On the regularity of the leafwise Poincaré metric

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Abstract

We prove that for a foliation of general type on a complex projective surface the curvature of the leafwise Poincaré metric is absolutely continuous.

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Let $X$ be a complex projective surface, possibly singular but with at most cyclic quotient singularities. Let $\mathcal{F}$ be a holomorphic foliation on $X$. We shall suppose that the singularities of $\mathcal{F}$ satisfy the following standing assumptions: (i) $\text{Sing}(\mathcal{F})$ are disjoint from $\text{Sing}(X)$, i.e., around a cyclic quotient singularity of $X$ the foliation is the quotient of a nonsingular one; (ii) $\text{Sing}(\mathcal{F})$ are reduced, in Seidenberg’s sense, i.e., locally generated by vector fields whose linear parts have eigenvalues $a, b$ with $a/b$ not a positive rational. We shall also suppose that the canonical bundle $K_\mathcal{F}$ of $\mathcal{F}$ is nef, that is $K_\mathcal{F} \cdot C \geq 0$ for every algebraic curve $C \subset X$. According to results of Seidenberg, Miyaoka and McQuillan [7], these assumptions on $\text{Sing}(\mathcal{F})$ and $K_\mathcal{F}$ are not at all restrictive: up to a birational transformation they are always satisfied, unless the foliation is birational to a $\mathbb{CP}^1$-bundle (an uninteresting case). Besides [7], we also refer to [3] for a general overview of the birational theory of foliations, and their Kodaira-type classification.

In this paper we pursue the study of the leafwise Poincaré metric, begun in [2]. Let us recall the context. Each leaf of $\mathcal{F}$ is a complex connected one-dimensional orbifold, injectively immersed in $X' = X \setminus \text{Sing}(\mathcal{F})$, and uniformized by the disc $D$ or by the affine line $C$. On each leaf we put its Poincaré metric (which is identically zero, by definition, if the leaf is parabolic, i.e., uniformized by $C$). This leafwise metric can be seen as a
(singular) hermitian metric on the canonical bundle $K_F$ of $F$, and the main result of [2] is: if there exists at least one hyperbolic leaf (uniformized by $D$) then the curvature $\Omega$ of this hermitian metric on $K_F$ is a closed positive current. See also below, Section 1, for a local reformulation of this result.

We can decompose this curvature $\Omega$ in two ways. First of all, according to Siu’s theorem [8] we have

$$\Omega = \Omega_{\text{alg}} + \Omega_{\text{res}},$$

where $\Omega_{\text{alg}} = \sum \lambda_j \delta_{C_j}$ is a finite (possibly empty) positive sum of integration currents $\delta_{C_j}$ over algebraic curves $C_j \subset X$, and $\Omega_{\text{res}}$ is a closed positive current with vanishing Lelong number outside a finite set $P \subset X$. Here the finiteness of the sum in $\Omega_{\text{alg}}$ comes from the fact that each $C_j$ is necessarily the closure of a parabolic leaf, hence their number is finite (mainly by Jouanolou’s theorem [5]), and the finiteness of $P$ comes from $P \subset \text{Sing}(F)$.

Secondly, we can take the Lebesgue’s decomposition of $\Omega$ into singular part and absolutely continuous part [4, p. 355]

$$\Omega = \Omega_{\text{sing}} + \Omega_{\text{ac}},$$

corresponding to the fact that the coefficients of $\Omega$ are (complex) measures which can be decomposed with respect to the Lebesgue measure on $X$. Generally speaking, for a closed positive current we have $\Omega_{\text{alg}} \leq \Omega_{\text{sing}}$, and the inequality may well be strict: on $C$, a singular measure is generally speaking far from being atomic. Moreover, whereas $\Omega_{\text{alg}}$ is always closed, $\Omega_{\text{sing}}$ can be nonclosed. Here we shall prove that in our special context we always have equality.

**Theorem.** Let $X$ be a complex projective surface with at most cyclic quotient singularities, and let $F$ be a holomorphic foliation on $X$ with $\text{Sing}(F)$ reduced and disjoint from $\text{Sing}(X)$, and with $K_F$ nef. Suppose that $F$ has at least one hyperbolic leaf. Then for the curvature $\Omega$ of the leafwise Poincaré metric we have

$$\Omega_{\text{alg}} = \Omega_{\text{sing}}.$$

Let us discuss this result in the context of Kodaira dimension of foliations [3,7]. When $\text{Kod}(F) = -\infty$ the result was already proved in [2, Proposition 5] (with equality to zero), as a step toward the classification of those foliations; the proof we give here is however different and independent, and perhaps more natural. When $\text{Kod}(F) = 0$ the result is empty, because in that case all the leaves are parabolic. When $\text{Kod}(F) = 1$ the result is trivial, due to the special structure of those foliations. Hence, the really interesting case is when $\text{Kod}(F) = 2$ (which happens for “most” foliations). In that case some estimates of McQuillan [7] give $\Omega_{\text{alg}} \equiv 0$, and therefore we obtain the following regularity statement.

**Corollary.** If moreover $F$ is of general type then $\Omega$ is absolutely continuous.

This allows to define the (punctual) wedge product $\Omega \wedge \Omega$, which by [4, Theorem 10.7] is an absolutely continuous positive measure on $X$ whose total mass bounds the selfintersection of the canonical bundle: $c_1^2(K_F) \geq \int_X \Omega \wedge \Omega$. We don’t know if equality
is here always realized. Anyway, we can at least claim that \( \int_X \Omega \wedge \Omega > 0 \), i.e., the positive measure \( \Omega \wedge \Omega \) is not identically zero. Indeed, if \( \Omega \wedge \Omega \equiv 0 \) then, following the last pages of [2], we can construct on \( X \) a second holomorphic foliation \( \mathcal{G} \) in the Kernel of \( \Omega \). This \( \mathcal{G} \) is tangent to \( \mathcal{F} \) along its parabolic leaves, at first order, so that we have \( K_{\mathcal{F}} = N_{\mathcal{G}}^* \otimes \mathcal{O}_X(D) \) for a suitable reduced divisor \( D \) which is \( \mathcal{G} \)-invariant. But then the logarithmic Castelnuovo–De Franchis–Bogomolov lemma says that \( \text{kod}(\mathcal{F}) = \text{kod}(N_{\mathcal{G}}^* \otimes \mathcal{O}_X(D)) \leq 1 \).

However, for a foliation of general type it is tempting to do a much stronger conjecture: the support of \( \Omega \wedge \Omega \) is the full \( X \). It is not difficult to see (using [2, Proposition 6]) that \( \text{Supp}(\Omega \wedge \Omega) \) is a closed subset invariant by \( \mathcal{F} \).

1. Some local computations

Take a point \( p \in X'' = X \setminus \{ \text{Sing}(\mathcal{F}) \cup \text{Sing}(X) \} \) and let \( (z, w) \in D \times D \) be local coordinates around \( p \) in which the foliation \( \mathcal{F} \) is expressed by the equation \( dz = 0 \). In this local chart, the leafwise Poincaré metric is represented by

\[
e^F_i \, dw \wedge d\bar{w},
\]

where \( F : D \times D \to [-\infty, +\infty) \) is a plurisubharmonic function [2] which satisfies moreover the “curvature \(-1\)” differential equation

\[
F_{ww} = e^F.
\]

The polar set \( \Sigma = \{ F = -\infty \} \) is possibly nonempty: it coincides with the trace of parabolic leaves on our local chart \( D \times D \). According to McQuillan [7, §V], the function \( F \) and its polar set \( \Sigma \) have the following regularity properties:

(a) \( \Sigma \) is an analytic subset of \( D \times D \), that is a discrete set of fibres;
(b) \( F \) is continuous on \( D \times D \setminus \Sigma \), that is \( e^F \) is continuous on \( D \times D \).

In fact, we shall need only \( F \in L^\infty_{\text{loc}} \) outside \( \Sigma \); but on the other hand the difficult part in [7, §V] consists in showing such a local boundedness property, the continuity being then a simple consequence via a Montel-type argument.

Let us consider the following two positive measures on \( D \times D \):

\[
\mu = F_z \delta_z, \\
\nu = (e^F)_z \delta_z
\]

and let us observe that, as distributions, they are related by

\[
\mu_{ww} = \nu
\]

because \( (F_{z\bar{z}})_{ww} = (F_{w\bar{w}})_{z\bar{z}} = (e^F)_{z\bar{z}} \) (it is perhaps worth noting that the derivative \( F_{w\bar{w}} \) appearing in \( F_{w\bar{w}} = e^F \) has to be understood, a priori, as a “classical” derivative, not as a distributional one, using the smoothness of \( F \) on the vertical fibres; but the fact that this classical derivative is bounded implies that it coincides with the distributional one). We
may assume, without loss of generality, that the chart $D \times D \subseteq X$ is well embedded up to the boundary, so that $\mu$ and $\nu$ have a finite total mass.

Take the Lebesgue’s decomposition of $\mu$ and $\nu$:

$$\mu = \mu_{\text{sing}} + \mu_{\text{ac}}, \quad \nu = \nu_{\text{sing}} + \nu_{\text{ac}}.$$ 

Our first aim is to prove that the above relation between $\mu$ and $\nu$ still holds between their singular parts: $(\mu_{\text{sing}})_{\pi} = \nu_{\text{sing}}$. In some sense this is obvious, since $F$ is smooth on the verticals, but a formal proof requires some care.

The measure $\mu$ can be disintegrated with respect to the projection $\pi : D \times D \to D$, $\pi(z, w) = z$:

$$\mu(\phi) = \int_{D} \mu(\phi(z, \cdot)) \, d\sigma(z) \quad \forall \phi \in C_{c}^{\infty}(D \times D),$$

where $\sigma = \pi_{*}\mu$ and $\mu^{(z)}$ is a positive probability measure on $D_{z} = \pi^{-1}(z)$ for $\sigma$-a.e. $z \in D$. From $\nu = \mu_{\pi}$ we then obtain

$$\nu(\phi) = \int_{D} \mu_{\pi}^{(z)}(\phi(z, \cdot)) \, d\sigma(z) \quad \forall \phi \in C_{c}^{\infty}(D \times D),$$

where $\mu_{\pi}^{(z)}$ is a distribution on $D_{z}$ for $\sigma$-a.e. $z \in D$. Remark that $z \mapsto \mu_{\pi}^{(z)}(\phi(z, \cdot)) = \mu^{(z)}(\phi_{\pi}(z, \cdot))$ is a bounded function, for every fixed $\phi$.

**Lemma 1.** For $\sigma$-a.e. $z \in D$, $\mu_{\pi}^{(z)}$ is a positive measure.

**Proof.** Take $\phi, \psi \in C_{c}^{\infty}(D)$, $\phi \geq 0$, $\psi \geq 0$, and consider $\phi$ as a function of $w$ and $\psi$ as a function of $z$. Then

$$\int_{D} \psi(z) \mu_{\pi}^{(z)}(\phi) \, d\sigma(z) = \nu(\psi \phi) \geq 0$$

and the arbitrariness of $\psi$ gives

$$\mu_{\pi}^{(z)}(\phi) \geq 0 \quad \text{for } \sigma\text{-a.e. } z \in D.$$ 

More precisely, there exists $E_{\phi} \subseteq D$, with $\sigma(E_{\phi}) = 0$, such that $\mu_{\pi}^{(z)}(\phi) \geq 0$ for every $z \in D \setminus E_{\phi}$. Take now $\{\phi_{n}\}_{n=1}^{\infty} \subseteq C_{c}^{\infty}(D)$, $\phi_{n} \geq 0$ for every $n$, dense in the space of positive smooth functions, and set $E = \bigcup_{n=1}^{\infty} E_{\phi_{n}}$. Then $\sigma(E) = 0$, and $\mu_{\pi}^{(z)}(\phi_{n}) \geq 0$ for every $z \in D \setminus E$ and for every $n$. By density we conclude that $\mu_{\pi}^{(z)}(\phi) \geq 0$ for every $z \in D \setminus E$ and for every positive smooth $\phi$. This means that $\mu_{\pi}^{(z)}$ is a measure for $\sigma$-a.e. $z \in D$. 

Therefore, for $\sigma$-a.e. $z \in D$ we can write

$$\mu^{(z)} = h^{(z)}(w) \, dw \wedge d\overline{w}$$

where $h^{(z)}$ is a positive subharmonic function on $D_{z}$. The submean inequality for each $h^{(z)}$ and the fact that $\mu^{(z)}(D_{z}) = 1$ for every $z$ show that the function $(z, w) \mapsto h^{(z)}(w)$ is
locally bounded from above (and of course also from below, being positive). This allows to compute easily the Lebesgue’s decomposition of $\mu$:

$$\mu_{\text{sing}}(\phi) = \int_D \mu^{(z)}(\phi(z, \cdot)) \, d\sigma_{\text{sing}}(z),$$

$$\mu_{\text{ac}}(\phi) = \int_D \mu^{(z)}(\phi(z, \cdot)) \, d\sigma_{\text{ac}}(z),$$

where $\sigma = \sigma_{\text{sing}} + \sigma_{\text{ac}}$ is the Lebesgue’s decomposition of $\sigma$. As a consequence of this we also find

$$(\mu_{\text{sing}})_{w,\pi}(\phi) = \int_D \mu^{(z)}_{w,\pi}(\phi(z, \cdot)) \, d\sigma_{\text{sing}}(z).$$

Let us now compute the Lebesgue’s decomposition of $\nu$.

**Lemma 2.**

$\nu_{\text{sing}} = e^F \mu_{\text{sing}}$.

**Proof.** It is sufficient to prove that for every $w \in D$ the subharmonic function $f(z) = F(z, w)$ satisfies

$$((e^f)_{\text{sing}} = e^f (f_{z\bar{z}})_{\text{sing}}.$$

Outside the poles, $f$ is locally bounded and hence locally of finite energy (i.e., locally integrable w.r. to the measure $f_{z\bar{z}}$). Therefore the first derivatives $f_z$ and $f_{\bar{z}}$ are locally square integrable [6, Chapter 1, §4]. By a standard regularization procedure we then obtain the chain-rule formula

$$((e^f)_{z\bar{z}} = e^f f_z f_{\bar{z}} + e^f f_{\bar{z}} z,$$

where $e^f f_z, f_{\bar{z}} \in L^1_{\text{loc}}$ is an absolutely continuous measure. Taking singular parts we obtain the desired formula, at least outside the poles of $f$. But these poles (which are discrete) are not charged neither by $((e^f)_{z\bar{z}})_{\text{sing}}$, for $e^f$ is bounded, nor by $e^f (f_{z\bar{z}})_{\text{sing}}$, for $e^f$ vanishes at those poles. Whence the equality everywhere. □

Using the disintegration formula before Lemma 1, the singular part $\nu_{\text{sing}}$ of $\nu$ can be written as

$$\nu_{\text{sing}}(\phi) = \int_D \mu^{(z)}_{w,\pi}(\phi(z, \cdot)) \, d\sigma_{\text{sing}}(z) + \tilde{\nu}(\phi),$$

where $\tilde{\nu}$ is a “residual” positive measure. Indeed, the integral above defines a singular measure $\gamma$ which is less than or equal to $\nu$, and so $\nu_{\text{sing}} \geq \gamma$. Remark that if $B \subset D \times D$ is a Borel set with $\sigma_{\text{ac}}(\tau(B)) = 0$ then $\gamma(B) = \nu(B)$ and so $\tilde{\nu}(B) = \nu_{\text{sing}}(B) - \gamma(B) = \nu_{\text{sing}}(B) - \nu(B) \leq 0$, i.e., $\tilde{\nu}(B) = 0$. On the other hand, by the previous lemma $\nu_{\text{sing}}$, and hence $\tilde{\nu}$, is absolutely continuous with respect to $\mu_{\text{sing}}$, and hence with respect to
\[ \sigma_{\text{sing}} \wedge i \, dw \wedge d\overline{w}. \] Thus \( \tilde{\nu}(B) = 0 \) for any Borel set \( B \subset D \times D \) with \( \sigma_{\text{sing}}(\pi(B)) = 0 \). It follows from these properties that \( \tilde{\nu} \) is in fact identically zero.

Therefore the Lebesgue’s decomposition of \( \nu \) is simply

\[
\nu_{\text{sing}}(\phi) = \int_{D} \mu^{(z)}_{w \overline{w}}(\phi(z, \cdot)) \, d\sigma_{\text{sing}}(z),
\]

\[
\nu_{\text{ac}}(\phi) = \int_{D} \mu^{(z)}_{w \overline{w}}(\phi(z, \cdot)) \, d\sigma_{\text{ac}}(z)
\]

and by comparison we find:

**Lemma 3.** \((\mu_{\text{sing}})_{w \overline{w}} = \nu_{\text{sing}}.\)

Remark that the regularity properties of \( F \) and \( \Sigma \) were used only in Lemma 2, in a rather weak way.

2. **Proof of the theorem**

Let us still work in the local chart \( D \times D \subset X'' \) of the previous section. The curvature \( \Omega \) is here expressed by

\[
\Omega = \frac{i}{2\pi} \partial \bar{\partial} F.
\]

The derivative \( F_{w \overline{w}} = e^F \) is an absolutely continuous measure, and so the singular part \( \Omega_{\text{sing}} \) does not contain the term in \( i \, dw \wedge d\overline{w} \). By positivity, \( \Omega_{\text{sing}} \) does neither contain the terms in \( i \, dz \wedge d\overline{w} \) and \( i \, dw \wedge d\overline{z} \), and therefore

\[
\Omega_{\text{sing}} = \frac{1}{2\pi} \mu_{\text{sing}} i \, dz \wedge d\overline{z}
\]

where, as in the previous section, \( \mu \) is the measure \( F_{z \overline{z}} \).

By Lemma 3 we have

\[
i \partial \bar{\partial} \Omega_{\text{sing}} = \frac{1}{2\pi} \nu_{\text{sing}}
\]

where, with a double abuse of notation, we have dropped the factor \( i \, dw \wedge d\overline{w} \wedge i \, dz \wedge d\overline{z} \).

By Lemma 2 we also have

\[
i \partial \bar{\partial} \Omega_{\text{sing}} = \frac{1}{2\pi} e^F \mu_{\text{sing}}.
\]

The first important consequence of these computations is that \( i \partial \bar{\partial} \Omega_{\text{sing}} \) is positive, at least outside \( \text{Sing}(F) \cup \text{Sing}(X) \). Using an extension theorem [1] we now check this positivity everywhere. Of course, the problem concerns only \( \text{Sing}(F) \), for \( \text{Sing}(X) \) can be treated by simply lifting to a local smooth cyclic covering, where \( F \) becomes nonsingular.

**Lemma 4.** \( i \partial \bar{\partial} \Omega_{\text{sing}} \) is a positive measure on \( X. \)
Proof. Instead of $\Omega_{\text{sing}}$, let us consider $\Omega_{\text{ac}} = \Omega - \Omega_{\text{sing}}$. It is a positive current on $X$ which satisfies $i\partial \bar{\partial} \Omega_{\text{ac}} \leq 0$ on $X' = X \setminus \text{Sing}(\mathcal{F})$, because $\Omega$ is closed. According to [1], $\Omega_{\text{ac}}|_{X'}$ can be extended to $X$ as a positive current $\tilde{\Omega}_{\text{ac}}$ with $i\partial \bar{\partial} \tilde{\Omega}_{\text{ac}} \leq 0$ (everywhere). Moreover, this extension can be choosen equal to the trivial extension by zeroes: the coefficients of $\Omega_{\text{ac}}|_{X'}$ are measures, and they are extended to $X$ without additional mass. Then the absolute continuity of $\Omega_{\text{ac}}$ shows that $\tilde{\Omega}_{\text{ac}} = \Omega_{\text{ac}}$ on the full $X$, whence $i\partial \bar{\partial} \Omega_{\text{ac}} \leq 0$ everywhere and therefore $i\partial \bar{\partial} \Omega_{\text{sing}} \geq 0$ everywhere. \hfill $\Box$

By Stokes’ theorem, the total mass of this positive measure must vanish, and consequently the measure itself must be identically zero:

$$i\partial \bar{\partial} \Omega_{\text{sing}} \equiv 0.$$

Therefore, returning to a local chart we see that

$$e^{\nu} \mu_{\text{sing}} \equiv 0,$$

which means that $\text{Supp}(\mu_{\text{sing}}) \subset \Sigma$, the trace of parabolic leaves. Globally, we see that $\text{Supp}(\Omega_{\text{sing}})$ is contained in the algebraic subset of $X$ filled by parabolic leaves and singularities. More precisely, we also see that $\text{Supp}(\Omega_{\text{sing}} - \Omega_{\text{alg}})$ is contained in $\text{Sing}(\mathcal{F})$.

Indeed, if $C \subset X'$ is a parabolic leaf then, along $C$, the pluriharmonic current $\Omega_{\text{sing}}$ has the form $g\delta_C$, for some harmonic function $g$ on $C$. But for every $p \in C$ the Lelong number of $\Omega$ at $p$ is obviously equal to $g(p)$, so that $g$ is in fact constant and $\Omega_{\text{sing}} = \Omega_{\text{alg}}$ along $C$.

Now, from

$$i\partial \bar{\partial} (\Omega_{\text{sing}} - \Omega_{\text{alg}}) \equiv 0$$

and

$$\text{Supp}(\Omega_{\text{sing}} - \Omega_{\text{alg}}) \text{ finite}$$

we deduce $\Omega_{\text{sing}} = \Omega_{\text{alg}}$, and this completes the proof of the theorem.

References