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Chain-condition methods in topology ☆

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Abstract

The special role of countability in topology has been recognized and commented upon very early in the development of the subject. For example, especially striking and insightful comments in this regard can be found already in some works of Weil and Tukey from the 1930s (see, e.g., Weil (1938) and Tukey (1940, p. 83)). In this paper we try to expose the chain condition method as a powerful tool in studying this role of countability in topology. We survey basic countability requirements starting from the weakest one which originated with the famous problem of Souslin (1920) and going towards the strongest ones, the separability and metrizability conditions. We have tried to expose the rather wide range of places where the method is relevant as well as some unifying features of the method.

Keywords: The countable chain condition; Shanin's condition; Property K; Strictly positive measures; Mappings onto Tychonoff cubes; Separability; Dense metrizable subspaces; Metrizably fibered spaces; Linearly fibered spaces

Introduction

A topological space X satisfies the *countable chain condition* (often called *Souslin's condition*) if every family of pairwise disjoint open subsets of X is countable. This is the weakest chain condition considered in this survey where we make an attempt to expose the classifying power of an array of other chain conditions that one can put on a given space before the ultimate one, the *separability* condition on X. For example, if one thinks of ccc

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as saying that every point-1 family of open subsets of X must be countable, or in short, 'point-1 = countable' one may ask about other such equalities:

$$\begin{aligned} \text{point-2} &= \text{countable} \\ \text{point-3} &= \text{countable} \\ &\vdots \\ \text{point-finite} &= \text{countable} \end{aligned}$$

and finally, one may ask for the strongest one of this kind:

$$\text{point-countable} = \text{countable}$$

and see whether one really gets different conditions. The answer depend on the class of spaces we are working with, and this what we meant when we said 'classifying power'. In any space X, the ccc is the same as the requirement that for any integer k, point-k families of open subsets of X are countable. To identify ccc and the requirement that point-finite families are countable one needs to assume something like compactness on X. This has been first established by Rosenthal [37,38] dealing with a problem in Banach space theory, and independently by Arhangel'skii [4] solving a problem from the theory of Moore spaces. Tall [57] noted that the argument given in [4] shows that the requirement that X be a Baire space is all that is needed to identify ccc and the condition 'point-finite = countable'. Already these simple results give us a clear indication that better spaces are likely to identify more chain conditions than the more pathological ones. For example, not even the class of compact spaces seem to be restrictive enough to identify the remaining chain condition listed above asserting that point-countable families of open subsets of X must be countable. This turned out to be a major new chain condition introduced long ago by Shanin [41] and it will therefore be referred to here as *Shanin's condition.* While productiveness of the ccc is questionable, and the productiveness of separability false, it turns out that Shanin's condition is always productive:

Theorem 1 (Shanin). *Shanin's condition is preserved in Tychonoff products of any number of factors.*

The theme 'point-countable = countable' in topology has proved to be quite a fruitful one as the following beautiful result of Mischenko [31] shows.

Theorem 2 (Mischenko). *Every point-countable basis of a compact space is in fact countable.*

Back to the new chain condition and the question which classes of spaces would identify Shanin's condition with either ccc or separability. This turned out to be a quite subtle matter as the following result shows.

Theorem 3 [52]. MA_{ω_1} *is equivalent to the statement that the ccc and Shanin's condition are equal restriction on a given compact space.*

The proof of Theorem 1 required a new combinatorial idea, a still prominent theme in combinatorics today (both finite and infinite; see, e.g., [93]):

Delta-System Lemma (Shanin). *Every uncountable family $\mathcal{F}$ of finite sets contains an uncountable subfamily $\mathcal{F}_0$ such that $E \cap F = \bigcap \mathcal{F}_0$ for every $E \neq F$ in $\mathcal{F}_0$.*

Interestingly, a proof of Theorem 2 also involves the Delta-System lemma, and the Solovay–Tennenbaum proof of the consistency of MA_{ω_1} uses this lemma at a crucial point (see [21,59]). While Shanin's theorem and the Delta-System lemma are easily seen to be just reformulations of each other, Mischenko's theorem and the Delta-System lemma are some sort of duals. To prove Mischenko's theorem consider the family of all finite minimal covers of the space by members of the basis and prove, using the Delta-System lemma, that it must be countable. Conversely, given an uncountable family $\mathcal{F}$ of finite sets, all of the same size, let $\mathcal{X}$ be its closure in $\{0,1\}^I$, where I is the union of $\mathcal{F}$ and where we identify sets with their characteristic functions. For a given $D \in \mathcal{X}$ there is a natural choice of a family $\mathcal{B}_D$ of basic open sets of the Cantor cube $\{0,1\}^I$ which separates D from its supersets and subsets in $\mathcal{X}$. Since $\mathcal{X}$ is not metrizable, by Mischenko's theorem, $\mathcal{B} = \bigcup_{D \in \mathcal{X}} \mathcal{B}_D$ cannot be point-countable. So there is a $D \in \mathcal{X}$ whose $\mathcal{B}_D$ is uncountable. Assuming D has a maximal size of a set with this property, it is easy to build a Delta-subsystem $\mathcal{F}_0$ of $\mathcal{F}$ with root $D = \bigcap \mathcal{F}_0$. We have made this short exposition in order to hint at a reappearing phenomenon of this subject. Seemingly quite different topological results tend to have common combinatorial essence. On the other hand, the discovery of these combinatorial results would have been much more difficult without the powerful topological intuition behind.

Today we know many more chain conditions and many remarkable results associated with them, so selecting a small but representing part of the theory was a quite demanding task. The process of selecting was in part made more difficult by our decision to concentrate on recent results rather than older ones. Let us now give a short overview of the content of this paper. In Section 1 we consider the problem of productiveness of the countable chain condition which historically was quite important for the development of the subject. In Section 2 we consider another great motivating source of this subject, the problem of the existence of strictly positive measures. It was this area that initiated the study of a whole new array of chain conditions which all, in some sense, resemble the dual form of the separability condition. In Sections 3–6 we study some special classes of compact spaces by examining which chain conditions they identify. In Section 8 the same study is presented but from a different angle involving some basic cardinal characteristics of the continuum. In Section 9 we present some applications of the chain condition method in studying compact subsets of function spaces.

1. The countable chain condition of products

Proving that a given space satisfies the countable chain condition can sometimes be quite difficult especially when the space is 'barely' ccc. For a quite long time it was not clear how to express the idea that some spaces are 'barely ccc'. One property that hints to the 'barely ccc' is the question of its productiveness which already appears in the Scottish-book of problems (see [72, Problem 192]), but whose importance was fully recognized only after the following result of Kurepa [24].

Theorem 1.1 (Kurepa). *The square of a Souslin continuum is not ccc.*

Proof. Let S be a given Souslin continuum and recursively pick sequences I_ξ, J_ξ, K_ξ $(\xi < \omega_1)$ of nonempty open intervals of S such that:

(1) $I_\xi, J_\xi \subseteq K_\xi$ and $I_\xi \cap J_\xi = \emptyset$,

(2) K_η contains no end-point of any I_ξ, J_ξ, or K_ξ for $\xi < \eta$.

Then $I_\xi \times J_\xi$ $(\xi < \omega_1)$ is a disjoint family of open rectangles of S^2. □

This elegant and rather simple argument, however, contains an idea which can vastly be generalized. We mention one result which uses the idea, a result would have been perhaps hard to discover in a different context:

Theorem 1.2 [75]. *Let X be a compactum such that not only X^2 but any of its subspaces satisfies the ccc. Then X is separable.*

Proof. Let π be the π-weight of X, the minimal cardinality of a π*-basis* of X, i.e., a family $\mathcal{P}$ of nonempty open subsets of X such that every nonempty open subset of X includes a member of $\mathcal{P}$. Choose recursively a sequence (F_ξ, G_ξ) $(\xi < \pi)$ such that

(1) F_ξ is a closed G_δ-subset of X with nonempty interior,

(2) G_ξ is an open F_σ-subset of X containing F_ξ,

(3) G_η contains no nonempty intersection of finitely many sets of the form F_ξ or $X \setminus G_\xi$ for $\xi < \eta$.

It is not hard to show that for every $\xi < \pi$, the product $F_\xi \times (X \setminus G_\xi)$ cannot be covered by finitely many products of the form $G_\eta \times (X \setminus F_\eta)$ for $\eta \neq \xi$. So, by compactness, for every $\xi < \pi$ we can pick a point (x_ξ, y_ξ) from the set

$$\big(F_\xi \times (X \setminus G_\xi)\big) \setminus \bigcup_{\eta \neq \xi} G_\eta \times (X \setminus F_\eta).$$

It follows that (x_ξ, y_ξ) $(\xi < \pi)$ is a discrete subspace of X^2, and so this finishes the proof. □

While Souslin continua are hard to find, a search for a space X that can be constructed without any appeal to additional set-theoretical assumptions but which would closely resemble Souslin continuum in some of its striking properties has resulted in the following example where $\bar{\mathfrak{c}}$ denotes the cofinality of the continuum.

Theorem 1.3 [54]. *There is a compactification $a\mathbb{N}$ of $\mathbb{N}$ with character of any point smaller than $\bar{\mathfrak{c}}$ such that the growth $a\mathbb{N} \setminus \mathbb{N}$ satisfies $\bar{\mathfrak{c}}$cc but its square does not.*

Corollary 1.4 ($\bar{\mathfrak{c}} = \omega_1$). *There is a first countable compact ccc space whose square is not ccc.*

This result shows that the problem of productiveness of the ccc property is not only related to the Souslin Problem (Theorem 1.1 above) but also to the Continuum Problem. In fact, the result says much more than that, it says that the phenomenon that products might have larger cellular families from their factors are always present even at levels which measure some characteristics of the continuum such as its cofinality. For example, we know one more characteristic of the continuum where the same phenomenon happens, the cardinal $\mathfrak{b}$, the minimal cardinality of a family of integer-valued functions on integers which is unbounded in the ordering of eventual dominance (see [53, Section 1]). The space of Theorem 1.3, and therefore that of Corollary 1.4, is constructed using also a well-ordering of the continuum as one of the parameters. The following remarkable result of Shelah [48] shows that some non-effective procedure is in fact necessary if one wants to produce such an example.

Theorem 1.5 (Shelah). *The countable chain condition of a Borel partial ordering is productive.*

Natural spaces tend to be associated, in one way or the other, with sets of reals in the sense that they have bases which can naturally be 'coded' as sets of reals. Theorem 1.5 says that whenever these sets of reals are Borel (i.e., natural), together with relations which correspond to relations of inclusion and disjointness, the spaces will have their countable chain condition productive. The following result gives us another clear indication of the usefulness of the idea of considering the productiveness of ccc.

Theorem 1.6 [43]. *The superextension λX of a compact space X satisfies the countable chain condition if and only if the infinite power of X does.*

One also has the following quite general result of this sort (for definitions see [60] or [43, Section 1]).

Theorem 1.7 [43]. *Let $\mathcal{F}$ be a normal functor of infinite degree. Then for every compact space X, $\mathcal{F}(X)$ satisfies the countable chain condition if and only if the infinite power of X does.*

The functors $\exp(X)$ and $P(X)$ are natural examples of normal functors of infinite degree. The results Theorems 1.6 and 1.7 are proved by relating the intersection properties of cellular families of open sets of $\mathcal{F}(X)$ to those of finite powers of X. The algebraic approach here was quite instrumental and, therefore, it is easiest to understand the result in the case of Stone spaces (an assumption which is easy to remove). Here is the crucial lemma in its algebraic form

Lemma 1.8 [43]. *Suppose $\vec{a}_\xi$ $(\xi \in I)$ is an uncountable sequence of n-tuples of pairwise disjoint nonzero elements of some Boolean algebra $\mathcal{B}$. Then either*

(1) *there is uncountable $J \subseteq I$ such that $\vec{a}_\xi$ $(\xi \in J)$ can be refined to a separated sequence $\vec{b}_\xi (\xi \in J)$, or*

(2) *there is uncountable $J \subseteq I$ such that for all $\xi \neq \eta$ in J either $\bigcup_{i=1}^n a_\xi^i$ is disjoint from $\bigcup_{i=1}^n a_\eta^i$ or is included in some a_η^j $(i = 1, \ldots, n)$, or vice versa with roles of ξ and η exchanged.*

Here 'separated sequence $\vec{b}_\xi$ $(\xi \in J)$' means that there is a fixed partition of unity

$$c^1 \cup c^2 \cup \cdots \cup c^n = 1$$

such that $b_\xi^i \subseteq c^i$ for all $\xi \in J$ and $i = 1, \ldots, n$. Also, '$\vec{b}_\xi$ $(\xi \in J)$ refines $\vec{a}_\xi$ $(\xi \in J)$' means simply that $b_\xi^i \subseteq a_\xi^i$ for all $\xi \in J$ and $i = 1, \ldots, n$.

To get an idea of how the lemma relates to Theorems 1.6 and 1.7 let us consider the case $\mathcal{F} = \exp$ of Theorem 1.7. So let us assume that $\exp(X)$ is ccc and let

$$P_\xi = a_\xi^1 \times \cdots \times a_\xi^n \quad (\xi < \omega_1)$$

be a given family of basic clopen subsets of some finite power X^n. By Lemma 1.8, refining the sequence, we may assume that either (1) or (2) holds. For $\xi < \omega_1$, set

$$U_\xi = \left\{ F \in \exp(X)\colon\ F \subseteq \bigcup_{i=1}^n a_\xi^i \text{ and } F \cap a_\xi^i \neq \emptyset \text{ for } i = 1, \ldots, n \right\}.$$

Since $\exp(X)$ is ccc pick $\xi \neq \eta$ such that U_ξ and U_η intersect. It is not hard to see that neither of the two cases of (2) can hold for these ξ and η. This shows that actually it must be that (1) holds, or in other words, that some fixed partition of unity

$$c^1 \cup \cdots \cup c^n = 1$$

separates the sequence $(a_\xi^1, \ldots, a_\xi^n)$ $(\xi < \omega_1)$. However, if this is the case, then it is not hard to see that for every ξ and η in ω_1 we have that

$$P_\xi \cap P_\eta \neq \emptyset \quad \text{if and only if} \quad U_\xi \cap U_\eta \neq \emptyset,$$

and so we are again done by the ccc property of $\exp(X)$.

Shanin's theorem (Theorem 1) gives us one sufficient condition for a product

$$X = \prod_{i \in I} X_i$$

to be ccc. That this can be useful is shown by the following interesting list of equivalences.

Theorem 1.9 (Noble and Ulmer, Schepin). *The following conditions are equivalent for any product X of uncountably many nontrivial factors*:

(1) *X is ccc,*

(2) *every regular-open subset of X depends on at most countably many coordinates,*

(3) *every continuous real-valued function defined on an open subspace of X depends on at most countably many coordinates.*

2. Strictly positive measures

A compact space X carries a *strictly positive measure* if there is a bounded Radon measure μ on X such that $\mu(U) > 0$ for all nonempty open $U \subseteq X$, or equivalently, for every open set U which belongs to some fixed π*-basis* of X. (Recall that a *Radon measure* μ on X is a measure defined on a σ-field of subsets of X which includes the family of all open subsets of X and which is *inner regular* with respect to the family of compact sets, i.e., $\mu(E) = \sup\{\mu(K)\colon K \subseteq E, K \text{ compact}\}$ for any measurable set E.) This concept was introduced by Rosenthal [37,38] in a course of studying the Banach spaces $C(X)$ and their conjugates. For example Rosenthal [37] shows that, if for a compact space X the Banach space $C(X)$ is isomorphic to a conjugate space, then X carries a strictly positive measure. Moreover, Rosenthal [37] has established the following reformulations of both ccc and this new condition showing thus their close relationship.

Theorem 2.1 (Rosenthal).
(1) *A compact space X is ccc if and only if every weakly compact subset of $C(X)$ is separable.*
(2) *A compact space X carries a strictly positive measure if and only if $C(X)^*$ contains a weakly compact total set.*

It is interesting that in proving (1) Rosenthal proves the following identification result for chain conditions in compact spaces:

Lemma 2.2. *In a compact ccc space point-finite families of open sets must be countable.*

Proof. Let $\mathcal{U}$ be a given uncountable family of open subsets of some compact ccc space X. If $\mathcal{U}$ is point-finite, using the ccc property of X we can easily conclude that there must be a nonempty open set $G \subseteq X$ such that every nonempty open subset of G has uncountably many different intersections with members of $\mathcal{U}$. Thus, we may as well assume $G = X$. For an integer k, let X_k be the set of all $x \in X$ which belong to at least k many members of $\mathcal{U}$. Our assumption $G = X$ means that each X_k is a dense open subset of X, so by the Baire category theorem $\bigcap_{k=0}^{\infty} X_k$ is nonempty. Any x from the intersection belongs to infinitely many members of $\mathcal{U}$, and this finishes the proof. □

To see how this is related to Theorem 2.1(1), consider a compact space X for which $C(X)$ contains a subset homeomorphic in its weak topology to the one-point compactification of an uncountable set (the general case reduces to this one using a deep result of [2]). Thus, we may assume that there is uncountable $\mathcal{F} \subseteq C(X)$ and $\varepsilon > 0$ such that $\|f\| > \varepsilon$ for all $f \in \mathcal{F}$ and such that every sequence of distinct elements of $\mathcal{F}$ converges weakly to zero. Then

$$\bigl\{x \in X\colon |f(x)| > \varepsilon\bigr\} \quad (f \in \mathcal{F})$$

is an uncountable point-finite family of open subsets of X.

More generally (see [11, Chapter 6]), we shall say that an arbitrary space X carries a *strictly positive measure* if there is a π-base $\mathcal{P}$ of X and a finite measure μ defined on the σ-field generated by $\mathcal{P}$ such that

$$\mu(U) > 0 \quad \text{for all } U \in \mathcal{P}.$$

It is not clear at all that this is a chain condition resembling any one considered so far, but we shall soon give a reformulation, due to J.L. Kelley [45], which shows that this condition has actually a quite natural place between ccc and separability. First of all note that every separable space X carries a strictly positive measure. For if $\{d_n\}_{n=1}^{\infty}$ is a sequence of elements of X which is dense in X then

$$\mu(A) = \Sigma\{2^{-n}\colon\ d_n \in A\}$$

defines a σ-additive measure defined on the power-set of X which is positive on open subsets of X. One of the crucial observations of Kelley is that one should really concentrate to strictly positive finitely additive measures, and then apply standard extension procedure to obtain σ-additive ones. This observation was based on the work of Horn and Tarski [33] who were studying strictly positive finitely additive measures on Boolean algebras rather than the corresponding Stone spaces. They observed that every σ-centered algebra carries a strictly positive measure (i.e., the dual of the fact that separable spaces carry strictly positive measure mentioned above) and have listed a number of chain conditions that follow from the existence of such a measure hoping that one of them would capture this notion. For example, by considering some of the properties of the family

$$\mathcal{B}_n = \{a \in \mathcal{B}\colon\ \mu(a) \geqslant 1/n\} \quad (n \in \mathbb{N}),$$

of subsets of a Boolean algebra $\mathcal{B}$ they have isolated the following two interesting chain conditions:

Definition 2.3. A Boolean algebra $\mathcal{B}$ satisfies *σ-finite chain condition* if it can be split into a sequence $\{\mathcal{B}_n\}$ of subsets none of which includes an infinite subset of pairwise disjoint elements. If we require that for each n every set of pairwise disjoint elements of $\mathcal{B}_n$ has size at most n, we get a stronger chain condition which can naturally be called *σ-bounded chain condition.*

It is interesting that so far there are no known examples showing that these two variations give us indeed different chain conditions. Isolating the σ-finite chain condition made it immediately clear that algebras which support a strictly positive measure have in fact a stronger property than ccc, a property considered before by Knaster [65] in connection with the Souslin Problem: Every uncountable subset $\mathcal{S}$ of $\mathcal{B}$ contains an uncountable subset in which no two elements are disjoint. Today this condition is known under the name *Knaster's condition*, or *Property K.* To see that σ-finite chain condition implies Knaster's condition, find an n such that $\mathcal{S} \cap \mathcal{B}_n$ is uncountable and apply the Dushnik–Miller partition relation $\omega_1 \to (\omega_1, \omega)^2$ to the disjointness graph of this set. The fact that every measure algebra has Knaster's property was observed long before by Marczewski–Szpilrajn who was also the one to prove the following interesting result even before Shanin proved his preservation result mentioned above in Theorem 1.

Theorem 2.4 (Szpilrajn [42]). *Knaster's condition is preserved in Tychonoff products of any number of factors.*

Proof. First note that Knaster's condition is preserved in products of two factors and then use the Delta-System lemma to reduce the general case to the case of products of finitely many factors. □

It should be noted that Szpilrajn [42] does not explicitly state the Delta-System lemma but his arguments do contain its proof. Shanin [41], very likely motivated by Szpilrajn's paper, considers Shanin's condition and proves the analogue of Szpilrajn's result for this case. However, he notices the independent interest of the Delta-System lemma and so he states it explicitly.

It turns out that the σ-bounded chain condition of Horn and Tarski [33] is not strong enough to capture the notion of existence of strictly positive measure. For this one needs a deeper insight into the intersection properties of sets of the form $\{a \in \mathcal{B}\colon \mu(a) \geqslant 1/n\}$ considered above, and that was the contribution of Kelley [45]. So let $\mathcal{B}_X$ be the field of sets generated by some π-basis $\mathcal{P}$ of X. For a nonempty $\mathcal{F} \subseteq \mathcal{B}_X$, let

$$I(\mathcal{F}) = \inf \frac{i(\vec{F})}{|\vec{F}|},$$

with infimum taken over all finite sequences $\vec{F} = \langle F_1, \ldots, F_n \rangle$ of (not necessarily distinct) elements of $\mathcal{F}$, where $|\vec{F}|$ is the length n of the sequence $\vec{F}$, and where

$$i(\vec{F}) = \max\left\{|J|\colon\ J \subseteq \{1, \ldots, n\},\ \bigcap_{j \in J} F_j \neq \emptyset\right\}.$$

The number $I(\mathcal{F})$ is the *intersection number* of the family $\mathcal{F}$. Note that

$$I(\mathcal{F}) = 1 \quad \text{iff} \quad \mathcal{F} \text{ is } \textit{centered},$$

i.e., $F_1 \cap \cdots \cap F_n \neq \emptyset$ for every finite sequence $F_1, \ldots, F_n$ of elements of $\mathcal{F}$. The reader may also wish to recheck that if μ is a finitely additive probability measure on $\mathcal{B}_X$, then for every $\varepsilon \in (0, 1]$, the family

$$\mathcal{F}_\varepsilon = \left\{F \in \mathcal{B}\colon\ \mu(F) \geqslant \varepsilon\right\}$$

has intersection number $\geqslant \varepsilon$. The following result of Kelley [45] is some sort of a converse to this.

Theorem 2.5 (Kelley). *For every nonempty family $\mathcal{F} \subseteq \mathcal{B}_X$ there is a finitely additive probability measure μ on $\mathcal{B}_X$ such that $\mu(F) \geqslant I(\mathcal{F})$ whenever $F \in \mathcal{F}$.*

If X were a compact space and $\mathcal{F}$ a nonempty family of open subsets of X, then the proof would actually give us a Radon probability measure μ of X with the same property.

Corollary 2.6. *For every nonempty $\mathcal{F} \subseteq \mathcal{B}_X$,*

$$I(\mathcal{F}) = \max_{\mu} \inf_{F \in \mathcal{F}} \mu(F),$$

where sup = max *is taken with respect to all finitely additive measures μ on $\mathcal{B}_X$.*

Corollary 2.7. *A space X supports a strictly positive measure if and only if the family of nonempty open subsets of X can be split into countably many subfamilies of positive intersection numbers.*

Spaces that carry strictly positive measures enjoy considerably finer intersection properties from the one introduced by Knaster. Out of a large body of such results we mention the following combination of $\omega_1 \to (\omega_1, \omega)^2$ and a classical result of Gillis [44] (see also [35]).

Theorem 2.8 (Gillis). *For every $\varepsilon > 0$ and every uncountable family $\mathcal{F}$ of measurable sets of some probability measure space (X, Σ, μ) such that $\mu(F) \geqslant \alpha > 0$ for all $F \in \mathcal{F}$ there is an uncountable subfamily $\mathcal{F}_0$ of $\mathcal{F}$ such that $\mu(E \cap F) \geqslant \alpha^2 - \varepsilon$ for all $E, F \in \mathcal{F}_0$.*

Gillis' result is based on the following simple lemma which allows generalization to k-intersection properties for any integer k.

Lemma 2.9. *If $n \geqslant 1 + (\alpha - \alpha^2)/\varepsilon$ then for every sequence $F_1, \ldots, F_n$ of measurable sets in some probability measure space (Σ, μ) such that $\mu(F_i) \geqslant \alpha$ for all i there exists $i \neq j$ such that $\mu(F_i \cap F_j) \geqslant \alpha^2 - \varepsilon$.*

For $k > 2$ passing from the finite to the infinite case is not automatic since then the partition relation $\omega_1 \to (\omega_1, \omega)^k$ is false. However, the proof of the kth analogue of Lemma 2.9 can be adjusted to give us the full generalization (see [35]):

Theorem 2.10 (Fremlin). *Let (X, μ) be a probability space and let $\mathcal{F}$ be an uncountable family of measurable sets all of measure $\geqslant \alpha > 0$. Then for every integer $k \geqslant 2$ and $\varepsilon > 0$ there is an uncountable family $\mathcal{F}_0 \subseteq \mathcal{F}$ such that $\mu(F_1 \cap \cdots \cap F_k) \geqslant \alpha^k - \varepsilon$ whenever $F_1, \ldots, F_k \in \mathcal{F}_0$.*

Going to all dimensions simultaneously in this result is a subtle matter (see [35, Problem 20]) as it is closely related to Martin's axiom restricted to measure algebras. For example, the statement that every uncountable family of elements of some measure algebra contains an uncountable centered subfamily is equivalent to the statement that the Haar group $\{0, 1\}^{\omega_1}$ cannot be covered by an ω_1-sequence of measure zero subsets. The study of strictly positive measures on topological spaces as well as on Boolean algebras continues to be a rich source of fascinating problems which connect to many other areas of mathematics (see [34,76]). Of course, one should not forget that the following result was known long before.

Theorem 2.11 (Haar). *Compact groups carry strictly positive measures.*

But it was not until 1940's that this was identified as a chain condition. For example, after the famous result of Ivanovskii, Kuzminov and Vilenkin that compact groups are dyadic (see, e.g., [61]) it became clear that the property K of a compact group (which one gets from the existence of a strictly positive measure) can be considerably improved using Shanin's theorem as follows.

Theorem 2.12 (Folklore). *Compact groups satisfy Shanin's condition.*

In the early 1980's Tkachenko [49] proved the following supplement to this result by a direct combinatorial argument.

Theorem 2.13 (Tkachenko). *All σ-compact groups satisfy the countable chain condition.*

Proof. Let $\mathcal{W}$ be a given uncountable family of open subsets of some σ-compact group G. Going to an uncountable subfamily of $\mathcal{W}$ we may assume that there is a compact set $K \subseteq G$ which intersects every member of $\mathcal{W}$. So for each $W \in \mathcal{W}$ we can fix $x_W \in K \cap W$ and an open symmetric neighborhood S_W of e such that both $x_W \cdot S_W^2$ and $S_W^2 \cdot x_W$ are included in W. Let n_W be an integer larger than a number of right translates, as well as number of left translates of S_W, needed to cover the compact set K. Thus we can find an infinite $\mathcal{W}_0 \subseteq \mathcal{W}$ and an integer n such that $n_W = n$ for all $W \in \mathcal{W}_0$. Applying Ramsey's theorem to a natural partition of $[\mathcal{W}_0]^2$ into n^2 cells, we get three different elements U, V, W of $\mathcal{W}_0$ such that x_U and x_V belong to the same right translate, $S_W \cdot g$, of S_W and such that x_V and x_W belong to the same left translate, $h \cdot S_U$, of S_U. It follows that, on one hand,

$$x_U \cdot x_V^{-1} \cdot x_W \in (S_W \cdot g) \cdot (S_W \cdot g)^{-1} \cdot x_W = S_W^2 \cdot x_W \subseteq W,$$

while on the other hand,

$$x_U \cdot x_V^{-1} \cdot x_W \in x_U \cdot (h \cdot S_U)^{-1} \cdot (h \cdot S_U) = x_U \cdot S_U^2 \subseteq U.$$

This shows that $\mathcal{W}$ is not a disjoint family, finishing the proof of Theorem 2.13. □

Note that the argument from the proof of Theorem 2.13 gives the stronger conclusion that every σ-compact group satisfies the σ-bounded chain condition of Horn and Tarski, and so in particular it satisfies the Knaster's condition. The fact that Tkachenko's argument had to be different has become clear much later:

Theorem 2.14 [62]. *The free topological group over the one-point compactification of a discrete space of size continuum does not carry a strictly positive measure and under the* *assumption* $\mathfrak{b} = \omega_1$ *it contains an uncountable family of open sets which cannot be refined to an uncountable centered subfamily.*

Thus, while compact groups satisfy the rather strong Shanin's condition, the compactly generated ones need not satisfy even a slight strengthening of Knaster's condition. (It should be noted that the consistency of this fact has been first established by Shakhmatov [50].) However, these results still leave unanswered the following interesting

Problem 2.15 (Tkachenko). Suppose H is a σ-compact group and $\mathcal{F}$ is an uncountable family of nonempty open subsets of H. Is there an uncountable $\mathcal{F}_0 \subseteq \mathcal{F}$ such that every three elements of $\mathcal{F}_0$ have nonempty intersection?

3. First-countable spaces

In this section we show that compact spaces with good local properties tend to identify some of the chain conditions considered so far. What are good local properties of a given space X? Of course, a good local property is that of being *first-countable*, i.e., having a countable local base at every point. A natural weakening of this condition is the condition of being *Fréchet–Urysohn*, which says that if a point $x \in X$ is in the closure of some set A then there is a sequence $\{x_n\}$ of elements of A that *converges* to A. If we require only that the sequence $\{x_n\}$ *accumulates* to x we get still a weaker condition called *countable tightness*. In the category of compact spaces there is a beautiful result of Shapirovskii (see [21, 3.14]) which shows that there is a deeper level in this formal game of weaker and weaker conditions: Every countably tight compact space X has *countable* π*-character*, i.e., for every $x \in X$ there is a countable collection $\mathcal{P}_x$ of open subsets of X (not necessarily neighborhoods of x) such that for every open U containing x there is $V \in \mathcal{P}_x$ such that $V \subseteq U$. We start the presentation with the following result of Shapirovskii (see [64, 75]) which is of independent interest.

Theorem 3.1 (Shapirovskii). *Compact countably tight spaces have point-countable* π*-base.*

Proof. The sequence (F_ξ, G_ξ) $(\xi < \pi)$ from the proof of Theorem 1.2 can of course be chosen in such a way that the interiors of F_ξ's form a π-basis of X. It suffices to show that F_ξ $(\xi < \pi)$ is a point-countable family of sets whenever X is a countably tight space. For suppose there is $A \subseteq \pi$ of order-type ω_1 such that F_α $(\alpha \in A)$ is a centered family. Using the property (3) of (F_ξ, G_ξ) $(\xi < \pi)$ one checks that the family

$$\{F_\alpha\colon \alpha \in I\} \cup \{X \setminus G_\beta\colon \beta \in J\}$$

has nonempty intersection for every pair of finite sets $I, J \subseteq A$ such that every ordinal from I is smaller than every ordinal from J. This is done by an easy induction on the size of the set J. By compactness of X, for each $\alpha \in A$, we can pick a point x_α from the intersection of the family

$$\{F_\xi\colon \xi \in A,\ \xi \leqslant \alpha\} \cup \{X \setminus G_\eta\colon \eta \in A,\ \eta > \alpha\}.$$

Let x be a complete accumulation point of the sequence x_α ($\alpha \in A$). Note that on one hand, $x \in F_\alpha \subseteq G_\alpha$ for all $\alpha \in A$, but on the other hand $x_\alpha \notin G_\beta$ whenever $\alpha < \beta$. It follows that no countable subsequence of x_α ($\alpha \in A$) accumulates to x, and so the space X is not countably tight at x. This finishes the proof. □

Corollary 3.2. *Shanin's condition is as strong as separability in the class of compact countably tight spaces.*

Corollary 3.2 is not the first such an identification result. Recall the following classical result of Knaster [65] which shows that ordered continua identify many more chain conditions.

Theorem 3.3 (Knaster). *Knaster's property is as strong as separability in the class of ordered continua.*

Proof. First note that, in the class of ordered continua, Knaster's property is equivalent to Shanin's condition. Note also that ccc ordered continua are first countable so that Corollary 3.2 applies. □

These results show that in locally nice spaces the gap between ccc and separability is really a gap between ccc and Shanin's condition. The Souslin hypothesis is an equivalence between ccc and separable in a very restricted class of spaces, ordered continua. Martin's axiom, invented during a course of solving Souslin hypothesis, is a similar identification statement in another class of spaces, compact spaces of π-weight smaller than the continuum. This equivalence was first established in our paper [52] with Velickovic and we shall use here some methods from that paper to further analyze MA. To understand MA one perhaps needs to analyze its strength when restricted to some nicer class of spaces as closely approximating ordered continua as possible.

Theorem 3.4. MA_{ω_1} *is equivalent to the statement that every compact first-countable ccc space is separable.*

Proof. It is well known that MA_{ω_1} implies that every compact ccc space satisfies Shanin's condition (see [14]) and is therefore separable by Corollary 3.2, so we are left to proving the reverse implication. Assume MA_{ω_1} is false. By the main result of [52] there is a family $\mathcal{K}$ of finite subsets of ω_1 such that:

(1) $[\omega_1]^1 \subseteq \mathcal{K}$,
(2) $E \subseteq F \in \mathcal{K}$ implies $E \in \mathcal{K}$,
(3) every uncountable subset of $\mathcal{K}$ contains two elements whose union belongs to $\mathcal{K}$,

(4) there is no uncountable $H \subseteq \omega_1$ such that $[H]^{<\omega} \subseteq \mathcal{K}$.

By [14, 21 N(e)] pick a one-to-one sequence a_ξ $(\xi < \omega_1)$ of infinite sets of integers such that for a finite set $F \subseteq \omega_1$,

(5) $F \in \mathcal{K}$ iff $\bigcap_{\xi \in F} a_\xi$ is infinite.

Now we proceed similarly as in [54, Example D]. Let U be the set of all pairs (t, n) where n is an integer, t a family of subsets of $n = \{0, \ldots, n-1\}$, and for every $k \leqslant n$,

(6) $|(\bigcap t) \cap k| \geqslant |\Delta_t \cap k|$,

where $\Delta_t = \{\min(a \Delta b)\colon a, b \in t,\ a \neq b\}$. For $\xi \in \omega_1$ and $(t, n) \in U$, set

$$U_\xi = \{(s, m) \in U\colon a_\xi \cap m \in s\},$$
$$U_{(t,n)} = \{(s, m) \in U\colon m \geqslant n \text{ and } s \restriction n = t\},$$

where $s \restriction n = \{a \cap n\colon a \in S\}$. Let $\mathcal{D}$ be the subalgebra of $\mathcal{P}(U)$ generated by

$$\{U_\xi\colon \xi \in \omega_1\} \cup \{U_{(t,n)}\colon (t, n) \in U\} \cup \text{Fin}$$

and let $\mathcal{J}$ be the ideal of $\mathcal{D}$ generated by the ideal Fin (finite subsets of U) together with all sets of the form

$$U_F = \bigcap_{\xi \in F} U_\xi,$$

where F is a finite subset of ω_1 which does not belong to $\mathcal{K}$, i.e., for which $a_F = \bigcap_{\xi \in F} a_\xi$ is finite. Observe that, if for a given $F \in \mathcal{K}$ the intersection U_F is infinite, then this in particular means that a_F is infinite and that $|a_F \cap k| \geqslant |\Delta_F \cap k|$ for all k (where $\Delta_F = \{\min(a_\xi \Delta a_\eta)\colon \xi, \eta \in F,\ \xi \neq \eta\}$). On the other hand, if $F \notin \mathcal{K}$ then sizes of the intersections $\bigcap s$ $((s, m) \in U_F)$ are uniformly bounded by the size of the finite set a_F. From this we can easily conclude that the set U_F cannot be covered (modulo Fin) by finitely many members of the form U_E for $E \notin \mathcal{K}$, i.e., it does not belong to $\mathcal{J}$. We shall use this observation in several places below.

Claim 1. *The algebra $\mathcal{D}/\mathcal{J}$ is ccc.*

Proof. Considering the elements of $\mathcal{D}/\mathcal{J}$ in terms of their representatives from $\mathcal{D}$ we are given an uncountable family $\mathcal{X}$ of basic elements of $\mathcal{D}/\mathcal{J}$. We need to find two distinct elements of $\mathcal{X}$ whose intersection is not in $\mathcal{J}$. A basic element of $\mathcal{D}/\mathcal{J}$ is a finite intersection of generating sets or their complements. Since U is a finitely branching tree, by shrinking, and still remaining outside $\mathcal{J}$, we may assume that all generators of the form $U_{(t,n)}$ appear positively. It is also easy to see that similarly any $-U_\xi$ can be eliminated by a further intersection with a generator of the form $U_{(t,n)}$. Since there are only countably many generators of the second form we may altogether ignore them. So, the problem reduces to the following case: $\mathcal{X} = \{U_F\colon F \in \mathcal{F}\}$ for some uncountable $\mathcal{F} \subset \mathcal{K}$. For $F \in \mathcal{F}$, set

$$\Delta_F = \{\Delta(a_\xi, a_\eta)\colon \xi \neq \eta \text{ in } F\},$$

$$a_F = \bigcap_{\xi \in F} a_\xi, \quad \ell_F = |F|, \quad m_F = \sup(\Delta_F) + 1,$$

$$n_F = \min\{n\colon |a_F \cap (m_F, n)| \geqslant \ell_F\},$$
$$\tau_F = \{a_\xi \cap n_F\colon \xi \in F\}.$$

(Note that for $F \in \mathcal{F}$, a_F is infinite so the number n_F exists.) Since there exist only countably many choices for the parameters, we can find (ℓ, m, n, τ) and uncountable $\mathcal{F}_0 \subseteq \mathcal{F}$ such that:

(7) $\langle \ell_F, m_F, n_F, \tau_F\rangle = \langle \ell, m, n, \tau\rangle$ for all $F \in \mathcal{F}_0$.

By (3) there exist $E \neq F$ in $\mathcal{F}_0$ such that $E \cup F \in \mathcal{K}$. The choice of the parameters and the fact that they are equal for E and F ensures that for all large-enough integers n, the pair (t_n, n) satisfies (6), where

$$t_n = \{a_\xi \cap n\colon \xi \in E \cup F\}.$$

It follows that $U_E \cap U_F = U_{E\cup F}$ is infinite so by the remark above it does not belong to the ideal $\mathcal{J}$. □

Claim 2. *Every ultrafilter $\mathcal{U}$ of $\mathcal{D}$ (and therefore every ultrafilter $\mathcal{V}$ of $\mathcal{D}/\mathcal{J}$) is countably generated.*

Proof. For a given integer n, let t_n be the unique subset of $\mathcal{P}(n)$ such that $U_{(t_n,n)} \in \mathcal{U}$. Note that $t_n \restriction m = t_m$ whenever $m \leqslant n$. Let

$$H = \{\xi \in \omega_1\colon a_\xi \cap n \in t_n \text{ for all } n\}.$$

Note that if H is infinite, the set

$$\Delta_H = \{\min(a_\xi \Delta a_\eta)\colon \xi, \eta \in H,\ \xi \neq \eta\}$$

is also infinite, so the condition (6) gives us easily that $\bigcap_{\xi\in F} a_\xi$ is infinite for every finite $F \subseteq H$. By (5) this means that $[H]^{<\omega} \subseteq \mathcal{K}$, so by (4) we conclude that H is countable. Therefore, to prove that $\mathcal{U}$ is countably generated it suffices to show that for every generator U_ξ with $\xi \notin H$ there is some n such that $U_\xi \cap U_{(t_n,n)}$ is finite. Clearly, the n that works is any n such that $a_\xi \cap n \notin t_n$. □

Claim 3. *The algebra $\mathcal{D}/\mathcal{J}$ contains an uncountable subset which cannot be refined to an uncountable centered subset.*

Proof. We have already seen that no generator U_ξ (nor $U_{(t,n)}$) can be covered (mod Fin) by finitely many elements of U_E ($E \notin \mathcal{K}$). So every U_ξ represents a positive element of the algebra $\mathcal{D}/\mathcal{J}$. It is also easy to see that U_ξ and U_η represent different elements of the quotient algebra $\mathcal{D}/\mathcal{J}$ whenever $\xi \neq \eta$. Suppose that for some $H \subseteq \omega_1$ the family U_ξ ($\xi \in H$) is centered in $\mathcal{D}/\mathcal{J}$, i.e., that the intersection $U_F = \bigcap_{\xi\in F} U_\xi$ does not belong to $\mathcal{J}$ for any finite $F \subseteq H$. By (6) and the definition of U_ξ we easily get that for every finite $F \subseteq H$,

(8) $|(\bigcap_{\xi\in F} a_\xi) \cap k| \geqslant |\Delta_H \cap k|$ for every k.

So if H is infinite then $\bigcap_{\xi\in F} a_\xi$ is infinite for every finite $F \subseteq H$. It follows that in this case, $[H]^{<\omega} \subseteq \mathcal{K}$. Applying (4) we conclude that H must be countable. This finishes the proof. □

Let $X = \mathrm{Ult}(\mathcal{D}/\mathcal{J})$, the Stone space of the quotient algebra $\mathcal{D}/\mathcal{J}$. Then X is ccc (Claim 1), nonseparable (Claim 3) and first-countable (Claim 2). This completes the proof of Theorem 3.4. □

The space $X = \mathrm{Ult}(\mathcal{D}/\mathcal{J})$ from the proof of Theorem 3.4 has some other interesting properties worth exposing. To see this, let $\mathcal{D}_0$ be the subalgebra of $\mathcal{D}$ generated by Fin and

$$U_{(t,n)} \quad ((t,n) \in U).$$

Let $X_0 = \mathrm{Ult}(\mathcal{D}_0/\mathcal{J})$. Then X_0 is a compact metric space and

$$\mathcal{U} \overset{\phi}{\mapsto} \mathcal{U} \cap (\mathcal{D}_0/\mathcal{J})$$

is a continuous map from X onto X_0. What we want to point out is that ϕ has metrizable fibers $\phi^{-1}(\mathcal{V})$. To see this, for a given integer n, let t_n be the unique subset of $\mathcal{P}(n)$ such that $U_{(t_n,n)}/\mathcal{J}$ belongs to $\mathcal{V}$. Then as before, we have that $t_n \restriction m = t_m$ for $n \leqslant m$ and that the set

$$H_{\mathcal{V}} = \{\xi \in \omega_1\colon\ a_\xi \cap n \in t_n \text{ for all } n\}$$

must be countable. It has also been shown above that any $\mathcal{U}/\mathcal{J}$ from $\phi^{-1}(\mathcal{V})$ has a local basis which involves only generators of the form

$$U_{(t_n,n)}\ (n \in \mathbb{N}), \quad \text{and} \quad U_\xi\ (\xi \in H_{\mathcal{V}})$$

so the fiber $\phi^{-1}(\mathcal{V})$ is second-countable. It follows that our space $X = \mathrm{Ult}(\mathcal{D}/\mathcal{J})$ belongs to a rather interesting class of compact spaces which Tkachuk [74] calls *metrizably fibered*, i.e., the class of compact spaces which map continuously, with metrizable fibers, onto metric compacta. This class of spaces looks quite restrictive but any known example of a reasonably nonpathological compactum belongs to this class. For example, it is still unknown whether any perfectly normal compactum belongs to this class unless it is constructed using some pathological additional set-theoretic axiom. It is clear that every metrizably fibered compactum is first-countable, so by Corollary 3.2 above, MA_{ω_1} implies that the class of metrizably fibered compacta cannot distinguish ccc from the separability. However, the example $X = \mathrm{Ult}(\mathcal{D}/\mathcal{J})$ shows that the converse of this implication is also true.

Theorem 3.5. MA_{ω_1} *is equivalent to the statement that every ccc metrizably fibered compactum is separable.*

The space $X = \mathrm{Ult}(\mathcal{D}/\mathcal{J})$, or some of its better versions, might have some other interesting properties. A discovery of any such pleasant property of X will shed some further light on the relationship between MA_{ω_1} and various forms of Souslin's hypothesis.

4. Hereditarily normal spaces

The separation axiom T_5 is yet another strong restriction satisfied by any ordered continuum and the purpose of this section is to analyze the corresponding form of Souslin's hypothesis stating that every compact T_5 ccc space is separable. The relation of this form of Souslin's hypothesis and Martin's axiom is still unclear, but we do have the following result announced by Shapirovskii [40] in a slightly weaker form.

Theorem 4.1. *If* MA_{ω_1} *holds then every compact* T_5 *ccc space has a countable* π*-basis.*

Proof. Let X be a given T_5 compact ccc space. Since no closed subset of X maps onto the Tychonoff cube $[0,1]^{\omega_1}$, by another well known result of Shapirovskii (see [21, 3.18] or Section 6 below), we know that the set

$$D = \{x \in X\colon\ x \text{ has countable } \pi\text{-character}\}$$

is dense in X. To prove the theorem it suffices to find a countable $D_0 \subseteq D$ dense in X. Suppose that such D_0 cannot be found. For each $x \in D$ fix a countable family $\mathcal{U}_x$ of open subsets of X forming a π-basis of x in X. By our assumptions it is easy to build an increasing sequence D_ξ $(\xi < \omega_1)$ of countable subsets of D such that, if for $\xi < \omega_1$ we let

$$\mathcal{U}_\xi = \bigcup_{x \in D_\xi} \mathcal{U}_x,$$

then we have the following conditions satisfied:

(1) $D_\xi \subseteq D_{\xi+1} \nsubseteq \overline{D}_\xi$,

(2) if for some finite $\mathcal{F} \subseteq \mathcal{U}_\xi$ the intersection $\bigcap \mathcal{F}$ is nonempty then so is the intersection $(\bigcap \mathcal{F}) \cap D_{\xi+1}$.

Let $Y = \overline{D}_{\omega_1}$, where $D_{\omega_1} = \bigcup_{\xi<\omega_1} D_\xi$. Set

$$\mathcal{U}_{\omega_1} = \bigcup_{x \in D_{\omega_1}} \mathcal{U}_x.$$

Then $\mathcal{U}_{\omega_1} \upharpoonright Y$ is a π-basis of Y so by (2), Y is also a ccc space. Using MA_{ω_1} we conclude that Y is separable (see [14, 43F(b)]) so let $\{d_n\}_{n=1}^\infty$ be a countable dense subset of Y. (Clearly we may assume that $d_n \notin \overline{D}_\xi$ for all n and ξ.) Pick an x in Y such that $x \notin \overline{D}_\xi$ for all ξ. Then for each ξ we can choose open neighborhood U_ξ of x in Y such that $\overline{U}_\xi \cap \overline{D}_\xi = \emptyset$. Moreover we can arrange that

(3) for every $\xi_1 \ldots, \xi_n$ there is η such that $\overline{U}_\eta \subseteq U_{\xi_i}$ for $i = 1, \ldots, n$.

Let $F = \bigcap_{\xi<\omega_1} U_\xi$. Then F is a closed set which avoids $\overline{D}_\xi$ for all ξ. Since $\{d_n\}$ is dense in Y, the family

$$I_\xi = \{n \in \mathbb{N}\colon\ d_n \in U_\xi \setminus F\} \quad (\xi \in \omega_1)$$

of infinite subsets of $\mathbb{N}$ has the finite intersection property. Using MA_{ω_1} we can find infinite $I \subseteq \mathbb{N}$ such that $I \setminus I_\xi$ is finite for all ξ. By our assumption that X is T_5, the subspace $G = X \setminus F$ is normal and d_n $(n \in I)$ is a discrete subset of G, so we can find a sequence of open sets $V_n \subseteq G$ such that $d_n \in V_n$, and such that the sequence V_n $(n \in I)$ is discrete in G. Since D_{ω_1} is dense in Y, for each $n \in I$ we can pick ξ_n such that

$$V_n \cap D_{\xi_n} \neq \emptyset.$$

Let $\eta < \omega_1$ be such that $\xi_n \leqslant \eta$ for all n. The subspace $\overline{D}_\eta$ is compact so the sequence

$$V_n \cap \overline{D}_\eta \quad (n \in I)$$

of nonempty subsets of $\overline{D}_\eta$ must have a complete accumulation point y in $\overline{D}_\eta$. But $\overline{D}_\eta$ is disjoint from F, and so y belongs to G contradicting the fact that V_n $(n \in I)$ is discrete in G. This finishes the proof of Theorem 4.1. □

Problem 4.2. Is there some standard fragment of MA_{ω_1} which is equivalent to the statement that every T_5 ccc compactum is separable?

A solution to this problem is likely to involve a much deeper understanding of T_5 compacta then we presently have. What we are really hoping for is an analogue of Theorem 3.1 for compact T_5 spaces, i.e., a structure theorem for this class of spaces (not involving MA_{ω_1} at all!) which would have Theorem 4.1 as one of its corollaries. The search for such structure results is the real reason behind other similar problems that we are going to ask below.

In [72] Velickovic showed that under OCA (see [53]) every separable compact T_5 space is countably tight so in this context, Theorem 4.1 relates to Corollary 3.2 above. Using an even stronger additional set-theoretical assumption, PFA, we have a quite strong grip on the structure of compact ccc T_5 spaces: They are all Fréchet–Urysohn and, therefore, of size at most continuum. In [72], Velickovic also showed that MA_{ω_1} does not suffice in proving that every compact separable T_5 space is countably tight, and so this adds to the interest in Theorem 4.1 above. We shall now see that there is another promising line of investigating the class of compact ccc T_5 spaces which gives, in a sense, even a stronger grip on their structure.

Theorem 4.3. (MA_{ω_1}) *There is a measure algebra which forces that every T_5 compact ccc space is hereditarily separable.*

Proof. Let I be an index-set of size equal to the first strong-limit cardinal of cofinality ω_1 and let $\mathcal{R}$ be the measure algebra of the Haar group $\{0,1\}^I$. In [66], we have proved that the forcing extension of $\mathcal{R}$ (or any other measure algebra) satisfies the following combinatorial property of independent interest:

(SM_{ω_1}) If F is a set-mapping which to every $\xi \in \omega_1$ associates a countable subset $F(\xi)$ of ω_1 which does not contain ξ, then either ω_1 can be decomposed into countably many subsets A with the property that $F(\xi) \cap A = \emptyset$ for all $\xi \in A$, or there is uncountable $B \subseteq \omega_1$ such that for every finite $C \subset B$ there are uncountably many $\eta \in \omega_1$ such that $C \subseteq F(\eta)$.

We shall be interested in the following consequence of SM_{ω_1} which follows easily from Proposition 1 of [66].

Lemma 4.4. (SM_{ω_1}) *If a compact space X is not separable then it contains an uncountable discrete subspace.*

Proof. Let Y be a subset of X well-ordered by $<_w$ in order-type ω_1 such that no $y \in Y$ is in the closure of $\{x \in Y\colon\ x <_w y\}$. So, for each y in Y, we can pick an open (in X) neighborhood U_y of y whose closure misses the closure of $\{x \in Y\colon\ x <_w y\}$. Having chosen U_y $(y \in Y)$ define a set-mapping F from Y into the family of countable subsets of Y as follows

$$F(z) = \{y \in Y\colon\ z \in U_y\}.$$

By SM_{ω_1} we have to consider the following two cases:

Case 1: There is an uncountable $Z \subseteq Y$ such that $y \notin F(z)$, or equivalently $z \notin U_y$, for every two elements $y <_w z$ in Z. Clearly, any such Z is an uncountable discrete subspace of X.

Case 2: There exists an uncountable $B \subseteq Y$ such that for every finite $C \subseteq B$ there exist uncountably many $z \in Y$ for which $C \subseteq F(z)$. In other words, the family

$$U_y \cap Y \quad (y \in B)$$

is strongly centered in the sense that the intersection of any finite subfamily of this family is uncountable. Let Z be the set of all elements z of Y such that:

(1) $Y_z = \{y \in Y\colon\ y <_w z\}$ is of a limit order type and $B \cap Y_z$ is unbounded in Y_z.

(2) $U_y \cap Y_z$ $(z \in B \cap Y_z)$ has the finite intersection property.

Note that Z is an uncountable subset of Y. For each $z \in Z$, we fix a point d_z belonging to the closure of every member of the family of sets from (2). Let Z_0 be the set of all $z \in Z$ which have immediate predecessor in Z, denoted by z^-. For $z \in Z_0$ let b_z be the minimal element of B above z. Then the sequence of neighborhoods

$$(X \setminus \overline{Y_{z^-}}) \cap (X \setminus \overline{U_{b_z}}) \quad (z \in Z_0)$$

separates the sequence of points d_z $(z \in Z_0)$. So we have found an uncountable discrete subspace of X also in this case. This finishes the proof. □

We shall also need the following simple fact.

Lemma 4.5. *If D is a discrete subspace of a T_5 space X then there is a one-to-one mapping from the power-set of D into the algebra of regular-open subsets of X.*

The following fine result of Shapirovskii is another key point of the argument and a fact about compact hereditarily normal ccc spaces which is clearly of independent interest (see [21, 3.21]).

Lemma 4.6 (Shapirovskii). *The regular-open algebra of any T_5 compact ccc space has size at most continuum.*

Proof. It suffices to show that the set D from the proof of Theorem 4.1 has a dense subset of size at most continuum. Thus, as in that proof, we construct an increasing sequence D_ξ

($\xi < \omega_1$) of subsets of D of size at most continuum such that if for $\xi < \omega_1$ we let $\mathcal{U}_\xi$ be as before, then

(1) if for some finite $\mathcal{F} \subseteq \mathcal{U}_\xi$ the intersection $\bigcap \mathcal{F}$ is nonempty then so is the intersection $(\bigcap \mathcal{F}) \cap D_{\xi+1}$,
(2) if for some countable $\mathcal{F} \subseteq \mathcal{U}_\xi$ the difference $X \setminus \overline{\bigcup \mathcal{F}}$ is nonempty then it contains a point from $D_{\xi+1}$.

Now it is quite easy to see that the union $D_{\omega_1} = \bigcup_{\xi<\omega_1} D_\xi$ must be dense in X. □

To finish the proof of Theorem 4.4, we note that the index-set I was chosen in such a way that the measure algebra $\mathcal{R}$ of $\{0,1\}^I$ forces that the power-set of ω_1 has size bigger than the continuum and so the combination of Lemmas 4.6, 4.5 and 4.4 gives us the desired conclusion.

It is interesting that in Theorem 7 of [23], Kunen and Tall present a similar scenario with SM_{ω_1} replaced by the statement that every compact ccc space satisfies Shanin's condition which today we know to be equivalent to MA_{ω_1} (see [52], or Theorem 3 of the Introduction). It follows, therefore that the two hypothesis of Theorem 7 of [23] contradict each other. It appears thus that the well known hypothesis $2^\omega < 2^{\omega_1}$ of Jones [46], which has proved to be quite useful in topology especially in questions involving normality, is incompatible with the hypothesis that compact ccc spaces satisfy Shanin's condition. Thus, we have to settle for some of its weakenings strong enough to have some applications like Lemma 4.4. We believe that studying the combinatorial statements similar to SM_{ω_1} which are forced by any measure algebra might lead to some advances in this area. Another possible line of attack to this set of problems (which seems though much less promising) is to prove that Jones' hypothesis is compatible with the assertion that ccc and Knaster's condition are equivalent restriction on a given compact space. There is a considerable strength in this weak form of MA_{ω_1} which has purely a Ramsey-theoretic nature making the following problem of great interest of its own (see [53]).

Problem 4.7. Is MA_{ω_1} equivalent to the assertion that every ccc compactum satisfies Knaster's condition?

5. Perfectly normal compacta

One of the main sources of interest in the class of compact T_5 spaces comes from a set of beautiful results of Katetov [79] about this separation axiom. For example, Katetov showed that if X and Y are infinite compact spaces such that $X \times Y$ is T_5 then X and Y must in fact be perfect. From this it follows immediately that if X is compact and if X^3 is T_5 then X is metric. Katetov [79] asked if the same can be concluded assuming only that X^2 is T_5. Today, this is a well known open problem known under the name of *Katetov's problem* (see [80]). We conjecture that a random forcing extension similar to the one of Theorem 4.4 above will give a positive answer to Katetov's problem.

Katetov's problem is just one of the attempts to understanding the class of perfect compacta and their close relationship to the class of metric compacta. It concentrates on the understanding what one needs to add to T_5 to get perfectness. In [81] Baturov considers another alternative. Rather than considering products, Baturov uses a notion introduced and extensively studied by Schepin [82,83]). Since in the case of normal spaces (a context in which we are working) Schepin's notion coincides with an older one introduced by Pelczynski [85], we follow Pelczynski and say that a space X has the *Bockstein separation property* if every two disjoint open sets are contained in disjoint open F_σ sets. An example of such a space is of course any product of separable metric spaces which is the content of Bockstein's theorem (see [84] and also Theorem 1.7 above). Schepin's class of spaces is the class of spaces in which regular-open sets are F_σ (see [82,83]). While compact T_5 spaces may contain uncountable discrete subspaces (even in the context of MA_{ω_1}) the following fact shows that the new class of spaces behaves from this point of view much the same way as the class of perfect compacta.

Theorem 5.1. *Let X be a compact space which has the Bockstein separation property hereditarily. Then every subspace of X satisfies the countable chain condition.*

Proof. Suppose X contains an uncountable discrete subspace D. Since the assumption on X is hereditary, we may assume that $X = \overline{D}$. Recursively on $\xi < \omega_1$ we build sequences D_ξ $(\xi < \omega_1)$, $\mathcal{U}_\xi$ $(\xi < \omega_1)$ and x_ξ $(\xi < \omega_1)$ such that

(1) D_ξ is a countable subset of D,
(2) $x_\xi \in D$ but $x_\xi \notin D_\eta$ for all ξ and η,
(3) $D_\xi \subseteq D_\eta$, for all $\xi < \eta$,
(4) $\mathcal{U}_\xi$ is a countable collection of open subsets of X,
(5) $\bigcup \mathcal{U}_\xi = X \setminus \overline{D}_\xi$,
(6) $\overline{U} \cap \overline{D}_\xi = \emptyset$ for all $U \in \mathcal{U}_\xi$,
(7) if $\mathcal{F} \subseteq \bigcup_{\xi \leqslant \eta} \mathcal{U}_\xi$ is finite and $(\bigcap \mathcal{F}) \cap D$ is uncountable, then $(\bigcap \mathcal{F}) \cap D_{\eta+1} \neq \emptyset$.

There are no problems in choosing these objects since for every ξ the closure $\overline{D}_\xi$ is a regular-closed subset of X, and therefore, by our assumption on X, a G_δ subset of X giving us a way to find the countable family $\mathcal{U}_\xi$ of open sets satisfying (5) and (6).

Claim. *The sequence x_ξ $(\xi < \omega_1)$ has a complete accumulation point which does not belong to the closure of the union of D_ξ's.*

Proof. The closure of

$$D_{\omega_1} = \bigcup_{\xi < \omega_1} D_\xi,$$

is a regular-closed set, so its complement is an F_σ-set in X. Pick a countable family $\mathcal{F}$ of closed subsets of X such that

$$X \setminus \overline{D}_{\omega_1} = \bigcup \mathcal{F}.$$

By (2), the union of $\mathcal{F}$ covers the sequence x_ξ $(\xi < \omega_1)$ so there is $F \in \mathcal{F}$ and uncountable $I \subseteq \omega_1$ such that $x_\xi \in F$ for all $\xi \in I$. Then any complete accumulation point of x_ξ $(\xi \in I)$ satisfies the conclusion of the claim. □

Fix x as in the claim. By (5), for every $\xi \in \omega_1$ there is $U_\xi \in \mathcal{U}_\xi$ containing x. Note that for every finite sequence $\xi_1, \ldots, \xi_n$ of elements of ω_1 the intersection

$$\left(\bigcap_{i=1}^{n} U_{\xi_i}\right) \cap D$$

is uncountable. Combining this with (6) and (7) and using compactness we conclude that

$$\left(\bigcap_{\xi \leqslant \eta} \overline{U}_\xi\right) \cap \overline{D}_{\eta+1}$$

is nonempty for each η. Therefore for each η we can fix an element y_η from this intersection. First of all, note that y_η $(\eta < \omega_1)$ is a discrete sequence since for a given η the set

$$\left(X \setminus \overline{D}_\eta\right) \cap \left(X \setminus \overline{U}_{\eta+1}\right)$$

is an open neighborhood of y_η which contains no y_ξ for $\xi \neq \eta$. Let Z be the set of all complete accumulation points of y_η $(\eta < \omega_1)$. Note that Z is a closed subset of X. If $z \in Z$ is an isolated point of Z then we would be able to select a subsequence y_η $(\eta \in J)$ of y_η $(\eta \in \omega_1)$ which has z as only complete accumulation point. Applying the argument from the proof of the claim to two uncountable disjoint subsequences of y_η $(\eta \in J)$ we would get a contradiction. It follows that Z contains no isolated points. Let F be a proper closed but not open relatively G_δ subset of Z. Then there is an infinite sequence $\{z_n\}$ of elements of $Z \setminus F$ which converges to F. By an easy application of the hereditary Bockstein separation property, we conclude that there is a sequence $\{W_n\}$ of pairwise disjoint open subsets of $X \setminus F$ such that $z_n \in W_n$ for all n and such that $\{W_n\}$ has no accumulation point in $X \setminus F$. Since each z_n is in the closure of D_{ω_1}, for each n there is ξ_n such that

$$W_n \cap D_{\xi_n} \neq \emptyset.$$

Let $\eta \in \omega_1$ be such that $\xi_n \leqslant \eta$ for all n. Then, since $\overline{D}_n$ is disjoint from Z, the sequence

$$\left\{W_n \cap \overline{D}_\eta\right\}$$

of pairwise disjoint subsets of $\overline{D}_\eta$ has no accumulation points in $\overline{D}_\eta$, contradicting the compactness of $\overline{D}_\eta$. This finishes the proof of Theorem 5.1. □

It should be noted that Theorem 5.1 was first proved by Baturov [81] using the assumption that the continuum is not bigger than the second uncountable cardinal. The same restriction on the continuum appears in the version of the following result that appears in [81].

Corollary 5.2. *If* MA_{ω_1} *holds then a compact space* X *is perfect if and only if every subspace of* X *has the Bockstein separation property.*

Proof. This is a combination of Theorem 5.1 and a result of Szentmiklossy [78]. □

We finish this section with the following natural problem.

Problem 5.3. Is there some standard fragment of MA_{ω_1} which is equivalent to the assertion that every perfectly normal compactum is separable?

6. Maps onto Tychonoff cubes

In this section we consider a class of compacta which includes those considered in Sections 3 and 4, the class of compacta which do not map onto the Tychonoff cube $[0,1]^{\omega_1}$. That this is indeed a 'local property' follows from another beautiful result of Shapirovskii [77].

Theorem 6.1 (Shapirovskii). *A compact space X maps onto $[0,1]^{\omega_1}$ if and only if it contains a closed subspace with no point of countable π-character.*

Proof. Going to a closed subspace of X we may assume that X does not have points of countable π-character. We construct recursively sequences (F_ξ, G_ξ) $(\xi < \omega_1)$ and σ_ξ $(\xi < \omega_1)$ such that

(1) F_ξ is a nonempty closed G_δ-subset of X,
(2) G_ξ is an open F_σ-subset of X which includes F_ξ,
(3) $\sigma_\xi \in \mathcal{C}_\xi$, where $\mathcal{C}_\xi$ is the set of all finite partial functions from $\{\alpha\colon \alpha < \xi\}$ into $\{0,1\}$,
(4) for every $\sigma \in \mathcal{C}_\xi$ whose domain is disjoint from that of σ_ξ, if the set $P_{\sigma_\xi\sigma}$ is nonempty then it must intersect both F_ξ and $X \setminus G_\xi$, where $\sigma_\xi\sigma$ is the union of σ_ξ and σ and where, for $\tau \in \mathcal{C}_\xi$, P_τ denotes the intersection of the family
$$\{F_\alpha\colon \tau(\alpha) = 0\} \cup \{X \setminus G_\alpha\colon \tau(\alpha) = 1\}.$$

To see that these sequences can be chosen, given (F_ξ, G_ξ) $(\xi < \eta)$ and σ_ξ $(\xi < \eta)$, note that for every $x \in X$, since its π-character is uncountable, there must be a pair (F_x, G_x) as in (1) and (2) such that $x \in \mathrm{int}(F_x)$ and such that $P_\tau \nsubseteq G_x$ for every $\tau \in \mathcal{C}_\eta$ with $P_\tau \neq \emptyset$. By compactness, choose finite sequence $x_1, \ldots, x_n$ of elements of X such that $\mathrm{int}(F_{x_i})$ $(i = 1, \ldots, n)$ covers X. If $(F_\eta, G_\eta) = (F_{x_1}, G_{x_1})$ satisfies (1)–(4) with $\sigma_\eta = \emptyset$, we are done; otherwise there is $\tau_1 \in \mathcal{C}_\eta$ such that $P_{\tau_1} \neq \emptyset$ and $P_{\tau_1} \cap F_{x_1} = \emptyset$. If $(F_\eta, G_\eta) = (F_{x_2}, G_{x_2})$ and $\sigma_\eta = \tau_1$ satisfy (1)–(4), we are done; otherwise we find $\tau_2 \in \mathcal{C}_\eta$ with domain disjoint from that of τ_1 such that $P_{\tau_1\tau_2} \cap F_{x_2} = \emptyset$ and $P_{\tau_1\tau_2} \neq \emptyset$, and so on. It is clear that we must get what we want before we reach stage n.

Having chosen this sequence, apply the Pressing Down lemma to find an unbounded $\Gamma \subseteq \omega_1$ and $\sigma \in \mathcal{C}_{\omega_1}$ such that $\sigma_\xi = \sigma$ for all $\xi \in \Gamma$. For each $\xi \in \Gamma$ choose a continuous function $f_\xi : X \to [0,1]$ such that $f_\xi^{-1}(0) = F_\xi$ and $f_\xi^{-1}(1) = X \setminus G_\xi$. Let $f : X \to [0,1]^\Gamma$ be the diagonal product of f_ξ $(\xi \in \Gamma)$. Then f is continuous and by (4) its range includes $\{0,1\}^\Gamma$. This finishes the proof. □

So, in this section we shall be working with the class of compact spaces X with the property that for every closed subspace Y of X there is $y \in Y$ with countable π-character relative to Y. The first result that we mention is due to Fremlin (see [14, 44A]) who based his proof on some ideas of Szentmiklossy [78].

Theorem 6.2 (Fremlin). *Assume* MA_{ω_1} *and suppose that a compact space* X *contains a subspace which is hereditarily ccc but not separable. Then* X *maps onto* $[0,1]^{\omega_1}$.

In other words, a slight (?) strengthening of the pathology 'ccc & non-separable' is as strong as the ultimate one, as far as the good local properties are concerned. Theorem 6.2 follows from a lemma which gives a more precise information. The proof that we give below is more direct from that of [14, 44A].

Lemma 6.3. *Assume* MA_{ω_1}, *let* X *be a regular space, and let* Y *be a nonseparable subspace of* X. *Then* Y *either contains an uncountable discrete subspace, or a nonempty subset* D *such that* $Z=\overline{D}$ *has no points of countable* π*-character.*

Proof. Clearly, we may assume that Y can be well-ordered by some $<_w$ in order type ω_1 so that for every $y \in Y$ there is an open neighborhood U_y of y (in X) such that

(1) $x \notin \overline{U}_y$ whenever $x <_w y$.

Let $\mathcal{P}$ be the set of all finite subsets p of Y such that

(2) $y \notin U_x$ for every $x < y$ in p.

If every uncountable $\mathcal{F} \subseteq \mathcal{P}$ contains two different elements p and q whose union is in $\mathcal{P}$, an application of MA_{ω_1} to $\mathcal{P}$ would give us the first alternative. So let us assume there is uncountable $\mathcal{F} \subseteq \mathcal{P}$ such that

(3) $p \cup q \notin \mathcal{P}$ for every $p \neq q$ in $\mathcal{F}$.

Using the Delta-System lemma we may assume that $\mathcal{F}$ consists of disjoint sets all of some fixed size $n \geqslant 1$. An element p of $\mathcal{F}$ has a natural enumeration according to $<_w$, so for $i=1,\ldots,n$, we let $p(i)$ denote the ith element of p according to this enumeration.

Let $\mathcal{G}_1=\mathcal{H}_1=\mathcal{F}$ and

$$D_1=\{p(1)\colon\ p \in \mathcal{H}_1\}.$$

Removing countably many points from D_1 we may assume that every relatively open subset of D_1 is uncountable (since, otherwise, one can easily select an uncountable discrete subspace of D_1). If $Z_1=\overline{D}_1$ contains no point of countable π-character we are done. Otherwise, we select a point $z_1 \in Z_1$ and a countable local π-base $\mathcal{V}_1$ of z_1 in Z_1. Using (1) there is uncountable $\mathcal{G}_2 \subseteq \mathcal{G}_1$ and $V_1 \in \mathcal{V}_1$ such that

(4) $V_1 \cap \overline{U}_y=\emptyset$ for all $y \in p \in \mathcal{G}_2$.

Let

$$\mathcal{H}_2=\{p \in \mathcal{H}_1\colon\ p(1) \in V_1\}.$$

By our assumption $\mathcal{H}_2$ is uncountable and

(5) $q(1) \notin \overline{U}_y$ for all $q \in \mathcal{H}_2$ and $y \in p \in \mathcal{G}_2$.

Now we proceed to

$$D_2=\{q(2)\colon\ q \in \mathcal{H}_2\},$$

the subspace $Z_2 = \overline{D}_2$, and so on. Obviously, this process must stop at some stage $< n$, or else, we get two uncountable subfamilies $\mathcal{G}_n$ and $\mathcal{H}_n$ of $\mathcal{F}$ such that

(6) $q(i) \notin \overline{U}_y$ for all $i = 1, \ldots, n$, $q \in \mathcal{H}_n$, and $y \in p \in \mathcal{G}_n$.

But this means that $p \cup q \in \mathcal{P}$ for every $p \in \mathcal{H}_n$ and $q \in \mathcal{G}_n$ contrary to our assumption (3) about $\mathcal{F}$. At the stage $i < n$ where the process has stopped we get a subspace $Z_i = \overline{D}_i$ with no point of countable π-character, i.e., the second alternative of the lemma. □

Theorem 6.2 leads naturally to the problem of determining under which conditions a compact ccc nonseparable space maps onto $[0,1]^{\omega_1}$. It would be desirable to have a result giving such a condition without the involvement of Martin's axiom or any additional set-theoretical assumption. The analysis might require a different way of constructing continuous maps onto large Tychonoff cubes from the one of Shapirovskii described above in Theorem 6.1. It should also be mentioned that the dual form of Theorem 6.2 (with 'separable' replaced by 'Lindelöf') is missing.

We finish this section with a recent result of D. Fremlin [15] which solves an old problem of R. Haydon.

Theorem 6.4 (Fremlin). *If* MA_{ω_1} *holds then a compact space* X *carries a nonseparable Radon measure if and only if* X *maps continuously onto* $[0,1]^{\omega_1}$.

It was known before (by results of Haydon, Kunen, van Mill and Plebanek [19,22,17, 63]) that under various assumptions (such as the assumption that $[0,1]^{\omega_1}$ can be covered by ω_1 measure zero sets) there exist compact space X and Radon measure μ on X such that the corresponding measure algebra is not separable while X does not map onto $[0,1]^{\omega_1}$. So, some assumption in Fremlin's theorem is needed.

7. The role of compactness

In Section 3 we have seen that under MA_{ω_1} compact first-countable ccc spaces are separable. This was one of the first topological applications of Martin's axiom and it is due to Juhasz [20]. However, in Section 3 we have also seen the rather unexpected fact, that this consequence of MA_{ω_1} is in fact one of its equivalents. In other words, we now know that the assertion that point-countable families of open subsets of first countable ccc spaces are countable is equivalent to MA_{ω_1}. Having in mind the large Tychonoff cubes, we see that some size-restriction like 'first countable' is needed here. How about compactness? In this section we list some examples which show that compactness is also an essential assumption in these results.

Theorem 7.1 (Bell [6]). *There is a first-countable* σ*-compact ccc nonseparable space.*

Theorem 7.2 (Bell [8]). *There is a first-countable countably-compact ccc nonseparable space.*

Proof (*Sketch*). Choose a sequence a_ξ $(\xi < \omega_1)$ of infinite sets of integers such that:
(1) $a_0 = \emptyset$,
(2) $a_\xi \setminus a_\eta$ is finite and $a_\eta \setminus a_\xi$ is infinite whenever $\xi < \eta$.
Let X be the following subspace of $2^{\mathbb{N}} \times \omega_1$:

$$\{(x, \eta)\colon a_\xi \setminus x^{-1}(1) \text{ is finite for all } \xi < \eta\}.$$

It is not hard to show that X is countably compact, first countable, ccc and nonseparable. □

If one allows the second coordinate η to be equal to ω_1, one gets a compactification γX of X which is no longer first countable and non-separable but it gives the following interesting fact which should be kept in mind any time one wants to pass from separability to the countable π-weight.

Theorem 7.3 (Bell [8]). *There is a compact separable space of uncountable π-weight which does not map continuously onto* $[0,1]^{\omega_1}$.

Proof. To show that γX does not map onto $[0,1]^{\omega_1}$, by a result of Shapirovskii (see [21, 3.18] and Theorem 6.1 above), it suffices to show that every closed subspace of γX contains a (relatively) G_δ point. □

We have already mentioned that in the class of compact spaces ccc is equivalent to the formally stronger chain condition asserting that point-finite collections of open sets are countable. The following example shows not only that this fails in a more general class of spaces but it also shows that this is not a productive chain condition. It should be noted that the nonproductiveness of this chain condition was first established by Watson and Zhou [67] but the stress here is, however, on a more restrictive class of spaces where the chain condition method was recently applied by Reed and his students (see [47]).

Theorem 7.4 [55]. *There is a first countable* (*or even a Moore*) *space X such that every point-finite family of open subsets of X is countable but X^2 fails to satisfy this chain condition.*

The space X of Theorem 7.4 is equal to some carefully chosen family of compact sets of reals equipped with the Ochan topology, a refinement of the usual Vietoris topology introduced long ago by Ochan [86] and used in more recent times by Pixley, Roy, van Douwen and others (see [91]). It is interesting that the space of Theorem 7.1 is a modification of the Ochan topology restricted to the collection of finite sets of reals. It should be noted, however, that the countable chain condition of all these spaces is quite strong, but in a different sense: all these spaces have σ-centered π-bases so their compactifications are separable.

8. Souslinean spaces

According to the results of Sections 3–6, compact ccc nonseparable spaces have to be quite complex unless we are using some strong set-theoretical assumptions to construct them. Trying to find out how 'small' they can be, is just another approach to the same old study of which topological properties identify which chain conditions. It turns out that this way of looking at the problem about chain conditions has some applications outside the intended area. For example, a ccc nonseparable space of the form $\gamma\mathbb{N}\setminus\mathbb{N}$ for some compactification $\gamma\mathbb{N}$ of $\mathbb{N}$, first constructed by Bell [7], has been put in a good use by van Mill (see [25] and [26, 4.3.3]) to construct some special points of $\beta\mathbb{N}\setminus\mathbb{N}$. It should be noted, however, that in all examples which we list below, the fact that the space is a growth of $\mathbb{N}$ is usually combinatorially easiest to establish. Making the spaces 'optimally small' will be our main concern here.

Theorem 8.1. *There is a ccc nonseparable growth of* $\mathbb{N}$ *of weight* $\mathfrak{p}$.

Proof. This is basically a reformulation of Theorem 4.5 of [52], a paper which made the real breakthrough in combinatorial analysis of Martin's axiom and its consequences. We start with a reformulation of the usual definition of the cardinal $\mathfrak{p}$ as the minimal cardinal such that:

There exist two families of infinite sets of integers A and B such that $A\cup B$ has size $\mathfrak{p}$ and the following conditions are satisfied:

(1) $a\cap b$ is infinite for every $a\in A$ and $b\in B$,

(2) B is totally ordered by $\subseteq^*$ and its coinitiality is uncountable,

(3) there is no c such that $c\subseteq^* b$ for all $b\in B$ and $c\cap a$ is infinite for all $a\in A$.

Fix such A and B, and assume $(|A|,|B|)$ is lexicographically minimal among all such pairs. Let T be the set of all triples (s,t,n) where n is an integer, s and t are families of subsets of n such that

(4) $|x\cap(\bigcap t)\cap k|\geqslant|\Delta_t\cap k|$ for all $x\in s$ and $k\leqslant n$,

where as before $\Delta_t=\{\min(x\Delta y)\colon x,y\in t,\ x\neq y\}$. For $a\in A$, $b\in B$ and $(s,t,n)\in T$, set

$$U_a=\bigl\{(u,v,m)\in T\colon a\cap m\in u\bigr\},$$

$$V_b=\bigl\{(u,v,m)\in T\colon b\cap m\in v\bigr\},$$

$$T_{(s,t,n)}=\bigl\{(u,v,m)\in T\colon m\geqslant n,\ u\restriction n=s,\ v\restriction n=t\bigr\}.$$

Let $\mathcal{B}$ be the subalgebra of $\mathcal{P}(T)/\mathrm{fin}$ generated by these sets.

Claim 1. *$\mathcal{B}$ is ccc.*

Proof. This is similar and in fact somewhat easier than the corresponding part of the proof of Theorem 3.4 above. □

Claim 2. *$\mathcal{B}$ is not σ-centered.*

Proof. Suppose $\mathcal{B}^+ = \bigcup_{i=0}^{\infty} \mathcal{B}_i$ where each $\mathcal{B}_i$ is a centered collection in $\mathcal{B}^+$. For every i, set

$$B_i = \{b \in B\colon\ V_b \in \mathcal{B}_i\},$$

$$c_i = \bigcap B_i,$$

$$\Delta_{B_i} = \left\{\min(x \Delta y)\colon\ x, y \in B_i,\ x \neq y\right\}.$$

By our assumption (2) there exists i such that B_i is coinitial with B, so by ignoring the rest, we may assume that B_i is coinitial with B for all i. It follows that each c_i satisfies the first requirement of (3). Fix an i, and assume that for some $a \in A$, the generator U_a belongs to $\mathcal{B}_i$. Then, since $\mathcal{B}_i$ is centered, the condition (4) can be applied to show that

(5) $|a \cap c_i \cap k| \geqslant |\Delta_{B_i} \cap k|$ for all k.

Since each Δ_{B_i} is infinite, we conclude that

(6) $a \cap c_i$ is infinite for all $i \in \mathbb{N}$ and $a \in A$ such that $U_a \in \mathcal{B}_i$.

If $\mathfrak{b} > \mathfrak{p}$ then the gap formed by $\{c_i\}$ and B can be interpolated by a single set c contradicting our assumption (3). On the other hand if $\mathfrak{b} = \mathfrak{p}$ then $\mathfrak{t} = \mathfrak{p}$ so (1)–(3) can be witnessed by $A = \{\mathbb{N}\}$ and some nonextendable tower B. By minimality assumption on the pair $(|A|, |B|)$ we conclude that in our case A consists of a single set a and a tower B which cannot be extended to any infinite subset of a. But note that $a \cap c_i$ is such an extension for any i for which B_i is coinitial with B, a contradiction. This completes the proof. □

Let $X = \mathrm{Ult}(\mathcal{B})$. Then X is a growth of some compactification of $\mathbb{N}$, it has weight $\mathfrak{p}$, and it is ccc and nonseparable by Claims 1 and 2. This completes the proof. □

Corollary 8.2 [52]. *Martin's axiom is equivalent to the statement that compact spaces of π-weight smaller than the continuum are separable.*

Proof. The direct implication is a well known application of Martin's axiom due to Hajnal and Juhasz [18] and Kunen (unpublished); see also [14, 43F(b)]. The converse implication is a combination of Theorem 8.1 and the well known characterization of $\mathfrak{p}$ due to Bell [5]; see also [14, 14C]. □

Theorem 8.3 [54]. *There is a ccc nonseparable growth of $\mathbb{N}$ of size continuum and character at any point smaller than $\mathfrak{t}$.*

Proof. This follows from the case $A = \{\mathbb{N}\}$ and $B =$ nonextendable tower of length $\mathfrak{t}$, in the previous construction. The fact that every ultrafilter of the algebra $\mathcal{B}$ is generated by $< \mathfrak{t}$ many sets is analogous (and easier) to the corresponding claim in the proof of Theorem 3.4 above. The fact that the corresponding space $X = \mathrm{Ult}(\mathcal{B})$ has size continuum follows from the well known cardinal equality $2^{<\mathfrak{t}} = \mathfrak{c}$ (see [12]). □

Theorem 8.4. *There is a ccc nonseparable growth X of $\mathbb{N}$ of countable π-character which admits a continuous map f onto a compact metric space such that every fiber of f is homeomorphic to an ordinal smaller than the additivity of the Lebesgue measure.*

Proof. For $i \in \mathbb{N}$ let $\mathbb{N}[i]$ denote the set of all integers of the form $2^i(2j+1)$. Let K be the set of all subsets x of $\mathbb{N}$ such that for every $i \in \mathbb{N}$ the section

$$x[i] = x \cap \mathbb{N}[i]$$

has at most i elements. Identifying sets with their characteristic functions, it is clear that K is closed, and therefore compact, subset of the Cantor set. Let

$$Z = \left\{x \in K\colon \lim_{i\to\infty} \frac{|x[i]|}{i} = 0\right\}.$$

By [92, Theorem 4] there is $A \subseteq Z$ which is well-ordered under $\subseteq^*$ in order-type equal to the additivity of the Lebesgue measure and which is unbounded in K, i.e., there is no $b \in K$ such that $a \subseteq^* b$ for all $a \in A$. We shall assume that A is closed under finite changes of its elements as far as they belong to Z. Set

$$T = \bigl\{(t,n)\colon n \in \mathbb{N},\ t \in K,\ \text{and } t \subseteq n = \{0,1,\dots,n-1\}\bigr\}.$$

To $a \in A$ and $(t,n) \in T$ we associate the following subsets of the tree T, respectively:

$$T_a = \bigl\{(s,m) \in T\colon a \cap m \subseteq s\bigr\},$$

$$T_{(t,n)} = \bigl\{(s,m) \in T\colon m \geqslant n,\ s \cap n = t\bigr\}.$$

Let $\mathcal{B}$ be the subalgebra of $\mathcal{P}(T)/\text{fin}$ generated by these two kinds of subsets of T.

Claim 1. *Every element of $\mathcal{B}^+$ contains a nonzero element of the form $T_a \cap T_{(t,n)}$.*

Proof. It suffices to show how to refine an element of $\mathcal{B}^+$ which is equal to the intersection of finitely many generators or their complements. Note that if F is a finite subset of A with the property that the intersection of T_a $(a \in F)$ is positive in $\mathcal{B}$ then $b = \bigcup F$ is an element of A being a finite modification of the maximal element of F and belonging to K. So in this case we have that

$$\bigcap_{a\in F} T_a = T_b.$$

Note now that the intersection of finitely many generators of the form $T_{(t,n)}$ is also equal to one of them. Finally note that if $T_b \cap T_{(t,n)}$ is not covered modulo a finite set by finitely many generators (of either form) then we can find an extension (s,m) of (t,n) such that $T_b \cap T_{(s,m)}$ is infinite and it avoids all these generators. □

Claim 2. *$\mathcal{B}$ is a ccc algebra* (*and in fact it satisfies Knaster's condition*).

Proof. In order to prove the ccc of $\mathcal{B}$ it suffices to consider an arbitrary uncountable family of elements of the canonical dense subset of $\mathcal{B}^+$ given by Claim 1. Since there exist only countably many objects of the form $T_{(t,n)}$, relativizing the proof to one of them, we may assume to have an uncountable family $\mathcal{F}$ of sets of the form T_a. We need to find two different elements T_a and T_b of $\mathcal{F}$ whose intersection is infinite. Going to an uncountable subfamily of $\mathcal{F}$ we may further assume to have an integer k and for each $i \leqslant k$ a set s_i such that for every T_a in $\mathcal{F}$:

(a) $a \cap \mathbb{N}[i] = s_i$ for $i \leqslant k$, and

(b) $|a[j]|/j \leqslant 1/2$ for $a \in F$ and $j > k$.

Then arbitrary T_a and T_b from $\mathcal{F}$ have an infinite intersection, since by (a) and (b) the pair $((a \cup b) \cap n, n)$ is in T for every n. □

Claim 3. *$\mathcal{B}$ is not σ-centered.*

Proof. Otherwise, since the cofinality of A under $\subseteq^*$ is not countable, we can find a cofinal subset $B \subseteq A$ such that T_b $(b \in B)$ is centered in $\mathcal{B}^+$. For $b \in B$, set

$$K_b = \{a \in K\colon\ (a \cap n, n) \in T_b \text{ for all } n \in \mathbb{N}\}.$$

Then K_b $(b \in B)$ is a centered family of compact subsets of K, so by compactness we can choose c in the intersection of this family. It follows that $b \subseteq c$ for all $b \in B$. Since b is $\subseteq^*$-cofinal in A, this gives us $a \subseteq^* c$ for all $a \in A$, a contradiction. □

Let $\mathcal{B}_0$ be the subalgebra of $\mathcal{B}$ generated by $T_{(t,n)}$ $((t, n) \in T)$. Identifying an ultrafilter $\mathcal{U}$ of $\mathcal{B}_0$ with the filter of $\mathcal{B}$ generated by $\mathcal{U}$, we make the following

Claim 4. *For every $\mathcal{U} \in \mathrm{Ult}(\mathcal{B}_0)$ the quotient algebra $\mathcal{B}/\mathcal{U}$ is an interval algebra over a well-ordered chain of order-type smaller than the additivity of Lebesgue measure.*

Proof. Note that $\mathcal{U}$ is uniquely determined by an element b of K in such a way that for every n, the generator $T_{(b\cap n,n)}$ is the only generator of level n which belongs to $\mathcal{U}$. The quotient algebra $\mathcal{B}/\mathcal{U}$ is therefore generated by T_a $(a \in A_b)$, where A_b is the set of all $a \in A$ such that $a \subseteq b$. It remains only to note that if $a_0 \subseteq^* a_1$ are two elements of A_b then for all sufficiently large n, the intersection $T_{(b\cap n,n)} \cap T_{a_1}$ is included in $T_{(b\cap n,n)} \cap T_{a_0}$. □

Claim 5. *Every $\mathcal{U} \in \mathrm{Ult}(\mathcal{B})$ has countable π-character.*

Proof. Let $\mathcal{U}_0 = \mathcal{U} \cap \mathcal{B}_0$ and let b be the element of K determined by $\mathcal{U}_0$ as in the above proof. Pick a $c \in A$ such that $c \not\subseteq^* b$. Let $A_b = \{a \in A\colon\ a \subseteq b\}$. Since A is totally ordered by $\subseteq^*$, we must have that $a \subseteq^* c$ for all $a \in A_b$. A typical element of $\mathcal{U}$ is equal to the intersection of the form $T_{(b,n)} \cap T_{a_0} \cap (\sim T_{a_1})$, where $n \in \mathbb{N}$ and $a_0 \subseteq^* a_1$ are in A_b. Since $a_0 \subseteq b$ and $a_0 \subseteq^* c$, we can find, in A, a finite modification c^* of c such that $a_0 \subseteq c^*$ and $c^* \cap n = b \cap n$. Since $a_1 \subseteq b$ and $c^* \not\subseteq b$ we can find $m \geqslant n$ such that $c^* \cap m \neq a_1 \cap m$. It follows that $T_{c^*} \cap T_{(c^*\cap m,m)}$ refines the given element of $\mathcal{U}$. Since there exist only countably many possibilities for c^* and m, this gives us a countable π-basis of $\mathcal{U}$ and finishes the proof of Claim 5 as well as the proof of Theorem 8.4. □

Corollary 8.5. *There is a ccc nonseparable growth of* $\mathbb{N}$ *of size continuum which does not map onto the Tychonoff cube* $[0,1]^{\omega_1}$.

Now we go to a combinatorially still finer characteristic associated to the continuum, the cardinal $\mathfrak{b}$, the minimal cardinality of a subset of $\mathbb{N}^{\mathbb{N}}$ which is unbounded in the ordering of eventual dominance.

Theorem 8.6 [54]. *There is a growth of* $\mathbb{N}$ *whose ccc is productive but which has a family of size* $\mathfrak{b}$ *of open subsets without a linked subfamily of the same size.*

Remark 8.7. An interesting example of a compact ccc nonseparable space that does not map onto $[0,1]^{\omega_1}$ was recently constructed by M. Bell [9] assuming that $\mathcal{P}(\mathbb{N})/\text{fin}$ contains some special kind of Hausdorff gaps. Bell's construction (reproduced in part above in Section 5) is based on an elegant theory of 'Total-Ideal-Spaces' over families $\mathcal{Q}$ of partial 0-1-functions on $\mathbb{N}$. Chain conditions of Total-Ideal-Spaces over families of partial functions $\mathcal{Q}$ are the same as those of $\mathcal{Q}$ viewed as partially ordered sets ordered by the inclusion. There is a general fact about posets $\mathcal{Q}$ of partial functions on $\mathbb{N}$ under the assumption of certain combinatorial principle OCA (see [68, Theorem 10.3*]): $\mathcal{Q}$ is ccc if and only if $\mathcal{Q}$ is σ-centered. This explains why the theory of Total-Ideal-Spaces can never give us small compact ccc nonseparable spaces without some additional set-theoretical assumptions, i.e., it can never give us results like Theorems 8.1, 8.3, 8.4, 8.5 and 8.6. However the relevance of gap-spaces in this context has been recently recuperated by Farah [58] who has constructed a kind of Hausdorff gaps in the quotient algebra of $\mathcal{P}(\mathbb{N})$ modulo an F_σ filter on $\mathbb{N}$ which then Moore [71] was able to use in producing a Souslinean space which does not map onto $[0,1]^{\omega_1}$, modifying a construction appearing in a previous version of this survey. The fact that this new construction works under the assumption of MA and non-CH has shattered our hope that the ultimate form of Souslin hypothesis, stating that any compact ccc nonseparable space maps onto $[0,1]^{\omega_1}$, is a consistent statement. This reconsideration has led us to Theorem 8.4, which we did not know at the time of our lectures in North Bay and whose proof, ironically, does not use gaps in quotient algebras at all, but only the simple technology already exposed in the same series of lectures.

9. Compact subsets of function spaces

The chain condition method has been first introduced to this area by H.P. Rosenthal [37, 38] who proved the following result which started a whole new theme of research in this subject.

Theorem 9.1 (Rosenthal). *Every weakly compact subset* K *of some Banach space which satisfies the ccc is separable.*

Proof. By Theorem 2.1(1) if K is ccc then every weakly compact subset of $C(K)$ is separable. By a well known result of Amir and Lindenstrauss [2], the Banach space

$C(K)$ is weakly compactly generated, and therefore, separable. It follows also that K is metrizable. □

Today we know much deeper reasons of why Theorem 9.1 is true. For example, using the result of Amir–Lindenstrauss, Rosenthal himself observed that every weakly compact subset K of a Banach space contains a σ-point-finite collection $\mathcal{F}$ of cozero subsets of K which separates any two points of K in the sense that for every $x \neq y$ in K there is $U \in \mathcal{F}$ which contains exactly one of the points (see [69]). Now, point-finite families in compact ccc spaces are countable (see Lemma 2.2), so if K is ccc the separating family $\mathcal{F}$ is countable and so K is second-countable. However, in [29], Namioka has discovered an even deeper reason:

Theorem 9.2 (Namioka). *Every weakly compact subset K of some Banach space contains a dense subset which is completely metrizable.*

This has turned out to be the right approach as similar results have been established for a wider and wider classes of compact spaces occurring in functional analysis. For example, one of the vast generalizations of Rosenthal's theorem is the following result of Gruenhage [32].

Theorem 9.3 (Gruenhage). *If K is a compact space for which $C(K)$ is weakly countably determined then K contains a dense completely metrizable subspace.*

A space Z is said to be 'countably determined' if it is a continuous image of a closed subset of some product of a compact space and a separable metric space. To relate this to Theorem 9.1 recall the well-known result of Talagrand [94] which says that if K is a weakly compact subset of some Banach space the function space $C(K)$ in its weak topology is a continuous image of a closed subset of some product of a compact space and the irrationals.

Note that if $\mathcal{C}$ is any one of these classes of compacta then it is closed under taking closed subspaces, so we conclude another interesting property of any K in $\mathcal{C}$: every Radon probability measure on K has a separable support. This is so because $K_\mu = \mathrm{supp}(\mu)$ also belongs to $\mathcal{C}$ and μ is a strictly positive measure on K_μ. So in particular, K_μ is ccc and therefore separable. The property that every Radon probability measure on K has a separable support is closely related to the weak Lindelöf property of the function space $C(K)$. In fact, it is equivalent to it in a class of compacta that includes all the classes considered so far:

Theorem 9.4 (Argyros, Mercourakis and Negrepontis). *The following two properties are equivalent for every compact subset K of some sigma-product of the unit interval*:

(a) *Every Radon probability measure on K has a separable support.*
(b) *The Banach space $C(K)$ is weakly Lindelöf.*

Proof. The implication from (a) to (b) is based on a well known result of Alster and Pol [1] and Gulko [70] which says that $C_p(K)$ is Lindelöf for every Corson compactum K.

First of all, $C(K)$ with the weak topology can naturally be identified with a closed subspace of $C_p(P(K))$, where $P(K)$ is the space of the Radon probability measures on K with the weak* topology, the essential part of the unit ball of $C(K)^*$. It turns out that under the assumption of (a) the space $P(K)$ is also a Corson compactum so (b) is an immediate consequence of the Alster–Pol–Gulko theorem. This fact was actually first exposed (without a proof) by R. Pol in [88]. To see this, recall that K can naturally be identified (via Dirac measures, $x \mapsto \delta_x$) with the set of extreme points of $P(K)$, so we are done by the following general fact (see [3]) which is of independent interest.

Lemma 9.5. *The following are equivalent for every Corson compactum* $K \subseteq \Sigma([0,1]^I)$.

(a) *Every Radon probability measure on* K *has a separable support.*

(c) *The closure of the convex hull of* K*, as taken in the Tychonoff cube* $[0,1]^I$*, is actually a subset of the sigma-product* $\Sigma([0,1]^I)$.

Proof. Suppose there is a point $x_0 \in \overline{\mathrm{conv}}(K)$ which has uncountably many non-zero coordinates. Then we can find a net c_ξ $(\xi \in D)$ of finite convex-combinations of elements of K converging to x_0. Each c_ξ can naturally be identified with a finitely-supported Radon probability measure μ_{c_ξ}. So, we can find a subnet c_η $(\eta \in E)$ such that the corresponding net μ_{c_η} $(\eta \in E)$ converges to some $\mu \in P(K)$. Suppose μ has a separable support, i.e., there is a countable set $A \subseteq I$ such that

$$\mathrm{supp}(\mu) \subseteq \{x \in K\colon\ x(i) = 0 \text{ for all } i \notin A\}.$$

Pick $j \in I \setminus A$ such that $x_0(j) > \varepsilon > 0$. Since c_η $(\eta \in E)$ converges to x_0, we have that $c_\eta(j) > \varepsilon$ for almost all $\eta \in E$. It follows that, on one hand, $\int \pi_j \,\mathrm{d}\mu = 0$, while on the other hand $\int \pi_j \,\mathrm{d}\mu_{c_\eta} = c_\eta(j) > \varepsilon$ for almost all $\eta \in E$, contradicting the fact that μ_{c_η} $(\eta \in E)$ converges to μ. This proves that (a) implies (c). The converse implication follows from the general fact that every Radon probability measure μ defined on a *convex* Corson compactum H has a separable support. This is an immediate consequence of the fact that every such μ is represented by a point x of H (see [87]) and therefore, using the Fréchet–Urysohn property of H, it follows that there is a sequence of finite convex-combinations of Dirac measures converging to μ. Clearly, the union of supports of these measures must be dense in $\mathrm{supp}(\mu)$. This finishes the proof of Lemma 9.5. □

The implication from (b) to (a) in Theorem 9.4 is in fact true for every compact space K. To see this, let μ be a given Radon probability measure on K, and going to $\mathrm{supp}(K)$, let us assume μ is strictly positive on K. For $x \in K$ and $n \in \mathbb{N}$, set

$$C_x^n = \{f \in C(K)\colon\ f(x) = 0 \text{ and } \textstyle\int f \,\mathrm{d}\mu \geqslant 1/n\}.$$

Then each C_x^n is a closed convex subset of $C(K)$ and for every n the intersection of C_x^n $(x \in K)$ is empty. Using the weak Lindelöf property of $C(K)$, for each n there is a countable set $D_n \subseteq K$ such that $\bigcap_{x \in D} C_x^n = \emptyset$. It is clear that $\bigcup_{n=1}^{\infty} D_n$ is dense in K. This finishes the proof of Theorem 9.4. □

The weak Lindelöf property of Banach spaces continues to be a source of vital and difficult problems in this area ever since the original paper of Corson [13] where the property was first considered. Theorem 9.4 indicates that in some of these problems the chain condition method is quite relevant. The paper [3] of Argyros et al. particularly emphasizes the usefulness of this method in showing the extreme complexity of the class of Corson-compact spaces as it is able (under certain additional set-theoretic assumptions) to distinguish basically all known chain conditions.

Let us now consider a class of compacta which is topologically much more complex than the class of Corson compacta, but whose importance in functional analysis is not smaller. This is the class of Rosenthal compacta, pointwise compact subsets of the first Baire class over the irrationals. That this class of spaces enjoys some of the pleasant properties of Corson compacta was established in a series of deep results of Bourgain, Fremlin, Talagrand and Godefroy (see [10,16]). For example, Bourgain, Fremlin and Talagrand have established that every Rosenthal compactum is a Fréchet space. Using this result, and his own result saying that the class of Rosenthal compacta is closed under the functor $P(K)$ (the space of Radon probability measures on K with the weak* topology), Godefroy has proved another analogue:

Theorem 9.6 (Godefroy). *Every Radon probability measure μ on a Rosenthal compactum K has a separable support.*

Proof. Every such μ is in the weak*-closure of the set of Radon probability measures on K with finite supports, so by the Bourgain–Fremlin–Talagrand theorem, there is a sequence $\{\mu_i\}$ of such measures with $\mu_i \to \mu$. Let S be the closure of the union of their supports. It is clear that S is a support of μ. □

A stronger version of this result was established by Bourgain (see [95, 14.2.2]): $L^1(K,\mu)$ is separable for every Radon probability measure μ on a Rosenthal compactum K. Note that in the class of Corson compacta K, the separability of $L^1(K,\mu)$ is clearly equivalent to the separability of the support of K, so we did not have to consider the problem of separability of $L^1(K,\mu)$ above. In this context it is also helpful to recall Fremlin's Theorem 6.4 which says that under MA_{ω_1}, if a compactum K supports a Radon probability measure μ for which $L^1(K,\mu)$ is not separable, then K maps onto $[0,1]^{\omega_1}$.

It is now quite natural to ask whether the class of Rosenthal compacta is pathological enough to be able to distinguish between the countable chain condition and separability, a problem first explicitly stated by Pol [30]. Starting from Pol's question we were recently able to prove a more general fact which even better fits the general picture.

Theorem 9.7 [56]. *Every Rosenthal compactum contains a dense metrizable subspace.*

Note the absence of complete metrizability in the conclusion of this result. This is not an accident since, for example, the split interval is a Rosenthal compactum in which all metrizable subspaces are countable and so none of them is dense G_δ. This also hints that the proof of Theorem 9.7 had to be quite different from the old Namioka-style arguments. The proof required an interesting synthesis of the chain-condition method discussed above in Section 3 with the method of forcing and absoluteness. Analyzing the algebraic properties of a carefully chosen point-countable π-basis of a compact countably tight ccc nonseparable space was quite instrumental in deciding that the method of forcing is relevant here. This method is also useful in understanding as well as relating some of the previous work since, for example, it gives another way to prove results like Theorems 9.2 and 9.3 above.

References

[1] K. Alster and R. Pol, On function spaces of compact subspaces of Σ-products of the real line, Fund. Math. 107 (1980) 135–143.

[2] D. Amir and J. Lindenstrauss, The structure of weakly compact sets in Banach spaces, Ann. of Math. 88 (1968) 35–46.

[3] S. Argyros, S. Mercourakis and S. Negrepontis, Functional analytic properties of Corson-compact spaces, Studia Math. 89 (1988) 197–229.

[4] A.V. Arhangel'skii, The property of paracompactness in the class of perfectly normal locally compact spaces, Soviet Math. Dokl. 12 (9) (1971) 1253–1257.

[5] M. Bell, On the combinatorial principle $P(c)$, Fund. Math. 114 (1981) 149–157.

[6] M. Bell, A normal first countable ccc nonseparable spaces, Proc. Amer. Math. Soc. 74 (1979) 151–155.

[7] M. Bell, Compact ccc nonseparable spaces of small weight, Topology Proc. 5 (1980) 11–25.

[8] M. Bell, Spaces of ideals of partial functions, in: Lecture Notes in Math. 1401 (Springer, Berlin, 1980) 1–4.

[9] M. Bell, A compact ccc nonseparable space from a Hausdorff gap and Martin's Axiom, Comment. Math. Univ. Carolin. 37 (1995) 589–594.

[10] J. Bourgain, D. Fremlin and M. Talagrand, Pointwise compact sets of Baire-measurable functions, Amer. J. Math. 100 (1978) 845–886.

[11] W.W. Comfort and S. Negrepontis, Chain Conditions in Topology (Cambridge Univ. Press, Cambridge, 1982).

[12] E.K. van Douwen, The integers and topology, in: Handbook of Set-Theoretic Topology (North-Holland, Amsterdam, 1984).

[13] H.H. Corson, The weak topology of a Banach space, Trans. Amer. Math. Soc. 101 (1961) 1–15.

[14] D.H. Fremlin, Consequences of Martin's Axiom (Cambridge Univ. Press, Cambridge, 1984).

[15] D.H. Fremlin, On compact spaces carrying Radon measures of uncountable Maharam type, 1996, Preprint.

[16] G. Godefroy, Compacts de Rosenthal, Pacific J. Math. 91 (1980) 293–306.

[17] K. Kunen and J. Van Mill, Measures on Corson compact spaces, Fund. Math. 197 (1995) 61–72.

[18] A. Hajnal and I. Juhasz, A consequence of Martin's axiom, Indag. Math. 33 (1971) 457–463.

[19] R.G. Haydon, On dual L^1-spaces and injective dual Banach spaces, Israel J. Math. 31 (1978).

[20] I. Juhasz, Martin's Axiom solves Ponomarev's problem, Bull. Acad. Polon. Sci. 18 (1970) 71–74.

[21] I. Juhasz, Cardinal Functions in Topology, Math. Centre Tract No. 123 (CWI, Amsterdam, 1980).

[22] K. Kunen, A compact L-space under CH, Topology Appl. 12 (1981) 283–287.

[23] K. Kunen and F. Tall, Between Martin's Axiom and Souslin's Hypothesis, Fund. Math. 102 (1979) 173–181.

[24] G. Kurepa, La condition de Souslin et une propriété caractéristique des nombres réels, C. R. Acad. Sci. Paris 231 (1950) 1113–1114.
[25] J. van Mill, Weak P-points in Čech–Stone compactifications, Trans. Amer. Math. Soc. 273 (1982) 657–678.
[26] J. van Mill, An introduction to $\beta\omega$, in: Handbook of Set-Theoretic Topology (North-Holland, Amsterdam, 1984).
[27] S. Koppelberg, General Theory of Boolean Algebras (North-Holland, Amsterdam, 1989).
[28] J.D. Monk, Cardinal Invariants on Boolean Algebras (Birkhauser, Boston, MA, 1996).
[29] I. Namioka, Separate continuity and joint continuity, Pacific J. Math. 51 (1974) 515–531.
[30] R. Pol, On pointwise and weak topology in function spaces, Preprint Nr. 4/84, Warszawa, 1984.
[31] A.S. Mischenko, Spaces with point-countable bases, Soviet Math. Dokl. 3 (1962) 855–858.
[32] G. Gruenhage, A note on Gulko compact spaces, Proc. Amer. Math. Soc. 100 (1987) 371–376.
[33] A. Horn and A. Tarski, Measures on Boolean algebras, Trans. Amer. Math. Soc. 64 (1948) 467–497.
[34] D.M. Fremlin, Measure algebras, in: Handbook on Boolean Algebras (North-Holland, Amsterdam, 1989).
[35] D.M. Fremlin, Large correlated families of positive random variables, Math. Proc. Cambridge Philos. Soc. 103 (1988) 147–162.
[36] D. Maharam, An algebraic characterization of measure algebras, Ann. of Math. 48 (1947) 154–167.
[37] H.P. Rosenthal, On injective Banach spaces and the spaces $C(S)$, Bull. Amer. Math. Soc. 75 (1969) 824–828.
[38] H.P. Rosenthal, On injective Banach spaces and the spaces $L^{\infty}(\mu)$ for finite measures μ, Acta Math. 124 (1970) 205–248.
[39] I. Juhasz, Consistency results in topology, in: Handbook of Mathematical Logic (North-Holland, Amsterdam, 1977).
[40] B.E. Shapirovskii, On π-character and π-weight in compact spaces, Soviet Math. Dokl. 16 (1975) 999–1004.
[41] N.A. Shanin, On products of topological spaces, Trudy Math. Inst. Akad. Nauk SSSR 24 (1948) 112.
[42] E. Marczewski-Szpilrajn, Remarque sur les products cartésiens d'espaces topologiques, Comptes Rendus (Doklady) Acad. Sci. URSS 31 (1941) 525–527.
[43] V.V. Fedorchuk and S. Todorcevic, Cellularity of covariant functors, Topology Appl. 76 (1990) 125–150.
[44] J. Gillis, Note on a property of measurable sets, J. London Math. Soc. 11 (1936) 139–141.
[45] J.L. Kelley, Measures on Boolean algebras, Pacific J. Math. 9 (1959) 1165–1177.
[46] F.B. Jones, Concerning normal completely normal spaces, Bull. Amer. Math. Soc. 43 (1937) 671–677.
[47] G.M. Reed, Set-theoretic problems in Moore spaces, in: Open Problems in Topology (North-Holland, Amsterdam, 1990).
[48] S. Shelah, How special are Cohen and Random forcings, Israel J. Math. 88 (1994) 159–174.
[49] M.G. Tkachenko, Souslin property in free topological groups, Math. Notes 34 (1983) 601–607.
[50] D.B. Shakhmatov, Precalibers of σ-compact topological groups, Math. Notes 39 (1986) 859–868.
[51] S. Todorcevic, Remarks on cellularity in products, Compositio Math. 55 (1989) 295–302.
[52] S. Todorcevic and B. Velickovic, Martin's axiom and partitions, Compositio Math. 63 (1987) 391–408.
[53] S. Todorcevic, Partition Problems in Topology (Amer. Math. Soc., Providence, RI, 1989).
[54] S. Todorcevic, Some compactifications of the integers, Math. Proc. Cambridge Philos. Soc. 112 (1992) 247–254.
[55] S. Todorcevic, Calibers of first countable spaces, Topology Appl. 57 (1994) 317–319.

[56] S. Todorcevic, Compact subset of the first Baire class, Preprint (1996).
[57] F. Tall, The countable chain condition versus separability-applications of Martin's Axiom, General Topology Appl. 4 (1974) 315–319.
[58] I. Farah, An F_σ Hausdorff gap in a quotient over an F_σ P-ideal, Preprint (1998).
[59] R.M. Solovay and S. Tennenbaum, Iterate Cohen extensions and Souslin's problem, Ann. of Math. 94 (1971) 201–245.
[60] E.V. Schepin, Functors and uncountable power of compacta, Russian Math. Surveys 36 (1981) 1–71.
[61] S. Todorcevic, Topics in Topology, Lecture Notes in Math. 1652 (Springer, Berlin, 1997).
[62] S. Todorcevic, Some applications of S and L combinatorics, Ann. New York Acad. Sci. 705 (1993) 130–167.
[63] G. Plebanek, Non-separable Radon measures on small compact spaces, Fund. Math. 153 (1997) 25–40.
[64] B.E. Shapirovskii, Cardinal invariants in compact Hausdorff spaces, Amer. Math. Soc. Transl. (2) 134 (1987) 93–118.
[65] B. Knaster, Sur une propriété caracteristique de l'ensemble des nombres réels, Mat. Sb. 16 (1945) 284–288.
[66] S. Todorcevic, Random set-mappings and separability of compacta, Topology Appl. 74 (1996) 265–274.
[67] S. Watson and Zhou Hao-Xuan, Caliber (ω_1, ω) is not productive, Fund. Math. 135 (1990) 1–4.
[68] S. Todorcevic and I. Farah, Some Applications of the Method of Forcing (Yenisei Publ. Co., Moscow, 1995).
[69] H.P. Rosenthal, The heredity problem of weakly compactly generated Banach spaces, Compositio Math. 28 (1974) 83–111.
[70] S.P. Gulko, On properties of some function spaces, Dokl. Akad. Nauk. SSSR 243 (1978) 839–842.
[71] J. Moore, A linearly fibered Souslinean space under MA, Preprint (1998).
[72] P. Nyikos, L. Soukup and B. Velickovic, Hereditary normality of $\gamma\mathbb{N}$-spaces, Topology Appl. 65 (1995) 9–19.
[73] R.D. Mauldin, ed., The Scottish Book (Birkhäuser, Boston, MA, 1981).
[74] V.V. Tkachuk, A glance at compact spaces which map onto metrizable ones, Topology Proc. 19 (1994) 321–334.
[75] S. Todorcevic, Free sequences, Topology Appl. 35 (1990) 235–238.
[76] N. Kalton, The Maharam Problem, Séminaire Initiation a l'Analyse (G. Choquet, G. Godefroy, M. Rogalski, J. Saint Reymond) 28e Année, no. 18 (1988/89) 13.
[77] B.E. Shapirovskii, Maps onto Tychonoff cubes, Russian Math. Surveys 53 (3) (1980) 145–153.
[78] Z. Szentmiklossy, S-spaces and L-spaces under Martin's Axiom, in: A. Csaszar, ed., Colloq. Math. Soc. Janos Bolyai 23 (North-Holland, Amsterdam, 1980).
[79] M. Katetov, Complete normality of Cartesian products, Fund. Math. 36 (1948) 271–274.
[80] G. Gruenhage and P. Nyikos, Normality of X^2 for compact X, Trans. Amer. Math. Soc. 340 (1993) 563–586.
[81] D.P. Baturov, On hereditarily normal compact spaces in which regular closed subsets are G_δ-sets, Topology Appl. 58 (1994) 151–155.
[82] E.V. Schepin, On topological products, groups, and a new class of spaces more general then metric spaces, Soviet Math. Dokl. 17 (1976) 152–153.
[83] E.V. Schepin, On k-metrizable spaces, Math. USSR Izv. 14 (1980) 407–440.
[84] M. Bockstein, Un théorème de séparabilite pour les produits topologiques, Fund. Math. 35 (1948) 242–246.
[85] A. Pelczynski, Linear extensions, linear averagings, and their applications to linear topological classification of spaces of continuous functions, Dissertationes Math. 68 (PWN, Warsaw, 1968).

[86] Y.S. Ochan, The space of all subsets of a topological space, Dokl. Akad. Nauk USSR 32 (1941) 101–109.

[87] R.R. Phelps, Lectures on Choquet's Theorem, Van Nostrand Mathematical Studies #7 (D. Van Nostrand Co., Princeton, NJ, 1966).

[88] R. Pol, Note on the space $P(S)$ of regular probability measures whose topology is determined by countable subsets, Pacific J. Math. 100 (1987) 185–201.

[89] M. Souslin, Problème 3, Fund. Math. 1 (1920) 223.

[90] J.W. Tukey, Convergence and Uniformity in Topology, Ann. of Math. Stud. 2 (Princeton Univ. Press, Princeton, NJ, 1940).

[91] E.K. van Douwen, The Pixley–Roy topology on spaces of subsets, in: G.M. Reed, ed., Set-Theoretic Topology (Academic Press, New York, 1977).

[92] D.H. Fremlin and K. Kunen, Essentially unbounded chains in compact sets, Math. Proc. Cambridge Philos. Soc. 1–9 (1991) 149–160.

[93] D.B. West, Extremal problems in partially ordered sets, in: I. Rival, ed., NATO Advanced Study Institute on Ordered Sets (D. Reidel, Dordrecht, 1982) 473–521.

[94] M. Talagrand, Espaces de Banach faiblement $\mathcal{K}$-analytiques, Ann. of Math. 110 (1979) 407–438.

[95] M. Talagrand, Pettis integrand and measure theory, Mem. Amer. Math. Soc. 307 (1984).

[96] A. Weil, Sur les Espaces à Structure Uniforme et sur la Topologie Générale, Actualités Sci. Indus. 551 (Hermann, Paris, 1938).