambda(x) \|^2 + U_\lambda(x)^2 \right) \, dx
\]

\[
= \frac{1}{p^2 \gamma^2 (p-1)} \left\{ -4\pi (2k + l) \frac{p}{2 - p} (1 - \log 8) - 8\pi (2k + l) \log \epsilon \right. 
\]

\[
- \frac{p}{2(2 - p)} \sum_{j=1}^{m} \left[ c_j^2 H(\xi_j, \xi_j) + \sum_{l \neq j} c_l c_j G(\xi_l, \xi_j) \right] + O(\log |\epsilon|^{-1}) \right\}. \tag{5.12}
\]
Finally, let us evaluate the second term in the energy
\[
\frac{\lambda}{p} \int_\Omega e^{(U_\lambda)^p} \, dx = \frac{\lambda}{p} \int_\Omega e^{\gamma p \left(1 + \frac{1}{p\gamma p}(V_\lambda(x))\right)^p} \, dx
\]
\[
= \frac{\lambda}{p} \sum_{j=1}^k \int_{B(\xi_j, \delta)} e^{\gamma p \left(1 + \frac{1}{p\gamma p}(V_\lambda(\xi_j))\right)^p} \, dx
\]
\[
+ \frac{\lambda}{p} \sum_{j=1}^k \int_{\Omega \setminus \bigcup_{j=1}^k B(\xi_j, \delta)} e^{\gamma p \left(1 + \frac{1}{p\gamma p}(V_\lambda(\xi_j))\right)^p} \, dx
\]
\[
:= I + II. \tag{5.13}
\]

First we observe that
\[
II = \lambda \Theta_\lambda(\xi) \tag{5.14}
\]
with \(\Theta_\lambda(\xi)\) a function, uniformly bounded, as \(\lambda \to 0\). On the other hand,
\[
I = \frac{1}{p^2 \gamma^{2(p-1)}} \sum_{j=1}^m \int_{B(\xi_j, \delta/\varepsilon)} e^{\gamma p \left(1 + \frac{1}{p\gamma p}(V_\lambda(y))\right)^p} dy
\]
\[
= \frac{1}{p^2 \gamma^{2(p-1)}} \sum_{j=1}^m \int_{B(\xi_j, \delta/\varepsilon)} e^{\gamma p \log \left|y - y_j\right|^2 \left(1 + \Theta_\lambda(\xi)\right)} \left(1 + O\left(\frac{1}{\gamma p}\right)\right) dy
\]
\[
= \frac{1}{p^2 \gamma^{2(p-1)}} 4\pi (2k + l) \left(1 + |\log \varepsilon|^2 \Theta_\lambda(\xi)\right), \tag{5.15}
\]
with \(\Theta_\lambda(\xi)\) a function, uniformly bounded, as \(\lambda \to 0\). From (5.13)–(5.15) we get
\[
\frac{\lambda}{p} \int_\Omega e^{(U_\lambda)^p} \, dx = \frac{1}{p^2 \gamma^{2(p-1)}} 4\pi (2k + l) \left(1 + |\log \varepsilon|^{-1} \Theta_\lambda(\xi)\right). \tag{5.16}
\]

Thus from (5.4), (5.5), (5.12) and (5.16), we obtain that
\[
F_\lambda(\xi) = \frac{1}{p^2 \gamma^{2(p-1)}} \left\{ -4\pi (2k + l) \frac{2 - p \log 8}{2 - p} - 8\pi (2k + l) \log \varepsilon 
- \frac{p}{2(2 - p)} \sum_{j=1}^m c_j H(\xi_j, \xi_j) + \sum_{l \neq j} c_l c_j G(\xi_l, \xi_j) + O(\varepsilon) \right\},
\]
which implies (3.10) by (1.8). This concludes the proof of Proposition 3.2. \(\square\)
Acknowledgments

This work is supported by a doctoral scholarship of the Center for Mathematical Modeling, University of Chile. I would like to thank Professors Manuel del Pino and Monica Musso for their guidance and help, thank Professor Monica Musso for reading the preprint of this paper and giving useful suggestions. I also give my thanks to the reviewer for the useful comments and suggestions.

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