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Generalized Convexity in Multiobjective Programming

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For the scalar programming problem, some characterizations for optimal solutions are known. In these characterizations convexity properties play a very important role. In this work, we study characterizations for multiobjective programming problem solutions when functions belonging to the problem are differentiable. These characterizations need some conditions of convexity. In differentiable scalar programming problems the concept of invexity is very important. We prove that it is also necessary for the multiobjective programming problem and give some characterizations of multiobjective programming problem solutions under weaker conditions. We define analogous concepts to those of stationary points and to the conditions of Kuhn–Tucker and Fritz–John for the multiobjective programming problem.

1. INTRODUCTION

In general, the vector optimization problem (VOP) is represented as the following vector-minimization problem:

$$
\begin{array}{lll}
(\text{VOP}) & \text{Minimize} & f(x) = \big(f_1(x), \dots, f_p(x)\big) \\
 & \text{subject to} & x \in S \subseteq \mathbb{R}^n.
\end{array}
$$

Unlike problems with a unique objective, in which there may exist an optimal solution to the effect that it minimizes the objective function, in the multiobjective programming problem there does not necessarily exist a point which may be optimal for all objectives. To this effect the solution concept introduced by Pareto [15] in which the concept of efficient points is defined must be understood.

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DEFINITION 1.1. A feasible point, $\bar{x}$, is said to be an efficient solution, if and only if there does not exist another $x \in S$ such that $f_i(x) \leq f_i(\bar{x})$ for $i = 1, \ldots, p$ with strict inequality holding for at least one i.

At times, locating the efficient points is quite costly. As a result, there appears a more general concept such as that of the weakly efficient solution, which, under certain conditions, presents topological properties that are not given in the set of efficient points [13].

DEFINITION 1.2. A feasible point, $\bar{x}$, is said to be a weakly efficient solution WEP, if and only if there does not exist another $x \in S$ such that $f_i(x) < f_i(\bar{x})$ for all $i = 1, \ldots, p$.

It is easy to verify that every efficient point is a weakly efficient point.

The following convention for equalities and inequalities will be used. If $x, y \in \mathbb{R}^n$, then

$x = y$ iff $x_i = y_i$ $\quad i = 1, \ldots, n$;

$x \leqq y$ iff $x_i \leq y_i$ $\quad i = 1, \ldots, n$;

$x \leq y$ iff $x_i \leq y_i$ $\quad i = 1, \ldots, n$,

with strict inequality holding for at least one i;

$x < y$ iff $x_i < y_i$ $\quad i = 1, \ldots, n$.

The study of the solutions of a multiobjective problem may be approached from two aspects: one, trying to relate them with the solutions to the scalar problems, whose resolution has been studied extensively and another, trying to locate conditions which are easier to deal with computationally and which guarantee efficiency. As much in one case as in the other, the convexity concept plays an important role as a fundamental condition in obtaining the desired results.

In the past few years extensive literature relative to the other families of more general functions to substitute the convex functions in the mathematical programming has grown immensely. Such functions are called generalized convex functions. Within these and because of their importance, we point out the invex functions, defined by Hanson [7] and Craven [4] and studied extensively by other authors [1], [10], [18], [17], [11], [19].

DEFINITION 1.3. Let $\theta: S \subseteq \mathbb{R}^n \to \mathbb{R}$ be a differentiable function on the open set S. Then θ is invex on S with respect to η if for all $x_1, x_2 \in S$ there exists $\eta(x_1, x_2) \in \mathbb{R}^n$ such that:

$$\theta(x_1) - \theta(x_2) \geq \eta(x_1, x_2)^T \nabla\theta(x_2).$$

There are simple extensions of invex functions to pseudoinvex and quasinvex functions, respectively.

DEFINITION 1.4. Let $\theta: S \subseteq \mathbb{R}^n \to \mathbb{R}$ be a differentiable function on the open set S. Then θ is pseudoinvex on S with respect to η if for all $x_1, x_2 \in S$ there exists $\eta(x_1, x_2) \in \mathbb{R}^n$ such that:

$$\eta(x_1, x_2)^T \nabla\theta(x_2) \geq 0 \Rightarrow \theta(x_1) - \theta(x_2) \geq 0.$$

It is clear that:

$$\text{convexity} \Rightarrow \text{pseudoconvexity} \Rightarrow \text{invexity} \Rightarrow \text{pseudoinvexity}.$$

For the scalar functions the class of invex functions and the class of pseudoinvex functions coincide [2].

DEFINITION 1.5. Let $\theta: S \subseteq \mathbb{R}^n \to \mathbb{R}$ be a differentiable function on the open set S. Then θ is quasinvex on S with respect to η if for all $x_1, x_2 \in S$ there exists $\eta(x_1, x_2) \in \mathbb{R}^n$ such that:

$$\theta(x_1) - \theta(x_2) \leq 0 \Rightarrow \eta(x_1, x_2)^T \nabla\theta(x_2) \leq 0.$$

And it is easy show that: invexity $\Rightarrow$ quasinvexity $\Leftarrow$ quasiconvexity.

Martin [12] studied how these concepts take part in the resolution of the scalar mathematical programming problems of the following forms:

$$\begin{array}{lll} (\mathrm{P}) & \text{Minimize} & \theta(x) \\ & \text{subject to} & x \in S \subseteq \mathbb{R}^n \end{array}$$

and

$$\begin{array}{llll} (\mathrm{CP}) & \text{Minimize} & \theta(x) & \\ & \text{subject to} & g_j(x) \leq 0 & j = 1, \dots, m \\ & & & x \in S \subseteq \mathbb{R}^n \end{array}$$

where $\theta: S \to \mathbb{R}$ and $g: S \to \mathbb{R}^m$ are differentiable functions on an open set $S \subseteq \mathbb{R}^n$.

A point $\bar{x} \in S$ is said to be a stationary point or critical point for θ if $\nabla\theta(\bar{x}) = 0$.

The point $(\bar{x}, \bar{u}) \in S \times \mathbb{R}^m$ with $\bar{x} \in S$ and $\bar{u}_j \geq 0$ for $j = 1, \dots, m$ is said to be a Kuhn–Tucker stationary point for (CP) if

$$\nabla\theta(\bar{x}) + \bar{u}^T \nabla g(\bar{x}) = 0,$$

$$\bar{u}^T g(\bar{x}) = 0.$$

$$g(x) \leq 0$$

For unconstrained problems, Martin [12] proved the following result.

THEOREM 1.1. *A function, θ, is invex in S if and only if every critical point of θ is a global minimizer of θ in S.*

However, for constrained problems the invexity defined by Hanson is a sufficient condition but not a necessary condition for every Kuhn−Tucker point to be a global minimizer.

Martin [12] defined a weaker invexity notion, called Kuhn−Tucker invexity or KT-invexity, which is both necessary and sufficient to establish the Kuhn−Tucker conditions.

DEFINITION 1.6. The problem (CP) is said to be KT-invex on the feasible set with respect to η if for any $x_1, x_2 \in S$ with $g(x_1) \leqq 0$ and $g(x_2) \leqq 0$ there exists $\eta(x_1, x_2) \in \mathbb{R}^n$ such that

$$\theta(x_1) - \theta(x_2) \geq \eta(x_1, x_2)^T \nabla\theta(x_2),$$
$$-\eta(x_1, x_2)^T \nabla g_i(x_2) \geq 0 \quad \forall i \in I(x_2).$$

where $I(x_2) = \{i\colon i = 1, \dots, m \text{ such that } g_i(x_2) = 0\}$.

For scalar constrained problems, Martin gave the following result.

THEOREM 1.2. *Every Kuhn−Tucker stationary point of problem (CP) is a global minimizer if and only if (CP) is KT-invex.*

2. UNCONSTRAINED MULTIOBJECTIVE PROGRAMMING PROBLEMS

In this section, we characterize the solutions for an unconstrained multiobjective programming problem. As in the scalar case, the concept of invexity function plays an important role. Definition 2.1 generalizes the concept of invexity for the p-dimensional case.

DEFINITION 2.1. Let $f\colon S \subseteq \mathbb{R}^n \to \mathbb{R}^p$ be a differentiable function on the open set S. Then f is a vector invex function on S with respect to η if for all $x_1, x_2 \in S$ there exists $\eta(x_1, x_2) \in \mathbb{R}^n$ such that:

$$f(x_1) - f(x_2) \geq \nabla f(x_2)\eta(x_1, x_2)$$

where $\nabla f(x_2) \in \mathscr{M}^{p\times n}$ whose rows are gradient vectors of each component function valued at the point x_2.

Since our purpose is to establish conditions for multiobjective problems, similar to those given by Kuhn−Tucker for the scalar problems, we need to define a concept analogous to the stationary point or critical point for the scalar function.

DEFINITION 2.2. A feasible point, $\bar{x} \in S$, is said to be a vector critical point VCP to (VOP) if there exists a vector $\lambda \in R^p$ with $\lambda \geq 0$ such that $\lambda^T \nabla f(\bar{x}) = 0$.

Scalar stationary points are those whose vector gradients are zero. For vector problems, the vector critical points are those such that there exists a non-negative linear combination of the gradient vectors of each component objective function, valued at that point, equal to zero.

Craven [3] established the following theorem for (VOP).

THEOREM 2.1. *Let $\bar{x}$ be a weakly efficient solution for problem (VOP). Then there exists $\bar{\lambda} \geq 0$ such that $\bar{\lambda}^T \nabla f(\bar{x}) = 0$.*

Then, every weakly efficient solution is a vector critical point, but to establish the reciprocal we need some convexity hypotheses.

THEOREM 2.2. *Let $\bar{x}$ be a vector critical point to problem (VOP) and f an invex function at $\bar{x}$ with respect to η, then $\bar{x}$ is a weakly efficient solution for (VOP).*

Proof. If $\bar{x}$ is a VCP, then there exists $\lambda \geq 0$ such that $\lambda^T \nabla f(\bar{x}) = 0$. By Gordan's theorem, the system

$$\nabla f(\bar{x})^T u < 0$$

does not have a solution at $u \in \mathbb{R}^n$.

From the invexity of f at $\bar{x}$ we have that $\forall x \in S$, there exists $\eta(x, \bar{x})$ such that

$$f(x) - f(\bar{x}) \geq \nabla f(\bar{x}) \eta(x, \bar{x}).$$

Then, there will not exist any $x \in S$ such that $f(x) < f(\bar{x})$. Therefore $\bar{x}$ is a weakly efficient solution for (VOP). ▮

The usual way to solve multiobjective programming problems is to relate its weakly efficient solutions to the optimal solutions for scalar problems whose resolution has already been studied. Ruíz and Rufián [16] have characterized weakly efficient solutions in the case of nondifferentiable functions. In this work we characterize these solutions when the functions are differentiable.

One of the most known scalar problems associated with multiobjective programming problems is the weighting problem whose formulation has the following form:

$$(P_\lambda) \quad \text{Minimize} \quad \lambda^T f(x)$$
$$\text{subject to} \quad x \in S \subseteq \mathbb{R}^n$$

where $\lambda \in \mathbb{R}^p$.

It has been proved that every solution of weighting scalar problem with $\lambda \geq 0$ is a weakly efficient solution, but the reciprocal is not always true [5].

THEOREM 2.3. *If f is invex on an open set S, then all weakly efficient solutions solve a weighting scalar problem with* $\lambda \geq 0$.

Proof. Let $\bar{x}$ be a WEP; then there exists $\lambda \in \mathbb{R}^p$ with $\lambda \geq 0$ such that $\lambda^T \nabla f(\bar{x}) = 0$.

On the other hand, if f is invex at $\bar{x}$ with respect to η, then $\lambda^T f$ is also an invex function with respect to η, so for any $x \in S$ there exists $\eta(x, \bar{x}) \in \mathbb{R}^n$ such that

$$\lambda^T f(x) - \lambda^T f(\bar{x}) \geq \lambda^T \nabla f(\bar{x}) \eta(x, \bar{x}) = 0.$$

Then $\lambda^T f(x) \geq \lambda^T f(\bar{x})$, $\forall x \in S$. And so, $\bar{x}$ is optimal solution for P_λ, with $\lambda \geq 0$. ∎

Under the invexity condition, we have seen that the vector critical points, the weakly efficient solutions and the optimal solutions for weighting scalar problems coincide.

Now we will show that the invexity condition is not only a sufficient but also a necessary condition for all these classes of points to be equivalent.

THEOREM 2.4. *Each vector critical point is a weakly efficient solution and solve a weighting scalar problem if and only if the objective function is invex.*

Proof. The sufficient part has already been proved [theorem 2.2, theorem 2.3].

Let us suppose that the three classes of points are equivalent; then the following system has no solution for any $\bar{x}$, $x \in S$

$$[\nu \mid \lambda] \left[\begin{array}{c|c} 0 & 1 \\ \nabla f(\bar{x}) & f(x) - f(\bar{x}) \end{array} \right] = 0$$

$$\nu > 0, \qquad \lambda \geq 0.$$

By Gordan's alternative theorem, the system

$$\left[\begin{array}{c|c} 0 & 1 \\ \nabla f(\bar{x}) & f(x) - f(\bar{x}) \end{array} \right] \begin{bmatrix} \eta \\ \xi \end{bmatrix} < 0$$

has solution at $\eta \in \mathbb{R}^n$ and $\xi \in \mathbb{R}$ $\forall \bar{x}, x \in S$. This implies that

$$\xi < 0$$

and

$$\nabla f(\bar{x})\eta + \xi(f(x) - f(\bar{x})) < 0.$$

Putting $\eta = \eta / -\xi$ we obtain that, for all $\bar{x}, x \in S$, there exists a vector function $\eta(x, \bar{x})$ such that

$$f(x) - f(\bar{x}) \geqq \nabla f(\bar{x})\eta(x, \bar{x}).$$

Therefore, f is invex on S. ▮

We have proved that if the vector objective function is invex, then all vector critical points are weakly efficient solutions [theorem 2.2]. That equivalence is true under weaker conditions. To prove this assertion, we first define the pseudoinvexity concept for vector functions.

DEFINITION 2.3. Let $f: S \subseteq \mathbb{R}^n \to \mathbb{R}^p$ be a differentiable function on the open set S. Then f is a vector pseudoinvex function on S with respect to η if for all $x_1, x_2 \in S$ there exists $\eta(x_1, x_2) \in \mathbb{R}^n$ such that:

$$f(x_1) - f(x_2) < 0 \Rightarrow \nabla f(x_2)\eta(x_1, x_2) < 0.$$

It is clear that if f is invex, f is pseudoinvex too.

THEOREM 2.5. *All vector critical points are weakly efficient solutions if and only if f is a vector pseudoinvex function on S.*

Proof. Let us suppose that all vector critical are weakly efficient solutions and let $\bar{x}$ be a WEP; then the system

$$f_i(x) - f_i(\bar{x}) < 0 \qquad i = 1, \dots, p$$

has no solution in $x \in S$.

On the other hand, if $\bar{x}$ is a VCP, then $\exists\lambda$ such that $\lambda^T \nabla f(\bar{x}) = 0$. Applying Gordan's theorem the next system has no solution at $u \in \mathbb{R}^n$

$$\nabla f_i(\bar{x})^T u < 0 \qquad i = 1, \dots, p$$

and the reciprocals are also true. Thus, if there exists $x \in S$ such that $f(x) < f(\bar{x})$, then there exists $\eta(x, \bar{x}) \in \mathbb{R}^n$ such that $\nabla f(\bar{x})\eta(x, \bar{x}) < 0$, And so, f is pseudoinvex on S.

Now, let us assume that f is pseudoinvex on S and suppose that $\bar{x}$ is a vector critical point but that it is not a weakly efficient solution. Then there exists another point $x \in S$ such that $f(x) < f(\bar{x})$; then $\nabla f(\bar{x})\eta(x, \bar{x}) < 0$.

On the other hand, there exists $\lambda \in \mathbb{R}^p$, $\lambda \leq 0$, such that $\lambda^T \nabla f(\bar{x}) = 0$. And this is a contradiction to Gordan's alternative theorem. ▮

That theorem coincides with the one proved by Martin for the scalar case [theorem 1.1] since in this case the invex and pseudoinvex functions coincide.

3. CONSTRAINED MULTIOBJECTIVE PROGRAMMING PROBLEMS

Consider the following constrained multiobjective programming problem (CVOP). Often, the feasible set can be represented by functional inequalities as in the following:

$$\text{(COVP)} \quad \text{Minimize} \quad f(x) = \big(f_1(x), \dots, f_p(x)\big)$$
$$\text{subject to} \quad g(x) \leqq 0$$
$$x \in S \subseteq \mathbb{R}^n$$

where $f: S \to \mathbb{R}^p$ and $g: S \to \mathbb{R}^m$ are differentiable functions on the open set $S \subseteq \mathbb{R}^n$.

In this section, we characterize weakly efficient solutions for the (CVOP) problem using concepts similar to Fritz–John and Kuhn–Tucker optimality condition concepts.

DEFINITION 3.1. A feasible point, $\bar{x} \in S$, is said to be a vector Fritz–John point (VFJP) to the problem (CVOP) if there exists a vector $(\overline{\lambda}, \overline{\mu}) \in \mathbb{R}^{p+m}$ with $(\overline{\lambda}, \overline{\mu}) \geq 0$ such that:

$$\overline{\lambda}^T \nabla f(\bar{x}) + \overline{\mu}^T \nabla g(\bar{x}) = 0 \tag{1a}$$

$$\overline{\mu}^t g(\bar{x}) = 0. \tag{1b}$$

DEFINITION 3.2. A feasible point, $\bar{x} \in S$, is said to be a vector Kuhn–Tucker point (VKTP) to the problem (CVOP) if there exists a vector $(\overline{\lambda}, \overline{\mu}) \in \mathbb{R}^{p+m}$ with $(\overline{\lambda}, \overline{\mu}) \geq 0$ and $\overline{\lambda} \neq 0$ such that:

$$\overline{\lambda}^T \nabla f(\bar{x}) + \overline{\mu}^T \nabla g(\bar{x}) = 0 \tag{2a}$$

$$\overline{\mu}^t g(\bar{x}) = 0. \tag{2b}$$

Observe that in definition 3.2 it is not necessary for $\overline{\lambda}$ to be strictly positive; it is sufficient that $\overline{\lambda} \neq 0$.

The following results extend the scalar case in a natural way. In fact, the above definitions coincide with the Fritz–John and Kuhn–Tucker conditions when f is a numerical function.

THEOREM 3.1. *Let $\bar{x}$ be a weakly efficient solution for problem $(CVOP)$; then there exist $\bar{\lambda}$ and $\bar{\mu}$ such that $\bar{x}$ is a vector Fritz–John point for $(CVOP)$.*

Proof. Let $I(\bar{x}) = \{j = 1, \dots, m$ such that $g_j(\bar{x}) = 0\}$. We will prove that there exist $\bar{\lambda} \in \mathbb{R}^p$ with $\bar{\lambda} \geq 0$, and $\bar{\mu}_j \geq 0$, for all $j \in I(\bar{x})$ holding

$$\sum_{i=1}^{p} \bar{\lambda}_i \nabla f_i(\bar{x}) + \sum_{j \in I(\bar{x})} \bar{\mu}_j \nabla g_j(\bar{x}) = 0. \tag{3}$$

Putting $\bar{\mu}_j = 0$ for all $j \notin I(\bar{x})$ we obtain (1a) and (1b).

Let us suppose that (3) has no solution, so there do not exist $\bar{\lambda}$ and $\bar{\mu}$ satisfying (3). By Motzkin's theorem, the system

$$\begin{aligned} \nabla f_i(\bar{x})^T u &< 0 \qquad i = 1, \dots, p \\ \nabla g_j(\bar{x})^T u &< 0 \qquad j \in I(\bar{x}) \end{aligned} \tag{4}$$

has a solution at $u \in \mathbb{R}^n$.

Then there exists $u \in \mathbb{R}^n$ such that for all i we have

$$0 > \nabla f_i(\bar{x})^T u = \lim_{h \to 0} \frac{f_i(\bar{x} + hu) - f_i(\bar{x})}{h}.$$

And so, for all i there exist h_0^i such that:

$$f_i\left(\bar{x} + h_0^i u\right) - f_i(\bar{x}) < 0.$$

Let $h_0 = \operatorname{Min}\{h_0^i \colon i = 1, \dots, p\}$ be; then

$$f_i(\bar{x} + h_0 u) - f_i(\bar{x}) < 0 \qquad i = 1, \dots, p. \tag{5}$$

On the other hand, let $J_1 = \{j \in I(\bar{x})$ such that $\nabla g_j(\bar{x})^T u < 0\}$ be. For all $j \in J_1$ we can find a h_1^j such that

$$g_j\left(\bar{x} + h_1^j u\right) - g_j(\bar{x}) = g_j\left(\bar{x} + h_1^j u\right) < 0. \tag{6}$$

Let us take $h_1 = \operatorname{Min}\{h_1^j \colon j \in J_1\}$.

As g_j with $j \notin I(\bar{x})$ is differentiable, and therefore continuous, and $g_j(\bar{x}) < 0$; then

$$\lim_{h \to 0} g_j(\bar{x} + hu) = g_j(\bar{x}) < 0.$$

For $h_2 = \operatorname{Min}\{h_2^j \colon j \notin I(\bar{x})\}$ we obtain that

$$g_j(\bar{x} + h_2 u) \leq 0 \qquad j \notin I(\bar{x}). \tag{7}$$

Let $z = \bar{x} + \bar{h}u$ be where $\bar{h} = \text{Min}\{h_0, h_1, h_2\}$. From (6) and (7), z is a feasible point for (CVOP), and so (5) is a contradiction to $\bar{x}$ being a weakly efficient point. ∎

If we add a constraint qualification, we can be sure that $\bar{\lambda}$ is not equal to zero.

THEOREM 3.2. *Let $\bar{x}$ be a weakly efficient solution for problem (CVOP) and the Kuhn–Tucker constraint qualification is satisfied at $\bar{x}$. Then there exist $\bar{\lambda}$ and $\bar{\mu}$ such that $\bar{x}$ is a vector Kuhn–Tucker point for problem (CVOP).*

Proof. Let us suppose that the following system has no solution

$$\lambda^T \nabla f(\bar{x}) + \mu_I g(\bar{x}) = 0,$$
$$\lambda \in \mathbb{R}^p, \qquad \lambda \geq 0 \qquad \mu_j \geq 0, \quad \forall j \in I(\bar{x}).$$

Then, there exists $u \in \mathbb{R}^n$ such that

$$\begin{aligned} \nabla f_i(\bar{x})^T u &< 0 \qquad i = 1, \dots, p, \\ \nabla g_j(\bar{x})^T u &< 0 \qquad j \in I(\bar{x}). \end{aligned} \tag{8}$$

Hence by the Kuhn–Tucker constraint qualification at $\bar{x}$, there exists a n-dimensional vector function e defined on $[0,1]$ such that $e(0) = \bar{x}$, $e(\tau) \in \{x \in S \mid g(x) \leqq 0\}$ for $0 \leq \tau \leq 1$, e is differentiable at $\tau = 0$, and $[de(0)]/d\tau = \xi u$ for some $\xi > 0$. Therefore,

$$\nabla f(\bar{x}) < 0 \Rightarrow \xi \nabla f(\bar{x}) u < 0, \quad \text{for some } \xi > 0 \Rightarrow$$
$$\Rightarrow \nabla f(e(0)) < 0 \Rightarrow \lim_{\tau \to 0} \frac{f(e(\tau)) - f(\bar{x})}{\tau} < 0.$$

Therefore, there exists τ_0 such that for all $\tau \leq \tau_0$ we have

$$f(e(\tau)) < f(\bar{x}),$$

and $e(\tau)$ is feasible for all $\tau \in [0,1]$. Then the above inequality is a contradiction to weak efficiency of $\bar{x}$. ∎

To establish reciprocals of the theorems 3.1 and 3.2, we need a generalized convexity hypothesis. As in the scalar case, we prove that KT-invexity for the optimization problem is sufficient for all vector Kuhn–Tucker points to be weakly efficient solutions.

We define a KT-invex multiobjective programming problem, properly.

DEFINITION 3.3. The problem (CVOP) is said to be a vector KT-invex problem on the feasible set with respect to η if for any $x_1, x_2 \in S$ with

$g(x_1) \leqq 0$ and $g(x_2) \leqq 0$ there exists $\eta(x_1, x_2) \in \mathbb{R}^n$ such that:

$$f(x_1) - f(x_2) \geq \nabla f(x_2)\eta(x_1, x_2)$$
$$-\nabla g_i(x_2)\eta(x_1, x_2) \geq 0 \quad \forall i \in I(x_2).$$

Now, we prove the following theorem for vector KT-invex problems.

THEOREM 3.3. *Every vector Kuhn–Tucker point is a weakly efficient solution if problem (CVOP) is KT-invex.*

Proof. Let $\bar{x}$ be a vector Kuhn–Tucker point for (CVOP) and let us suppose that this problem is KT-invex. We will see that $\bar{x}$ is a WEP.

If there exists another feasible point x such that $f(x) < f(\bar{x})$, then

$$0 > f(x) - f(\bar{x}) \geqq \nabla f(\bar{x})\eta(x, \bar{x}) \Rightarrow \bar{\lambda}^T \nabla f(\bar{x})\eta(x, \bar{x}) < 0, \quad \forall \bar{\lambda} \geq 0 \tag{9}$$

Since $\bar{x}$ was assumed a VKTP,

$$\bar{\lambda}^T \nabla f(\bar{x})\eta(x, \bar{x}) + \sum_{j \in I(\bar{x})} \bar{\mu}_j \nabla g_j(\bar{x})\eta(x, \bar{x}) = 0. \tag{10}$$

From (9) and (10) we have that

$$\sum_{j \in I(\bar{x})} \bar{\mu}_j \nabla g_j(\bar{x})\eta(x, \bar{x}) > 0. \tag{11}$$

As the problem (CVOP) is KT-invex, then

$$-\nabla g_j(\bar{x})\eta(x, \bar{x}) \geq 0 \quad \forall j \in I(\bar{x}).$$

Since $\bar{\mu}_j \geq 0$,

$$-\bar{\mu}_j \nabla g_j(\bar{x})\eta(x, \bar{x}) \geq 0 \quad \forall j \in I(\bar{x});$$

therefore

$$\sum_{j \in I(\bar{x})} \bar{\mu}_j \nabla g_j(\bar{x})\eta(x, \bar{x}) \leq 0.$$

This is a contradiction to (11). ∎

As for unconstrained multiobjective programming problems, all optimal solutions for weighting scalar problems with $\lambda \geq 0$ are weakly efficient solutions, but the reciprocal is not always true. We use the above theorem to prove that if the problem (CVOP) is KT-invex, then all weakly efficient solutions can be found as solutions for a scalar problem. Thus, under the

KT-invexity condition and if constraint qualification is satisfied, vector Kuhn–Tucker points, weakly efficient points, and optimal solutions for weighting problems coincide.

THEOREM 3.4. *If problem (CVOP) is KT-invex and the Kuhn–Tucker constraint qualification is satisfied at all weakly efficient solutions, then every weakly efficient solution solves a weighting scalar problem.*

Proof. Let be $\bar{x}$ a WEP; then there exists $\bar{\lambda} \geq 0$ and $\bar{\mu} \geqq 0$ such that

$$\bar{\lambda}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x}) = 0$$

$$\bar{\mu}^T g(\bar{x}) = 0.$$

Or, equivalently,

$$\bar{\lambda}^T \nabla f(\bar{x}) + \bar{\mu}_I^T \nabla g_I(\bar{x}) = 0$$

where $I = \{i = 1, \dots, m$ such that $g_i(\bar{x}) = 0\}$. By KT-invexity, for all feasible points, $x \in S$, we have

$$f(x) - f(\bar{x}) \geqq \nabla f(\bar{x}) \eta(x, \bar{x})$$

$$0 \geqq \nabla g_I(\bar{x}) \eta(x, \bar{x}).$$

Or, equivalently,

$$\bar{\lambda}^T f(x) - \bar{\lambda}^T f(\bar{x}) \geqq \bar{\lambda}^T \nabla f(\bar{x}) \eta(x, \bar{x})$$

$$0 \geqq \sum_{j \in I(\bar{x})} \bar{\mu}_j \nabla g_j(\bar{x}) \eta(x, \bar{x}).$$

Adding the above inequalities

$$\bar{\lambda}^T f(x) - \bar{\lambda}^T f(\bar{x}) \geq \bar{\lambda}^T \nabla f(\bar{x}) \eta(x, \bar{x}) + \sum_{j \in I(\bar{x})} \bar{\mu}_j \nabla g_j(\bar{x}) \eta(x, \bar{x}) = 0,$$

for all $x \in S$ such that $g(x) \leqq 0$. So, $\bar{x}$ is an optimal solution for $P_{\bar{\lambda}}$. ∎

Now, we prove an analogous theorem for vector Kuhn–Tucker points.

THEOREM 3.5. *Every vector Kuhn–Tucker point solves a weighting scalar problem if (CVOP) is KT-invex.*

Proof. Let $\bar{x}$ be a vector Kuhn–Tucker point; then there exist $\bar{\lambda} \geq 0$, and $\bar{\mu} \geqq 0$ such that

$$\bar{\lambda}^T \nabla f(\bar{x}) + \bar{\mu}^T \nabla g(\bar{x}) = 0$$

$$\bar{\mu}^T g(\bar{x}) = 0.$$

Hence $(\bar{x}, \overline{\mu})$ holds Kuhn–Tucker conditions for the scalar programming problem $P_{\bar{\lambda}}$. Moreover, if problem (CVOP) is KT-invex, then problem $P_{\bar{\lambda}}$ is KT-invex and therefore $\bar{x}$ is an optimal solution for $P_{\bar{\lambda}}$ [theorem 1.2]. ∎

We prove our second main result.

THEOREM 3.6. *Every vector Kuhn–Tucker point is a weakly efficient point and solves a weighting scalar problem if and only if the problem (CVOP) is KT-invex.*

Proof. In theorem 3.3 and theorem 3.5 we have proved that if the problem (CVOP) is KT-invex, every VKTP is a WEP and solves a P_λ. Let us prove the reciprocal.

If every VKTP is a WEP and solves a P_λ, then the following system has no solution at $\nu > 0$, $\lambda \geq 0$, for all $\bar{x}, x \in S$, $g(\bar{x}) \leqq 0$ and $g(x) \leqq 0$

$$[\nu \mid \lambda]\left[\begin{array}{c|c} 0 & 1 \\ \nabla f(\bar{x}) & f(x) - f(\bar{x}) \end{array}\right] + y\left[\nabla g_I(\bar{x}) \mid 0\right] = 0.$$

Then, by Motzkin's alternative theorem, the following system has a solution, for all $\bar{x}, x \in S$ with $g(\bar{x}) \leqq 0$ and $g(x) \leqq 0$,

$$\left[\begin{array}{c|c} 0 & 1 \\ \nabla f(\bar{x}) & f(x) - f(\bar{x}) \end{array}\right]\begin{bmatrix} \eta \\ \xi \end{bmatrix} < 0,$$

$$\left[\nabla g_I(\bar{x}) \mid 0\right]\begin{bmatrix} \eta \\ \xi \end{bmatrix} \leqq 0,$$

where $\eta \in \mathbb{R}^n$ and $\xi \in \mathbb{R}$. Or, equivalently, the system

$$\xi < 0$$

$$\nabla f(\bar{x})\eta + \xi(f(x) - f(\bar{x})) < 0$$

$$\nabla g_I(\bar{x})\eta \leqq 0$$

has solution at $\eta \in \mathbb{R}^n$ and $\xi \in \mathbb{R}$.

And so, for all $\bar{x}, x \in S$ with $g(x) \leqq 0$ and $g(\bar{x}) \leqq 0$, there exists a vector function $\eta(x, \bar{x})$ such that

$$f(x) - f(\bar{x}) \geq \nabla f(\bar{x})\eta(x, \bar{x})$$

$$-\nabla g_I(\bar{x})\eta(\bar{x}) \geqq 0.$$

Thus, the multiobjective programming problem (CVOP) is KT-invex on the feasible set. ∎

As for the unconstrained problems, the KT-invexity is an unnecessarily strong condition to all weakly efficient solutions for (CVOP) to be a vector Kuhn–Tucker point for problem (CVOP). Let us now define a weaker generalized convex condition for a vector programming problem.

DEFINITION 3.4. The problem (CVOP) is said to be a vector KT-pseudoinvex problem with respect to η if for any $x_1, x_2 \in S$ with $g(x_1) \leqq 0$ and $g(x_2) \leqq 0$ there exists $\eta(x_1, x_2) \in \mathbb{R}^n$ such that:

$$f(x) < f(\bar{x}) \Rightarrow \nabla f(\bar{x})\eta(x, \bar{x}) < 0$$
$$-\nabla g_i(\bar{x})\eta(x, \bar{x}) \geqq 0 \qquad i \in I(\bar{x}).$$

And now we prove that this condition is necessary and sufficient for the set of vector Kuhn–Tucker points and the set of weakly efficient point to be the same.

THEOREM 3.7. *Every vector Kuhn–Tucker point is weakly efficient for (CVOP) if and only if problem (CVOP) is a KT-pseudoinvex problem.*

Proof. Let us suppose that every vector Kuhn–Tucker point is a weakly efficient solution. If $\bar{x}$ is a vector Kuhn–Tucker point, then there exists $\bar{\lambda} \geq 0$ and $\bar{\mu} \geqq 0$ such that

$$\bar{\lambda}^T \nabla f(\bar{x}) + \sum_{j \in I(\bar{x})} \bar{\mu}_j \nabla g(\bar{x}) = 0. \tag{12}$$

By Motzkin's alternative theorem, the following system does not have any solution

$$\begin{aligned} &\nabla f_i(\bar{x})^T u < 0 \qquad i = 1, \dots, p \\ &\nabla g_j(\bar{x})^T u \leq 0, \qquad j \in I(\bar{x}). \end{aligned} \tag{13}$$

If $\bar{x}$ is a VKTP, then $\bar{x}$ is a WEP and therefore the system

$$\begin{aligned} f_i(x) - f_i(\bar{x}) &< 0 \qquad i = 1, \dots, p \\ g(x) &\leqq 0 \end{aligned} \tag{14}$$

does not have any solution.

If $\bar{x}$ is not a VKTP, then (12) does not have any solution. This implies that (13) has a solution. If $\bar{x}$ is not a VKTP, then $\bar{x}$ is not a WEP and (14) does not have any solution. Then, for all $x \in S$ with $g(x) \leqq 0$, if $f_i(x) < f_i(\bar{x})$, there exists $\eta(x, \bar{x})$ such that

$$\nabla f_i(\bar{x})\eta(x, \bar{x}) < 0 \qquad i = 1, \dots, p$$
$$-\nabla g_j(\bar{x})\eta(x, \bar{x}) \geq 0 \quad \forall j \in I(\bar{x}).$$

Then we have

$$\forall x, \bar{x} \in S \qquad g(x) \leqq 0, \qquad g(\bar{x}) \leqq 0$$
$$f(x) - f(\bar{x}) < 0 \Rightarrow \nabla f(\bar{x})\eta(x, \bar{x}) < 0$$
$$-\nabla g_j(\bar{x})\eta(x, \bar{x}) \geq 0, \quad \forall j \in I(\bar{x}).$$

Therefore (CVOP) is a KT-pseudoinvex problem.

The proof of the reciprocal is analogous to the proof of theorem 3.3. ∎

4. CONCLUSION

In this work, we have analyzed the resolution of the differentiable multiobjective programming problems from two points of view. We have studied the relation with the solutions of associated scalar problems, as well as the search for the conditions of optimality which are easiest to handle.

As much in one case as in the other, the convexity conditions on the functions of the problem are fundamental. Just as in scalar programming, convexity may be substituted by more general conditions. We have studied the multiobjective programming problems under different invexity definitions.

Invexity allows us to give the necessary and sufficient conditions for locating the solutions to the general problem starting from the solutions of a scalar problem or verifying certain conditions of optimality of the type defined by Kuhn−Tucker and Fritz−John. Some results may therefore be concluded which were left incomplete in the literature.

This work shows that the results published in the past few years for scalar programming are transferable to the multiobjective programming, making it possible to enclose everything within the general scheme which we have constructed here.

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