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Congruence Relations Characterizing the Representation Ring of the Symmetric Group

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Using the theory of Burnside rings, various canonical families of congruence relations are exhibited which hold for characters of the symmetric group and characterize the ring generated by them as a subring of the ring of integer valued class functions. As a corollary, the irreducible characters are characterized as integer valued class functions $\chi: \Sigma_n \to \mathbb{Z}$ which satisfy those congruence relations and the standard conditions $\chi(1) > 0$ and $\sum_{\sigma \in \Sigma_n} \chi(\sigma)^2 = n!$. © 1986 Academic Press, Inc.

1. RESTRICTED BURNSIDE RINGS

Let G denote a finite group and let $\mathscr{U}$ denote a set of subgroups of G which is closed with respect to conjugation and intersection and with $G \in \mathscr{U}$. We define a $(G, \mathscr{U})$-set as a finite, left G-set S with $G_s =: \{g \in G \mid gs = s\} \in \mathscr{U}$ for all $s \in S$.

Our conditions on $\mathscr{U}$ imply that for any $U \in \mathscr{U}$ the set $G/U = \{gU \mid g \in G\}$ of cosets of U in G is a $(G, \mathscr{U})$-set and that for any two $(G, \mathscr{U})$-sets S_1 and S_2 the G-sets $S_1 \cup S_2$ and $S_1 \times S_2$ are $(G, \mathscr{U})$-sets. So the isomorphism classes of $(G, \mathscr{U})$-sets form a commutative half ring $\Omega^+(G, \mathscr{U})$ which—using the standard Grothendieck ring construction—gives rise to an associated Grothendieck ring $\Omega(G, \mathscr{U})$, the Burnside ring of G with respect to $\mathscr{U}$.

Note that $\Omega(G, \mathscr{U})$ coincides with the Burnside ring $\Omega(G)$ of G if $\mathscr{U}$ coincides with the set $\mathscr{S}(G)$ of all subgroups of G. Using standard arguments concerning Burnside rings the following statements are easily verified:

(i) $\Omega(G, \mathscr{U})$ is generated freely as an additive group by the isomorphism classes of transitive $(G, \mathscr{U})$-sets, i.e., of G-sets of the form G/U with $U \in \mathscr{U}$: so its rank equals the number $k = k_{\mathscr{U}}$ of conjugacy classes of subgroups in $\mathscr{U}$.

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(ii) For any subgroup $V \leqslant G$ of G, whether in $\mathscr{U}$ or not, the mapping

$$\chi_V: S \mapsto \#\{s \in S \mid V \leqslant G_s\}$$

which associates with any $(G, \mathscr{U})$-set S the number of elements in $S^V =: \{s \in S \mid V \leqslant G_s\}$, the set of V-invariant elements in S, induces a homomorphism—also denoted by χ_V or, more precisely, by $\chi_V^{\mathscr{U}}$—from $\Omega(G, \mathscr{U})$ into $\mathbb{Z}$.

(iii) Any homomorphism from $\Omega(G, \mathscr{U})$ into $\mathbb{Z}$ is of this form.

(iv) For $V, W \leqslant G$ one has $\chi_V^{\mathscr{U}} = \chi_W^{\mathscr{U}}$ if and only if $\bar{V} =: \bigcap_{V \leqslant U \in \mathscr{U}} U$ is conjugate to $\bar{W} =: \bigcap_{W \leqslant U \in \mathscr{U}} U$. So one has precisely $k = k_{\mathscr{U}}$ different homomorphisms from $\Omega(G, \mathscr{U})$ into $\mathbb{Z}$ which after choosing a system $\{U_1, U_2, ..., U_k\}$ of representatives of conjugacy classes of subgroups of $\mathscr{U}$ with $|U_1| \geqslant |U_2| \geqslant \cdots \geqslant |U_k|$ may be denoted by $\chi_1 = \chi_{U_1}$, $\chi_2 = \chi_{U_2}, ..., \chi_k = \chi_{U_k}$.

(v) The product $\chi =: \prod_{i=1}^k \chi_i: \Omega(G, \mathscr{U}) \to \prod_{i=1}^k \mathbb{Z}$ of the k different homomorphisms from $\Omega(G, \mathscr{U})$ into $\mathbb{Z}$ is injective and maps $\Omega(G, \mathscr{U})$ onto a subring of finite index $\prod_{i=1}^k (N_G(U_i): U_i)$ of $\prod_{i=1}^k \mathbb{Z}$, this way identifying $\prod_{i=1}^k \mathbb{Z}$ with the integral closure $\tilde{\Omega}(G, \mathscr{U})$ of $\Omega(G, \mathscr{U})$ in its total quotient ring $\check{\Omega}(G, \mathscr{U}) \cong \mathbb{Q} \otimes_{\mathbb{Z}} \Omega(G, \mathscr{U}) \cong \prod_{i=1}^k \mathbb{Q}$.

Moreover, as we observed several years ago already in the case $\mathscr{U} = \mathscr{S}(G)$ of standard Burnside rings, there are canonical families of congruence relations by which the image $\chi(\Omega(G, \mathscr{U})) \subseteq \prod_{i=1}^k \mathbb{Z}$ can be characterized as a subgroup of $\prod_{i=1}^k \mathbb{Z}$. More precisely, if $V \trianglelefteq W \leqslant G$ and if S is a G-set, then the Burnside Lemma, applied with respect to the W/V-set S^V, implies

$$\sum_{\bar{w} \in W/V} \chi_{\langle \bar{w} \rangle}(S) \equiv 0 \,(\mathrm{mod}(W:V)),$$

where $\langle \bar{w} \rangle$ denotes the subgroup of $W \leqslant G$, generated by the coset $\bar{w} \subseteq W$. So, if $x \in \Omega(G, \mathscr{U})$, if $\chi(x) = (a_1, a_2, ..., a_k) \in \prod_{i=1}^k \mathbb{Z}$ and if for any $i \in \{1, 2, ..., k\}$ we denote by $n(i; V, W)$ the number of cosets $\bar{w} \in W/V$ for which $\overline{\langle \bar{w} \rangle} = \bigcap_{\bar{w} \subseteq U \in \mathscr{U}} U$ is conjugate to U_i, then

$$\sum_{i=1}^k n(i; V, W) \cdot a_i \equiv 0 \ \mathrm{mod}(W:V)).$$

Note that for $V = U_i$ one has $n(i; V, W) = 1$ and for $j \neq i$ one has $n(j; V, W) \geqslant 0$ only if $|U_j| \gneqq |U_i|$, so the above relation determines a_i uniquely modulo $(W:V)$ in terms of the a_j with $j < i$. Hence, if we choose $V = U_1, U_2, ..., U_k$ and $W = N_G(V)$, then the corresponding k relations

$$\sum_{i=1}^k n(i; U_j, N_G(U_j)) \cdot a_i \equiv 0 \ (\mathrm{mod}(N_G(U_j) : U_j)$$

$(j=1, 2, \ldots, k)$ determine a subgroup of $\prod_{i=1}^{k} \mathbb{Z}$ which contains $\chi(\Omega(G, \mathscr{U}))$ and has the same index $\prod_{j=1}^{k} (N_G(U_j) : U_j)$ as $\chi(\Omega(G, \mathscr{U}))$ in $\prod_{i=1}^{k} \mathbb{Z}$, so this subgroup must coincide with $\chi(\Omega(G, \mathscr{U}))$. Similarly, if $\mathbb{Z}_p = \{n/m \in \mathbb{Q} \mid (m, p) = 1\}$ for some prime p, then $\mathbb{Z}_p \cdot \chi(\Omega(G, \mathscr{U})) \cong \Omega_p(G, \mathscr{U}) =: \mathbb{Z}_p \otimes_{\mathbb{Z}} \Omega(G, \mathscr{U})$ coincides with the subgroup of $\prod_{i=1}^{k} \mathbb{Z}_p$ for which the k relations

$$\sum_{i=1}^{k} n(i; U_j, N_G(U_j)_p) \cdot a_i \equiv 0 \pmod{(N_G(U_j)_p : U_j)}$$

$(j=1, \ldots, k)$ hold with $N_G(U_j)_p$ denoting a subgroup of G between U_j and $N_G(U_j)$ for which $N_G(U_j)_p/U_j$ is a Sylow-p-subgroup of $N_G(U_j)/U_j$.

So $\chi(\Omega(G, \mathscr{U}))$ is characterized as a subgroup of $\prod_{i=1}^{k} \mathbb{Z}$ by those relations, where j runs from 1 to k and p runs through all prime divisors of $|G|$.

In case $\mathscr{U} = \mathscr{S}(G)$ one can derive essentially all (representation theoretically or topologically) interesting basic arithmetic properties of $\Omega(G)$ from its description by those congruences including its prime ideal spectrum, the structure of its group of units and its Picard and divisor class group (cf. [4, 1, 2]) and it is of course easy to generalize most of these results to the case of restricted Burnside rings, i.e., to rings of the type $\Omega(G, \mathscr{U})$ with $\mathscr{U} \subsetneqq \mathscr{S}(G)$. But rather than to work out these generalizations I would like to discuss some interesting choices of G and $\mathscr{U}$.

2. Coxeter Groups

Let $B = (b_{ij})_{i,j=1,\ldots,N}$ denote a Coxeter matrix (i.e., a matrix with $b_{ij} = b_{ji} \in \mathbb{N} = \{1, 2, \ldots\}$ and $b_{ij} = 1$ if and only if $i = j$ for $i, j = 1, \ldots, N$) for which the associated Coxeter group $G = G_B =: \langle \sigma_1, \ldots, \sigma_N \mid (\sigma_i \sigma_j)^{b_{ij}} = 1$ for $i, j = 1, \ldots, N \rangle$ is finite. For each subset $I \subseteq \{1, \ldots, N\}$ let $G_I =: \langle \sigma_i \mid i \in I \rangle \leqslant G$ and let $\mathscr{U} = \mathscr{U}_B =: \{\tau G_I \tau^{-1} \mid \tau \in G, I \subseteq \{1, \ldots, N\}\}$. It is clear from the definition that $\mathscr{U}$ is closed with respect to conjugation and it follows from the theory of Coxeter groups that it is also closed with respect to conjugation. More precisely, if $\tau G_I \tau^{-1}$, $\sigma G_J \sigma^{-1} \in \mathscr{U}$ for some $\tau, \sigma \in G$ and if $\alpha \in G_I$ and $\beta \in G_J$ are chosen so that $\rho = \alpha^{-1} \tau^{-1} \sigma \beta \in G_I \tau^{-1} \sigma G_J$ is an element of shortest length in the double coset $G_I \tau^{-1} \sigma G_J$ then $\tau G_I \tau^{-1} \cap \sigma G_J \sigma^{-1} = (\tau\alpha) G_I (\tau\alpha)^{-1} \cap (\sigma\beta) G_J (\sigma\beta)^{-1}$ equals $(\tau\alpha) G_K (\tau\alpha)^{-1}$ with $K = \{i \in I \mid$ there is some $j \in J$ with $\sigma_i = \rho \sigma_j \rho^{-1}\}$.

It might be an interesting exercise to determine the structure of the associated restricted Burnside ring $\Omega(B) = \Omega(G_B, \mathscr{U}_B)$ for the various finite Coxeter groups and to relate its structure to the various other properties of Coxeter groups.

As we will see in the next section, in case of the symmetric group Σ_n which is isomorphic to G_B for

$$B = \begin{pmatrix} 1 & 3 & 2 & 2 & \cdots & 2 \\ 3 & 1 & 3 & 2 & \cdots & 2 \\ 2 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 2 \\ & & & \ddots & \ddots & 3 \\ 2 & \cdots & & 2 & 3 & 1 \end{pmatrix}_{(n-1)\times(n-1)},$$

the associated restricted Burnside ring $\Omega(B)$ is isomorphic to the representation ring of Σ_n and so, applying the foregoing results, we get rather nice descriptions of this ring as a subring of the ring of integer valued class functions in terms of congruence relations.

3. The Symmetric Group

Let Σ_n denote the (full) symmetric group on $\{1, 2,..., n\}$. For each set theoretic partition $\Pi = \{I_1,..., I_k\}$ of $\{1, 2,..., n\}$, i.e., for each system $\{I_1,..., I_k)$ of disjoint subsets $I_1,..., I_k \subseteq \{1, 2,..., n\}$ with $\bigcup_{i=1}^k I_i = \{1, 2,..., n\}$ let $U_\Pi =: \{\sigma \in \Sigma_n \mid \sigma(I_i) = I_i \text{ for } i = 1,..., k\}$ denote the subgroup of permutations of $\{1,..., n\}$ which "respect" the subsets in Π. Let $\mathscr{U}_n = \{U_\Pi \mid \Pi$ a partition of $\{1, 2,..., n\}\}$. It is easily seen that $\Sigma_n = U_{\{\{1,2,...,n\}\}} \in \mathscr{U}_n$ and that $\mathscr{U}_n$ is closed with respect to conjugation and intersection since $\tau U_{\{I_1,...,I_k\}} \tau^{-1} = U_{\{\tau I_1,..., \tau I_k\}}$ and since $U_{\{I_1,...,I_k\}} \cap U_{\{J_1,...,J_h\}} = U_{\{I_i \cap J_j \mid i=1,...,k; j=1,...,l\}}$. So we may form $\Omega_n =: \Omega(\Sigma_n, \mathscr{U}_n)$. Since U_Π is conjugate to $U_{\Pi'}$ if and only if the associated number theoretic partitions $\pi = \{\#I \mid I \in \Pi\}$ and $\pi' = \{\#J \mid J \in \Pi'\}$ coincide we see that the rank of Ω_n coincides with the number of partitions of n, i.e., with the class number of Σ_n and hence with the rank of its representation ring $R(\Sigma_n)$ (cf. [3] for a thorough study of $R(\Sigma_n)$ and further references). Moreover, we have a canonical mapping $\Omega_n \to R(\Sigma_n)$ which associates to each $(\Sigma_n, \mathscr{U})$-set S the associated permutation representation $\mathbb{C}[S]$ and it follows easily from the classical theory of representation of Σ_n that this map is surjective. So the equality of ranks implies that this map is an isomorphism. Moreover, for each $\sigma \in \Sigma_n$ one has $\chi_{\langle\sigma\rangle}(S) = \chi_{\mathbb{C}[S]}(\sigma)$, i.e., the evaluation of the character of the permutation representation $\mathbb{C}[S]$ at σ coincides with the value of $\chi_{\langle\sigma\rangle}(S)$, i.e., with the number of σ-invariant elements in S, and one has $\overline{\langle\sigma\rangle} = U_\Pi$ where Π is the cycle partition of σ. Hence, if for any partition $\pi = (1^{k_1}, 2^{k_2},..., n^{k_n})$ of n, i.e., for each system of nonnegative integral numbers $(k_1, k_2,..., k_n)$ with $\sum_{i=1}^n i \cdot k_i = n$, we denote by U_π the subgroup of Σ_n

consisting of all permutations $\sigma \in \Sigma_n$ with $\sigma(\{1, 2,..., \sum_{i=1}^{j-1} i \cdot k_i + j \cdot k_j'\}) = \{1, 2,..., \sum_{i=1}^{j-1} i \cdot k_i + j \cdot k_j'\}$ for all $j = 1,..., n$ and $0 \leqslant k_j' < k_j$;—so

$$U_\pi = U_\Pi$$

for

$$\Pi = \left\{ \left\{ \sum_{i=1}^{j-1} i \cdot k_i + j \cdot k_j', \sum_{i=1}^{j-1} i \cdot k_i + j \cdot k_j' + 1,..., \sum_{i}^{j-1} = ik_i + jk_j' + (j-1) \right\} \middle| \right.$$
$$\left. j = 1,..., n; 0 \leqslant k_j' < k_j \right\}$$

and $(N_{\Sigma_n}(U_\pi): U_\pi) = N(\pi) =: k_1! \cdot k_2! \cdot \cdots \cdot k_n!$—and if for any two partitions π, π' of n we denote by $n(\pi, \pi')$ the number of elements $\bar{\sigma}$ in $N_{\Sigma_n}(U_\pi)/U_\pi$ for which the number-theoretic partition of n associated with the partition of $\{1,..., n\}$ into domains of transitivity with respect to $\langle \bar{\sigma} \rangle$ coincides with π', then we may conclude that a mapping χ from the set Part(n) of number theoretic partitions of n into $\mathbb{Z}$ is a generalized character of Σ_n if and only if for each partition π of n one has

$$\sum_{\pi'} n(\pi; \pi') \cdot \chi(\pi') \equiv 0 \pmod{N(\pi)}.$$

Similarly, one can characterize $\mathbb{Z}_p \otimes R(\Sigma_n)$ as a subring of $\prod_{\pi \in \mathrm{Part}(n)} \mathbb{Z}_p$ using Sylow-p-subgroups of $N_{\Sigma_n}(U_\pi)/U_\pi$.

So it remains to compute more explicitly the coefficients $n(\pi, \pi')$ and the coefficients $n_p(\pi, \pi')$ which occur in these congruence relations. Unfortunately, this is mainly a matter of notations. So let us recall that there are precisely $K(\pi) =: n!/\prod_{i=1}^n i^{k_i} \cdot k_i!$ permutations σ in Σ_n for which the number-theoretic partition π_σ of n associated to the cycle decomposition partition Π_σ of $\{1,..., n\}$ induced from σ coincides with $\pi = (1^{k_1}, 2^{k_2},..., n^{k_n})$. So for any such σ we have $K(\pi) = (\Sigma_n : C_{\Sigma_n}(\sigma))$ and for $\pi = (1^n)$ we have $n(\pi, \pi') = K(\pi')$. Moreover, if we define a super partition $(\pi; \pi_1,..., \pi_n)$ of n of type π to be a partition $\pi = (1^{k_1}, 2^{k_2},..., n^{k_n})$ of n together with partitions $\pi_1, \pi_2,..., \pi_n$ of $k_1, k_2,..., k_n$, respectively, then the super partitions of type π correspond in a canonical one-to-one fashion to the conjugacy classes of elements in $N_{\Sigma_n}(U_\pi)/U_\pi \cong \Sigma_{k_1} \times \Sigma_{k_2} \times \cdots \times \Sigma_{k_n}$ and for any coset $\bar{\sigma} \in N_{\Sigma_n}(U_\pi)/U_\pi$ corresponding to a super partition $(\pi; \pi_1,..., \pi_n)$ of type π with $\pi_i = (1^{h_1^i}, 2^{h_2^i},...)$ the partition π' of n corresponding to the decomposition of $\{1,..., n\}$ into domains of transitivity with respect to $\langle \bar{\sigma} \rangle$ equals $\pi' = \pi'(\pi; \pi_1,..., \pi_n) =: (1^{k_1'}, 2^{k_2'},..., n^{k_n'})$ with $k_i' = \sum_{j|i} h_{i/j}^j = \sum_{a \cdot b = i} h_b^a$. So we have $n(\pi; \pi') = \sum K(\pi_1) \cdot K(\pi_2) \cdot \cdots \cdot K(\pi_n)$ where the sum is taken over all super partitions $(\pi; \pi_1,..., \pi_n)$ of type π with $\pi'(\pi; \pi_1,..., \pi_n) = \pi'$. In other words we have proved

THEOREM 1. *A mapping* $\chi\colon \mathrm{Part}(n) \to \mathbb{Z}$ *of the set* $\mathrm{Part}(n)$ *of (number-theoretic) partitions of* $n \in \mathbb{N}$ *is a generalized character of* Σ_n *if and only if for any partition* $\pi = (1^{k_1}, 2^{k_2}, \ldots, n^{k_n})$ *of* n *one has*

$$\sum_{(\pi_1,\ldots,\pi_n)} \prod_{i=1}^{n} \frac{k_i!}{\prod_j j^{h_j^i} \cdot h_j^i!} \cdot \chi(1^{h_1^1}, 2^{h_2^1 + h_1^2}, \ldots, n^{\sum_{a \cdot b = n} h_b^a}) \equiv 0 \left(\prod_{i=1}^{n} k_i! \right),$$

where the sum is taken over all systems $(\pi_1, \ldots, \pi_n)$ *of partitions* $\pi_i = (1^{h_1^i}, 2^{h_2^i}, \ldots)$ of k_i $(i = 1, \ldots, n)$.

In case $n = 2$ we have the two partitions (1^2) and (2^1) and the only nontrivial relation

$$\sum_{\pi_1 = (1^{h_1^1}, 2^{h_2^1}) \in \mathrm{Part}(2)} \frac{2!}{h_1^1!\, 2^{h_2^1} \cdot h_2^1!}\, \chi(1^{h_1^1}, 2^{h_2^1}) \equiv 0 \pmod 2,$$

i.e.,

$$\chi(1^2) + \chi(2^1) \equiv 0 \pmod 2.$$

In case $n = 3$ we have the three partitions (1^3), $(1^1, 2^1)$, and (3^1) and the only nontrivial relation

$$\sum_{\pi = (1^{h_1}, 2^{h_2}, 3^{h_3}) \in \mathrm{Part}(3)} \frac{3!}{h_1! \cdot 2^{h_2} \cdot h_2! \cdot 3^{h_3} h_3!} \cdot \chi(\pi) \equiv 0 \pmod 6,$$

i.e., $\chi(1^3) + 3 \cdot \chi(1^1, 2^1) + 2 \cdot \chi(3^1) \equiv 0 \pmod 6$.

In case $n = 4$ we have the five partitions (1^4), $(1^2, 2^1)$, $(1^1, 3^1)$, (2^2), (4^1) and we get the following three nontrivial relations:

$$\chi(1^4) + 6 \cdot \chi(1^2, 2^1) + 8 \cdot \chi(1^1, 3^1) + 3 \cdot \chi(2^2) + 6 \cdot \chi(4^1) \equiv 0 \pmod{24},$$

$$\chi(1^2, 2^1) + \chi(2^2) \equiv 0 \pmod 2$$

and

$$\chi(2^2) + \chi(4^1) \equiv 0 \pmod 2.$$

And in case $n = 5$ we have the seven partitions (1^5), $(1^3, 2^1)$, $(1^2, 3^1)$, $(1^1, 2^2)$, $(1^1, 4^1)$, $(2^1, 3^1)$, (5^1), so we get the following four nontrivial congruence relations:

$$\chi(1^5) + 10 \cdot \chi(1^3, 2^1) + 20 \cdot \chi(1^2, 3^1) + 15 \cdot \chi(1^1, 2^2) + 30 \cdot \chi(1^1, 4^1) + 20 \cdot \chi(2^1, 3^1) + 24 \cdot \chi(5^1) \equiv 0 \pmod{120}$$

$$\chi(1^3, 2^1) + 3 \cdot \chi(1^1, 2^2) + 2 \cdot \chi(2^1, 3^1) \equiv 0 \pmod 6,$$

$$\chi(1^2, 3^1) + \chi(2^1, 3^1) \equiv 0 \pmod 2,$$

$$\chi(1^1, 2^2) + \chi(1^1, 4^1) \equiv 0 \pmod 2.$$

Using the Chinese remainder theorem these congruences are easily seen to be equivalent to the following system of congruences:

$$\chi(1^5) \equiv \chi(5^1) \pmod 5,$$
$$\chi(1^3, 2^1) \equiv \chi(2^1, 3^1) \pmod 3,$$
$$\chi(1^5) \equiv \chi(1^2, 3^1) \pmod 3,$$
$$\chi(1^5) + 2\chi(1^3, 2^1) \equiv \chi(1^1, 2^2) + 2\chi(1^1, 4^1) \pmod 8,$$
$$\chi(1^3, 2^1) \equiv \chi(1^1, 2^2) \equiv \chi(1^1, 4^1) \pmod 2,$$
$$\chi(1^2, 3^1) \equiv \chi(2^1, 3^1) \pmod 2.$$

But, as pointed out already, there is a more direct way to derive prime power congruences characterizing $R(\Sigma_n)$: considering the set $\mathrm{Part}_p(n)$ of partitions of n into powers of p—so we can write $\pi = (1^{f_0}, p^{f_1}, (p^2)^{f_2}, \ldots)$—we may denote for $\pi \in \mathrm{Part}_p(n)$ by $K_p(\pi)$ the number of permutations σ in some Sylow-p-subgroup of Σ_n with $\pi_\sigma = \pi$. Using the fact that for $m = [n/p]$ the centralizer of $\sigma = (1, 2, \ldots, p)(p+1, p+2, \ldots, 2p) \cdots ((m-1) \cdot p+1, (m-1)p+2, \ldots, mp)$ in Σ_n contains a p-Sylow-subgroup of Σ_n and straightforward computations one can prove the inductive formula

$$K_p(1^{f_0}, p^{f_1}, (p^2)^{f_2}, \ldots)$$
$$= p^m \cdot \sum_{\substack{a_0, a_1, a_2, \ldots \\ b_0, b_1, b_2, \ldots}} p^{-\Sigma_i (a_i + b_i)} \cdot (p-1)^{\Sigma b_i}$$
$$\cdot \prod_i \binom{a_i + b_i}{a_i} \cdot K_p(1^{a_0 + b_0}, p^{a_1 + b_1}, (p^2)^{a_2 + b_2}, \ldots)$$

where the sum is taken over all non-negative integers $a_0, a_1, a_2, \ldots, b_0, b_1, b_2, \ldots,$ with $a_0 = [f_0/p]$, $pa_1 + b_0 = f_1$, $pa_2 + b_1 = f_2, \ldots$. Note that $\Sigma(a_i + b_i) \cdot p^i = \Sigma a_i p^i + \Sigma(f_{i+1} - pa_{i+1}) p^i = a_0 + (1/p) \sum_{i \geqslant 1} f_i p^i = [(1/p) \sum_{i \geqslant 0} f_i p^i] = [n/p] = m$, so the above formula reduces the computation of $K_p(\pi)$ for $\pi \in \mathrm{Part}_p(n)$ to the computation of $K_p(\pi')$ for $\pi' \in \mathrm{Part}_p([n/p])$. In particular, for $a, b, c \in \mathbb{N}$ and $b, c < p$ we get $K_p(1^{ap+c}, p^b) = (p-1)^b \cdot \binom{a+b}{a}$, for $a, b, c \in \mathbb{N}$, $p \leqslant b < 2p$ and $c < p$ we get

$$K_p(1^{ap+c}, p^b)$$
$$= (p-1)^b \cdot \binom{a+b}{a} + p^{p-1} \cdot (p-1)^{b-p+1} \cdot \binom{a+b-p}{a} \cdot \left[\frac{a+b}{p}\right]$$

and for $a, f_0, f_1, \dots, fr \in \mathbb{N}$ and $f_0, f_1, f_2, \dots, f_r < p$ we get

$$K_p(1^{ap+f_0}, p^{f_1}, (p^2)^{f_2}, \dots, (p^r)^{f_r}) = p^{f_1+f_2p+\cdots+f_rp^{r-1}} \cdot \left(\frac{p-1}{p}\right)^{f_1+f_2+\cdots+f_r} \cdot \binom{a+f_1}{a_1} \cdot K_p(1^{a+f_1}, p^{f_2}, (p^2)^{f_3}, \dots, (p^{r-1})^{f_r}),$$

so by induction we get – with $n = f_0 + f_1 \cdot p + f_2 \cdot p^2 + \cdots + f_r \cdot p^r$ and with $a_1 = a$,

$$a_2 = \left[\frac{a_1+f_1}{p}\right], \dots, a_{i+1} = \left[\frac{a_i+f_i}{p}\right];$$

the formula

$$K_p(1^{ap+f_0}, p^{f_1}, \dots, (p^r)^{f_r}) = (n!)_p \cdot \left(\frac{p-1}{p}\right)^{\sum_{i=1}^r i \cdot f_i} \cdot \prod_{i=1}^r \binom{a_i+f_i}{a_i}.$$

Using the notation $K_p(\cdots)$ we can state the canonical p-adic version of Theorem 1 as

THEOREM 2. *A mapping χ: Part$(n) \to \mathbb{Z}_p$ is a $\mathbb{Z}_p$-linear combination of characters of Σ_n if and only if for any partition $\pi = (1^{k_1}, 2^{k_2}, \dots, n^{k_n})$ of n one has*

$$\sum_{(\pi_1,\dots,\pi_n)} \sum_{i=1}^n K_p(\pi_i) \cdot \chi(\pi'(\pi; \pi_1, \dots, \pi_n)) \equiv 0 \quad \left(\bmod \left(\prod_{i=1}^n k_i!\right)_p\right)$$

where the sum is taken over all systems $(\pi_1, \dots, \pi_n)$ of p-power partitions π_i of k_i and

$$\left(\prod_{i=1}^n k_i!\right)_p = \prod_{i=1}^n p^{\sum_{a=1}^\infty [k_i/p^a]}$$

denotes the p-part of $\prod_{i=1}^n k_i!$

So for $n = 10$ and $p = 5$ the only partitions π which give rise to 5-power congruence relations are (1^{10}), $(1^8, 2^1)$, $(1^7, 3^1)$, $(1^6, 2^2)$, $(1^6, 4^1)$, $(1^5, 2^1, 3^1)$, $(1^5, 5^1)$, and (2^5). To write down explicitly the associated congruences we have to compute $K_5(1^{10}) = 1$, $K_5(1^5, 5^1) = 8$, $K_5(5^2) = 16$,

$K_5(1^8)=1$, $K_5(1^3,5^1)=4$, $K_5(1^7)=K_5(1^6)=K_5(1^5)=1$, $K_5(1^2,5^1)=K_5(1^1,5^1)=K_5(5^1)=4$, so we get

$$\begin{aligned}
&\chi(1^{10})+8\chi(1^5,5^1)+16\chi(5^2)\equiv 0 \pmod{25},\\
&\chi(1^8,2^1)\equiv\chi(1^3,2^1,5^1) \pmod 5,\\
&\chi(1^7,3^1)\equiv\chi(1^2,3^1,5^1) \pmod 5,\\
&\chi(1^6,2^2)\equiv\chi(1^1,2^2,5^1) \pmod 5,\\
&\chi(1^6,4^1)\equiv\chi(1^1,4^1,5^1) \pmod 5,\\
&\chi(1^5,2^1,3^1)\equiv\chi(2^1,3^1,5^1)\ \mathrm{mod}(5),\\
&\chi(1^5,5^1)\equiv\chi(5^2)\ \mathrm{mod}(5),\\
&\chi(2^5)\equiv\chi(10^1)\ \mathrm{mod}(5).
\end{aligned}$$

Finally, let us remark that a mapping χ: Part$(n)\to\mathbb{Z}$ is an irreducible character of Σ_n if and only if it is a generalized character and satisfies

$$\chi(1^n)>0 \qquad \text{and} \qquad \sum_{\pi\in \mathrm{Part}(n)} K(\pi)\cdot\chi(\pi)^2=n!,$$

so we can use the above description of generalized characters to get similar descriptions of irreducible characters, i.e., we get

THEOREM 3. *A mapping* χ: Part$(n)\to\mathbb{Z}$ *is an irreducible character of* Σ_n *if and only if* $\chi(1)>0$, $\sum_{\pi\in\mathrm{Part}(n)}K(\pi)\cdot\chi(\pi)^2=n!$ *and in addition for any* $\pi=(1^{k_1},2^{k_2},\ldots,n^{k_n})\in\mathrm{Part}(n)$ *one has*

$$\sum_{\pi_i\in\mathrm{Part}(k_i)}^{i=1,\ldots,n}\ \prod_{i=1}^{n} K(\pi_i)\cdot\chi(\pi'(\pi;\pi_1,\ldots,\pi_n))\equiv 0 \quad \left(\mathrm{mod}\ \prod_{i=1}^{n} k_i!\right)$$

or, equivalently, for any $\pi=(1^{k_1},2^{k_2},\ldots,n^{k_n})$ *and any prime* $p\leqslant n$ *one has*

$$\sum_{\pi_i\in\mathrm{Part}_p(k_i)}^{i=1,\ldots,n}\ \prod_{i=1}^{n} K_p(\pi_i)\cdot\chi(\pi'(\pi;\pi_1,\ldots,\pi_n))\equiv 0 \quad \left(\mathrm{mod}\left(\prod_{i=1}^{n} k_i!\right)_p\right).$$

We want to show how Theorem 3 can be used to determine all irreducible characters of Σ_6. At first we have to list all 11 partitions of 6: (1^6), $(1^4,2^1)$, $(1^3,3^1)$, $(1^2,2^2)$, $(1^2,4^1)$, $(1^1,2^1,3^1)$, $(1^1,5^1)$, (2^3), $(2^1,4^1)$, (3^2), (6^1).

Next for $p=2,3,5$ we have to compute $K_p(\pi)$ of all p-power partitions π of the exponents $k_1,k_2,\ldots$, occurring in the above list:

$K_2(1^6)=1,\ K_2(1^4, 2^1)=3,\ K_2(1^2, 2^2)=3+3=5,\ K_2(1^2, 4^1)=2,$
$K_2(2^3)=1+2=3,\ K_2(2^1, 4^1)=2;\ K_2(1^4)=1,\ K_2(1^2, 2^1)=2,$
$K_2(2^2)=1+2=3,\ K_2(4^1)=2;\ K_2(1^3)=1,\ K_2(1^1, 2^1)=1;$
$K_2(1^2)=1,\ K_2(2^1)=1;\ K_3(1^6)=1,\ K_3(1^3, 3^1)=2\cdot 2=4,$
$K_3(3^2)=4;\ K_3(1^4)=1,\ K_3(1^1, 3^1)=2;\ K_3(1^3)=1,$
$K_3(3^1)=2;\ K_5(1^6)=1,\ K_5(1^1, 5^1)=4.$

These coefficients lead to the following congruences:

$$\chi(1^6)+3\cdot\chi(1^4, 2^1)+5\cdot\chi(1^2, 2^2)+2\cdot\chi(1^2, 4^1)+3\cdot\chi(2^3)$$
$$+2\cdot\chi(2^1, 4^1)\equiv 0\ (16),$$
$$\chi(1^4, 2^1)+2\cdot\chi(1^2, 2^2)+3\cdot\chi(2^3)+2\cdot\chi(2^1, 4^1)\equiv 0\ (8)$$
$$\chi(1^3, 3^1)+\chi(1^1, 2^1, 3^1)\equiv 0\ (2),$$
$$\chi(1^2, 2^2)+\chi(2^3)+\chi(1^2, 4^1)+\chi(2^1, 4^1)\equiv 0\ (4),$$
$$\chi(1^2, 4^1)+\chi(2^1, 4^1)\equiv 0\ (2),\ \chi(2^3)+\chi(2^1, 4^1)\equiv 0\ (2),$$
$$\chi(3^2)+\chi(6^1)\equiv 0\ (2);$$
$$\chi(1^6)+4\cdot\chi(1^3, 3^1)+4\cdot\chi(3^2)\equiv 0\ (9),$$
$$\chi(1^4, 2^1)+2\cdot\chi(1^1, 2^1, 3^1)\equiv 0\ (3),$$
$$\chi(1^3, 3^1)+2\cdot\chi(3^2)\equiv 0\ (3),$$
$$\chi(2^3)+2\cdot\chi(6^1)\equiv 0\ (3);$$
$$\chi(1^6)+4\cdot\chi(1^1, 5^1)\equiv 0\ (5).$$

Using the mod 4-congruence, we can replace the mod 8-congruence by

$$\chi(1^4, 2^1)+\chi(2^3)\equiv 2\cdot\chi(1^2, 4^1)\ (8);$$

similarly, we can replace the mod 9-congruence by

$$2\chi(1^6)\equiv\chi(1^3, 3^1)+\chi(3^2) \bmod 9.$$

Now assume that χ is irreducible, i.e., that $\chi(1)\geqslant 1$ and that

$$\langle\chi\mid\chi\rangle=:\chi(1^6)^2+15\cdot\chi(1^4, 2^1)^2+45\cdot\chi(1^2, 2^2)^2+90\cdot\chi(1^2, 4^1)^2$$
$$+15\cdot\chi(2^3)^2+90\cdot\chi(2^1, 4^1)^2+40\cdot\chi(1^3, 3^1)^2+40\cdot\chi(3^2)^2$$
$$+144\cdot\chi(1^1, 5^1)^2+120\cdot\chi(1^1, 2^1, 3^1)^2+120\cdot\chi(6^1)^2=720.$$

If $\chi(1^6)=1$, then one knows anyway that either $\chi=\chi_0$ with $\chi_0(\pi)=:1$ for all $\pi\in\text{Part}(6)$ or $\chi=\chi_1$ with $\chi_1(1^{k_1},2^{k_2},\ldots)=:(-1)^{k_2+k_4+k_6}$ for all $\pi=(1^{k_1},2^{k_2},\ldots)\in\text{Part}(6)$, though this can also be deduced from our congruences by realizing that these congruences imply

$$\chi(1^6)\equiv\chi(1^4,2^1)\equiv\chi(1^2,2^2)\equiv\chi(1^2,4^1)\equiv\chi(2^3)\equiv\chi(2^3)\equiv\chi(2^1,4^1)\bmod 2$$

and

$$\chi(1^6)\equiv\chi(1^3,3^1)\equiv\chi(3^2)\bmod 3,$$

so $\chi(1^6)=1$ implies $|\chi(\pi)|\geqslant 1$ for all $\pi\in\text{Part}(6)$ except perhaps $\pi=(1^1,2^1,3^1)$ or $\pi=(6^1)$, the only two non-p-power partitions of 6. But $\chi(1^1,2^1,3^1)\equiv\chi(1^3,3^1)\bmod 2$ and $\chi(6^1)\equiv\chi(3^2)\bmod 2$, so we have either also $|\chi(1^1,2^1,3^1)|\geqslant 1$ and $|\chi(6^1)|\geqslant 1$ or we have $\chi(1^3,3^1)\equiv 0$ (2) or $\chi(3^2)\equiv 0$ (2), which would imply $|\chi(1^3,3^1)|\geqslant 2$ or $|\chi(3^2)|\geqslant 2$, respectively, whereas $\langle\chi\mid\chi\rangle=720$ together with $|\chi(\pi)|\geqslant 1$ for all p-power partitions π of 6 implies $|\chi(1^3,3^1)|\leqslant 2$ and $|\chi(3^2)|\leqslant 2$. So the above congruences give $\chi(1^3,3^1)$, $\chi(3^2)\in\{1,-2\}$, so $2=2\chi(1^6)\equiv\chi(1^3,3^1)+\chi(3^2)$ (9) implies $\chi(1^3,3^1)=\chi(3^2)=1$ and therefore $|\chi(\pi)|\geqslant 1$ for all partitions π of 6, i.e., $|\chi(\pi)|=1$ for all $\pi\in\text{Part}(6)$ in view of $\langle\chi\mid\chi\rangle=720$. So we have

$$\begin{aligned}&\chi(1^6)=\chi(1^1,5^1)=\chi(1^3,3^1)=\chi(3^2)=1,\\&\chi(1^4,2^1)=\chi(1^1,2^1,3^1)=\pm 1,\\&\chi(2^3)=\chi(6^1)=\pm 1,\\&\chi(1^2,2^2),\chi(1^2,4^1),\chi(2^1,4^1)=\pm 1.\end{aligned}$$

But now $\chi(1^4,2^1)+\chi(2^3)\equiv 2\chi(1^2,4^1)$ (8) implies

$$\chi(1^4,2^1)=\chi(2^3)=\chi(1^2,4^1)=\chi(1^1,2^1,3^1)=\chi(6^1)=\pm 1,$$

so $\chi(1^2,2^2)+\chi(1^2,4^1)+\chi(2^3)+\chi(2^1,4^1)\equiv 0$ (4) implies

$$\chi(1^2,2^2)=\chi(2^1,4^1)=\pm 1.$$

Finally, the mod 16-congruence implies $\chi(1^2,2^2)=\chi(2^1,4^1)=+1$, so we get indeed $\chi=\chi_0$ or $\chi=\chi_1$.

Next we show that $\chi(1^6)\equiv\pm 2(5)$ is impossible. Otherwise $\chi(1^1,5^1)\equiv\chi(1^6)(5)$ together with $\langle\chi\mid\chi\rangle=720$ implies

$$\chi(1^1,5^1)=\pm 2$$

and

$$\langle\chi\mid\chi\rangle-144\cdot\chi(1^1,5^1)^2=144,$$

so we must have $\chi(1^6) \equiv 0\ (2)$ since otherwise $\chi(\pi) \equiv 1\ (2)$ for all $\pi \in \text{Part}_2(6)$ and therefore $\sum_{\pi \in \text{Part}_2(6)} K(\pi) \cdot \chi(\pi)^2 \geqslant 256 > 144 = \langle \chi \mid \chi \rangle - 144 \cdot \chi(1^1, 5^1)^2$.

Similarly, we must have

$$\chi(1^3, 3^1) = \chi(3^2) = 0,$$

since

$$\chi(1^3, 3^1) \equiv \chi(1^1, 2^1, 3^1) \quad (2),$$
$$\chi(3^2) \equiv \chi(6^1) \qquad (2),$$

and $40 \cdot \chi(1^3, 3^1)^2 + 120 \cdot \chi(1^1, 2^1, 3^1)^2 \leqslant 144$ as well as

$$40 \cdot \chi(3^2)^2 + 120\chi(6^1)^2 \leqslant 144.$$

So we get $2\chi(1^6) \equiv 0\ (9)$, i.e., $\chi(1^6) \equiv 0\ (9)$. Together, these congruences imply that 18 divides $\chi(1^6)$ in contradiction to $0 < \chi(1^6)^2 \leqslant 144$.

So it remains to study the cases $\chi(1^6) = 4, 5, 6, 9, 10, 11, \ldots$. In case $\chi(1^6) = 4$ we get $\chi(1^1, 5^1) = -1$ and $\langle \chi \mid \chi \rangle - \chi(1^6)^2 - 144 \cdot \chi(1^1, 5^1)^2 = 560$. Moreover $\chi(1^3, 3^1) \equiv \chi(1^6) \not\equiv 0\ (3)$ and $\chi(1^3, 3^1) \equiv \chi(1^1, 2^1, 3^1)\ (2)$ together implies $40 \cdot \chi(1^3, 3^1)^2 + 120 \cdot \chi(1^1, 2^1, 3^1)^2 \geqslant 160$.

Moreover, $40 \cdot \chi(1^3, 3^1)^2 + 120 \cdot \chi(1^1, 2^1, 3^1)^2 > 160$ would imply

$$40 \cdot \chi(1^3, 3^1)^2 + 120 \cdot \chi(1^1, 2^1, 3^1)^2 \geqslant 640$$

in contradiction to

$$40 \cdot \chi(1^3, 3^1)^2 + 120 \cdot \chi(1^1, 2^1, 3^1)^2 \leqslant 560,$$

so we have

$$40 \cdot \chi(1^3, 3^1)^2 + 120 \cdot \chi(1^1, 2^1, 3^1) = 160.$$

Similarly, we get

$$40 \cdot \chi(3^2)^2 + 120 \cdot \chi(6^1)^2 = 160,$$

so we have

$$15 \cdot \chi(1^4, 2^1)^2 + 45 \cdot \chi(1^2, 2^2)^2 + 90 \cdot \chi(1^2, 4^1)^2 + 15 \cdot \chi(2^3)^2 + 90 \cdot \chi(2^1, 4^1) = 240.$$

Together with $\chi(\pi) \equiv 0\ (2)$ for $\pi \in \text{Part}_2(6)$ this implies $\chi(1^2, 4^1) = \chi(2^1, 4^1) = 0$, so we are left with

$$\chi(1^4, 2^1)^2 + 3 \cdot \chi(1^2, 2^2)^2 + \chi(2^3)^2 = 16,$$

so we have either $\chi(1^4, 2^1)=0$ or $\chi(2^3)=0$, but not both. But $\chi(1^4, 2^1)+\chi(2^3)\equiv 2\cdot\chi(1^2, 4^1)=0$ (8), so in any case we have $\chi(1^4, 2^1)\equiv\chi(2^3)\equiv 0$ (8) which together with $-4\leqslant\chi(1^4, 2^1)$, $\chi(2^3)\leqslant +4$ leads to $\chi(1^4, 2^1)=\chi(2^3)=0$, i.e., $3\cdot\chi(1^2, 2^2)^2=16$ which is impossible to solve. So $\chi(1^6)=4$ is ruled out.

In case $\chi(1^6)=5$ we have $\chi(1^1, 5^1)=0$ and $\sum_{\pi\in\mathrm{Part}_2(6)} K(\pi)\cdot\chi(\pi)^2\geqslant 280$, so we have $40\cdot\chi(1^3, 3^1)^2+120\cdot\chi(1^1, 2^1, 3^1)^2+40\cdot\chi(3^2)^2+120\cdot\chi(6^1)^2\leqslant 440$. Again, $\chi(1^3, 3^1)\not\equiv 0\not\equiv\chi(3^2)$ (3) implies $40\cdot\chi(1^3, 3^1)^2+120\cdot\chi(1^1, 2^1, 3^1)^2=160$ and $40\cdot\chi(3^2)^2+120\cdot\chi(6^1)^2=160$, i.e., $(\chi(1^3, 3^1), \chi(1^1, 2^1, 3^1))$, $(\chi(3^2), \chi(6^1))\in\{(\pm 1, \pm 1), (\pm 2, 0)\}$, so

$$\chi(1^3, 3^1)\equiv\chi(3^2)\equiv\chi(1^6)=5(3)$$

and

$$1\equiv 2\cdot\chi(1^6)\equiv\chi(1^3, 3^1)+\chi(3^2)\ (9)$$

implies $(\chi(1^3, 3^1),\ \chi(1^1, 2^1, 3^1),\ \chi(3^2),\ \chi(6^1))\in\{(-1, \pm 1, 2, 0), (2, 0, -1, \pm 1)\}$. Moreover, we get $\sum_{\pi\in\mathrm{Part}_2(6)} K(\pi)\,\chi(\pi)^2=400$ which implies together with $\chi(\pi)\equiv 1$ (2) for $\pi\in\mathrm{Part}_2(6)$ the relations $\chi(1^2, 2^2)$, $\chi(1^2, 4^1)$, $\chi(2^1, 4^1)=\pm 1$ and $15\cdot\chi(1^4, 2^1)^2+15\cdot\chi(2^3)^2=150$, i.e., $(\chi(1^4, 2^1),\ \chi(2^3))\in\{(\pm 1, \pm 3), (\pm 3, \pm 1)\}$, so $\chi(1^4, 2^1)+\chi(2^3)\equiv 2\chi(1^2, 4^1)\equiv\pm 2(8)$ implies $(\chi(1^4, 2^1), \chi(2^3))\in\{\pm(1, -3), \pm(-3, 1)\}$.

Now assume $\chi(1^3, 3^1)=-1$. By multiplying χ with χ_1 we may also assume $\chi(1^1, 2^1, 3^1)=1$ and we have anyway $\chi(3^2)=2$ and $\chi(6^1)=0$. $1=\chi(1^1, 2^1, 3^1)\equiv\chi(1^4, 2^1)$ (3) implies $\chi(1^4, 2^1)=1$, $\chi(2^3)=-3$ and $\chi(1^2, 4^1)=-1$, so $\chi(1^6)+3\chi(1^4, 2^1)+5\chi(1^2, 2^2)+2\chi(1^2, 4^1)+3\cdot\chi(2^3)+2\cdot\chi(2^1, 4^1)=5+3+5\cdot\chi(1^2, 2^2)-2-9+2\chi(2^1, 4^1)=5\chi(1^2, 2^2)+2\chi(2^1, 4^1)-3\equiv 0$ (16) implies $\chi(1^2, 2^2)=+1$, $\chi(2^1, 4^1)=-1$ and it is easily checked that this gives indeed a solution χ_2 of our congruences as well as $\chi_3=:\chi_2\cdot\chi_1$.

We may as well assume $\chi(1^3, 3^1)=2$, $\chi(1^1, 2^1, 3^1)=0$, $\chi(3^2)=-1$, $\chi(6^1)=+1$ in which case a similar reasoning leads to $\chi(2^3)=1$, $\chi(1^4, 2^1)=-3$, $\chi(1^2, 4^1)=-1$, $5-9+5\cdot\chi(1^2, 2^2)-2+3+2\chi(2^1, 4^1)=5\chi(1^2, 2^2)+2\chi(2^1, 4^1)-3\equiv 0$ (16), i.e., $\chi(1^2, 2^2)=1$, $\chi(2^1, 4^1)=-1$ and hence to a further solution χ_4 and its companion $\chi_5=:\chi_4\cdot\chi_1$.

In case $\chi(1^6)=6$ one gets $\chi(1^1, 5^1)=1$ and therefore $\langle\chi\mid\chi\rangle-\chi(1^6)^2-144\chi(1^1, 5^1)^2=540$. So we have $-3\leqslant\chi(1^3, 3^1)$, $\chi(3^2)\leqslant 3$ and $(\chi(1^3, 3^1),\ \chi(3^2))\neq(\pm 3, \pm 3)$. Since $0\equiv\chi(1^6)\equiv\chi(1^3, 3^1)\equiv\chi(3^2)$ (3) and $3\equiv 2\chi(1^6)\equiv\chi(1^3, 3^1)+\chi(3^2)$ (9), this leads to $(\chi(1^3, 3^1), \chi(3^2))\in\{(0, 3), (3, 0)\}$ and therefore—using once again $\chi(1^3, 3^1)\equiv\chi(1^1, 2^1, 3^1)$ (2) and $\chi(3^2)\equiv\chi(6^1)$ (2)—it leads to $40\chi(1^3, 3^1)^2+120\chi(1^1, 2^1, 3^1)^2+40\cdot\chi(3^2)^2+120\cdot\chi(6^1)^2\geqslant 360+120=480$. But if this sum would exceed 480 it would exceed it by at least 480 which is impossible. So we have $(\chi(1^1, 2^1, 3^1), \chi(6^1)) \in \{(0, \pm 1), (\pm 1, 0)\}$ and $15\chi(1^4, 2^1)^2 + 45 \cdot \chi(1^2, 2^2)^2 + 90\chi(1^2, 4^1)^2 + 15\chi(2^3)^2 + 90\chi(2^1, 4^1)^2 = 60$ together with $\chi(\pi) \equiv 0$ (2) for $\pi \in \mathrm{Part}_2(6)$ which implies $\chi(1^2, 2^2) = \chi(1^2, 4^1) = \chi(2^1, 4^1) = 0$ and $(\chi(1^4, 2^1), \chi(2^3)) \in \{(0, \pm 2), (\pm 2, 0)\}$ in contradiction to $\chi(1^4, 2^1) + \chi(2^3) \equiv 2 \cdot \chi(1^2, 4^1) = 0$ (8). So, $\chi(1^6) = 6$ is ruled out. Since $\chi(1^6) = 7, 8$ has been ruled out alredy, the next case which has to be considered is $\chi(1^6) = 9$, where we have $\chi(1^1, 5^1) = -1$ and hence $\langle \chi \mid \chi \rangle - \chi(1^6)^2 - 144 \cdot \chi(1^1, 5^1)^2 = 495$. Again, we get $-3 \leqslant \chi(1^3, 3^1)$, $\chi(3^2) \leqslant 3$, $(\chi(1^3, 3^1), \chi(3^2)) \neq (\pm 3, \pm 3)$, and $\chi(1^3, 3^1) \equiv \chi(3^2) \equiv \chi(1^6) \equiv 0$ (3) as well as $0 \equiv 2\chi(1^6) \equiv \chi(1^3, 3^1) + \chi(3^2) \bmod 9$ which leads to $\chi(1^3, 3^1) = \chi(3^2) = 0$. Moreover, $\chi(\pi) \equiv 1(2)$ for $\pi \in \mathrm{Part}_2(6)$ implies $\sum_{\pi \in \mathrm{Part}_2(6), \pi \neq 1^6} K(\pi)\, \chi(\pi)^2 \geqslant 255$ and therefore $120\chi(1^1, 2^1, 3^1)^2 + 120 \cdot \chi(6^1)^2 \leqslant 240$ which—in view of $\chi(1^1, 2^1, 3^1) \equiv \chi(1^3, 3^1) \equiv 0$ (2) and $\chi(6^1) \equiv \chi(3^2) \equiv 0$ (2)—leads to $\chi(1^1, 2^1, 3^1) = \chi(6^1) = 0$. So we are left the equation

$$15 \cdot \chi(1^4, 2^1)^2 + 45 \cdot \chi(1^2, 2^2)^2 + 90 \cdot \chi(1^2, 4^1)^2 + 15 \cdot \chi(2^3)^2 + 90 \cdot \chi(2^1, 4^1)^2 = 495,$$

where all $\chi(\cdots)$-values have to be odd. This implies $\chi(1^2, 4^1), \chi(2^1, 4^1) = \pm 1$ and leaves us with

$$\chi(1^4, 2^1)^2 + 3\chi(1^2, 2^2)^2 + \chi(2^3)^2 = 21$$

whose only odd solutions are $(\pm 3, \pm 1, \pm 3)$. Checking congruences and using that again by perhaps multiplying with χ_1 we may assume w.l.o.g. that $\chi(1^2, 4^1) = 1$ we get from $\chi(1^4, 2^1) + \chi(2^3) \equiv 2 \cdot \chi(1^2, 4^1) = 2$ (8) $\chi(1^4, 2^1) = \chi(2^3) = -3$, so $9 + 3 \cdot (-3) + 5 \cdot \chi(1^2, 2^2) + 2 \cdot 1 + 3 \cdot (-3) + 2 \cdot \chi(2^1, 4^1) \equiv 0\ (16)$ implies $\chi(1^2, 2^2) = \chi(2^1, 4^1) = 1$. Again, it is easily checked that this gives indeed another solutions χ_6 together with its companion χ_7.

One may now go on to check the case $\chi(1^6) = 10, 11, 14, 15, 16, \ldots$, to find the remaining characters χ_8, χ_9, and χ_{10}, but of course it is more natural now to use the Kronecker products $\chi_2 \cdot \chi_4$ and $\chi_2 \cdot \chi_2$ to conclude that the three remaining characters are $\chi_8 = \chi_2 \cdot \chi_4 - \chi_6$ (of degree 16), $\chi_9 = \chi_2 \cdot \chi_2 - \chi_1 - \chi_7 - \chi_3$ and $\chi_{10} = \chi_1 \cdot \chi_9$ (both of degree 10).

In any case, we hope that this exercise has shown that the above congruences do not only provide structural insight (from which one may, e.g., try to deduce the structure of the Picard group of $R(\Sigma_n)$, cf. [2]) but may also be helpful to derive a better understanding of the numerical evaluation of the characters of the symmetric group.

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