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A new double projection algorithm for variational inequalities☆

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Abstract

We present a modification of a double projection algorithm proposed by Solodov and Svaiter for solving variational inequalities. The main modification is to use a different Armijo-type linesearch to obtain a hyperplane strictly separating current iterate from the solutions of the variational inequalities. Our method is proven to be globally convergent under very mild assumptions. If in addition a certain error bound holds, we analyze the convergence rate of the iterative sequence. We use numerical experiments to compare our method with that proposed by Solodov and Svaiter.

MSC: 90C25; 90C33

Keywords: Variational inequalities; Projection algorithm; Armijo linesearch

1. Introduction

We consider the following variational inequality to find $x^* \in C$ such that

$$\langle F(x^*), y - x^* \rangle \geqslant 0 \quad \text{for all } y \in C, \tag{1}$$

where C is a nonempty closed convex set in $\mathbb{R}^n$ and F is a continuous mapping from $\mathbb{R}^n$ into itself, and $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^n$. Let S denote the solution set of the variational inequality.

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Throughout this paper, we assume that S is nonempty and F has the property

$$\langle F(y), y - x^* \rangle \geqslant 0 \quad \text{for all } y \in C \text{ and all } x^* \in S. \tag{2}$$

The property (2) holds if F is monotone or more generally pseudomontone on C in the sense of Karamardian [3].

Projection-type algorithms have been extensively studied in the literature, see [13] and the references therein. As one of the efficient methods, the algorithm introduced in [10] consists of two steps. First, a hyperplane is constructed which strictly separates current iterate from the solutions of the problem (1). The construction of this hyperplane requires an Armijo-type linesearch. Then the next iterate is produced by projecting the current iterate onto the intersection of the feasible set C and the hyperplane. This method is also called double-projection algorithm due to the fact that one needs to implement double projections in each iteration. In a similar way, we introduce a different double-projection algorithm for variational inequalities. The main difference of our method from that of [10] is the procedure of Armijo-type linesearch (see (3) in the next section). Moreover, we also prove that there is a close link between the natural residual function and the distance from the current iterate to the intersection of the feasible set C and the hyperplane produced by the algorithm (see expression (12) and Lemma 2.2). This observation makes our convergence analysis have a more direct feature. We also present numerical tests to compare our method and that in [10].

To devise algorithms, for variational inequalities, some researchers consider extragradient projection methods. These kind of methods are first proposed by Korpelevich [4]. We refer the reader to [12] and [13] for some recent developments. As a contrast to the extragradient projection methods, the double-projection methods are developed in a different ways, including the way generating the next iterate and the argument on the convergence analysis.

The organization of this paper is as follows. In the next section, we present the algorithm details and prove several critical lemmas for convergence analysis in Section 3. Numerical results are reported in the last section.

2. Algorithm and preliminary results

Let Π_C denote the projector onto C and let $\mu > 0$ be a parameter. A well-known fact is that the solution set S of the problem (1) coincides with the roots of the natural residual function $r_\mu(\cdot)$ which is defined by

$$r_\mu(x) := x - p(x, \mu), \quad \text{where } p(x, \mu) := \Pi_C(x - \mu F(x)).$$

Algorithm 2.1. Choose $x_0 \in C$ and three parameters $\sigma > 0$, $\mu \in (0, 1/\sigma)$ and $\gamma \in (0, 1)$. Set $i = 0$.

Step 1. Compute $r_\mu(x_i)$. If $r_\mu(x_i) = 0$, stop; else go to Step 2.

Step 2. Compute $z_i = x_i - \eta_i r_\mu(x_i)$, where $\eta_i = \gamma^{k_i}$, with k_i being the smallest nonnegative integer satisfying

$$\langle F(x_i) - F(x_i - \gamma^k r_\mu(x_i)), r_\mu(x_i) \rangle \leqslant \sigma \|r_\mu(x_i)\|^2. \tag{3}$$

Step 3. Compute $x_{i+1} = \Pi_{C_i}(x_i)$, where $C_i := C \cap H_i$ with $H_i = \{v : h_i(v) \leqslant 0\}$ being a hyperplane defined by the function

$$h_i(v) := \langle \eta_i r_\mu(x_i) + F(z_i), v - z_i \rangle + \eta_i(1 - \eta_i)\|r_\mu(x_i)\|^2 - \eta_i \mu \langle F(x_i), r_\mu(x_i) \rangle. \tag{4}$$

Let $i = i + 1$ and return to Step 1.

It can be seen that the linesearch in step 2 is well defined. Indeed, since $\gamma \in (0, 1)$ and F is continuous, $F(x_i) - F(x_i - \gamma^k r_\mu(x_i))$ and hence $\langle F(x_i) - F(x_i - \gamma^k r_\mu(x_i)), r_\mu(x_i) \rangle$ converge to zero as k tends to ∞. On the other hand, as a consequence of step 1, $r_\mu(x_i) > 0$ (otherwise, the procedure stops). Therefore there exists a nonnegative integer k_i satisfying (3).

Now let us compare the above algorithm with algorithms in [10]. In the step of the Armijo-type linesearch, [10] uses a different procedure which replaces (3) by the following one:

$$\langle F(x_i - \gamma^k r_\mu(x_i)), r_\mu(x_i) \rangle \geqslant \sigma \|r_\mu(x_i)\|^2, \tag{5}$$

where the parameter $\sigma > 0$ is required to be strictly less than 1, and μ is assumed to be equal to 1 in their Algorithm 2.1 or changes according to the value of η_i in each iteration in their Algorithm 2.2. The choice of the hyperplane in step 3 is also different from that in [10]. To devise extragradient projection algorithm for variational inequality, [12] considers the following Armijo-type linesearch procedure: k_i is the smallest nonnegative integer k satisfying

$$\langle F(x_i) - F(x_i - \gamma^k r_\mu(x_i)), r_\mu(x_i) \rangle \leqslant \sigma \|r_\mu(x_i)\|^2, \tag{6}$$

with $\sigma \in (0, 1)$ being a parameter. It can be seen that this linesearch has the same expression with (3). However, (6) requires σ be strictly less than 1 which is crucial for the convergence analysis in [12], while the parameter σ in our algorithm can take any positive scalar. In fact, our numerical experiments in the last section take $\sigma = 4$.

In the rest of this section, we prove several lemmas which are important for the convergence analysis in the next section.

Lemma 2.1. *For every $x \in C$,*

$$\langle F(x), r_\mu(x) \rangle \geqslant \mu^{-1} \|r_\mu(x)\|^2. \tag{7}$$

Proof. Note that $p(x, \mu)$ is defined to be $\Pi_C(x - \mu F(x))$ and $r_\mu(x) = x - p(x, \mu)$. It follows that

$$\langle x - p(x, \mu) - \mu F(x), y - p(x, \mu) \rangle \leqslant 0 \quad \text{for all } y \in C;$$

in particular, taking $y = x$, we obtain the desired inequality. □

Lemma 2.2. *Let x^* solve the variational inequality* (1) *and the function h_i be defined by* (4). *Then $h_i(x_i) \geqslant \eta_i(\mu^{-1} - \sigma)\|r_\mu(x_i)\|^2$ and $h_i(x^*) \leqslant 0$. In particular, if $r_\mu(x_i) \neq 0$ then $h_i(x_i) > 0$.*

Proof. Since $z_i = x_i - \eta_i r_\mu(x_i)$,

$$\begin{aligned} h_i(x_i) &= \langle \eta_i r_\mu(x_i) + F(z_i), x_i - z_i \rangle + \eta_i(1-\eta_i)\|r_\mu(x_i)\|^2 - \eta_i \mu \langle F(x_i), r_\mu(x_i)\rangle \\ &= \eta_i \langle F(z_i), r_\mu(x_i)\rangle + \eta_i \|r_\mu(x_i)\|^2 - \eta_i \mu \langle F(x_i), r_\mu(x_i)\rangle \\ &\geqslant \eta_i(1-\mu)\langle F(x_i), r_\mu(x_i)\rangle + \eta_i(1-\sigma)\|r_\mu(x_i)\|^2 \\ &\geqslant \eta_i(\mu^{-1} - \sigma)\|r_\mu(x_i)\|^2, \end{aligned}$$

where the first inequality follows from (3) and the last one follows from Lemma 2.1. If $r_\mu(x_i) \neq 0$ then $h_i(x_i) > 0$ because $\mu < 1/\sigma$. It remains to be proved that $h_i(x^*) \leqslant 0$. Since $r_\mu(x_i) = x_i - p(x_i, \mu)$ *and* $p(x_i, \mu) = \Pi_C(x_i - \mu F(x_i))$, we have

$$\langle r_\mu(x_i) - \mu F(x_i), x^* - x_i + r_\mu(x_i)\rangle \leqslant 0;$$

on the other hand, assumption (2) implies that

$$\langle \mu F(x_i), x^* - x_i \rangle = \mu \langle F(x_i), x^* - x_i \rangle \leqslant 0.$$

Adding the last two expressions, we obtain that

$$\langle r_\mu(x_i), x^* - x_i + r_\mu(x_i)\rangle - \mu \langle F(x_i), r_\mu(x_i)\rangle \leqslant 0.$$

It follows that

$$\begin{aligned} \langle \eta_i r_\mu(x_i) + F(z_i), x^* - z_i \rangle &= \langle \eta_i r_\mu(x_i) + F(z_i), x^* - x_i + \eta_i r_\mu(x_i)\rangle \\ &= \eta_i^2 \|r_\mu(x_i)\|^2 + \eta_i \langle r_\mu(x_i), x^* - x_i \rangle + \langle F(z_i), x^* - z_i \rangle \\ &\leqslant \eta_i^2 \|r_\mu(x_i)\|^2 + \eta_i \mu \langle F(x_i), r_\mu(x_i)\rangle - \eta_i \|r_\mu(x_i)\|^2. \end{aligned}$$

Thus $h_i(x^*) \leqslant 0$ is verified. □

Lemma 2.3. *Let C be a closed convex set in* $\mathbb{R}^n$, *h be a real-valued function on* $\mathbb{R}^n$, *and K be the set* $\{x \in C : h(x) \leqslant 0\}$. *If K is nonempty and h is Lipschitz continuous on C with modulus* $\theta > 0$, *then*

$$\operatorname{dist}(x, K) \geqslant \theta^{-1} \max\{h(x), 0\} \quad \textit{for all } x \in C, \tag{8}$$

where $\operatorname{dist}(x, K)$ *denotes the distance from x to K.*

Proof. Clearly (8) holds for all $x \in K$. Hence, it suffices to show that (8) holds for every $x \in C \backslash K$. Let $x \in C$ but $x \notin K$. Since K is closed, there exists $y(x) \in K$ such that $\|x - y\| = \operatorname{dist}(x, K)$. It follows from the Lipschitz continuity of h that

$$|h(x) - h(y(x))| \leqslant \theta \|x - y(x)\| = \theta \operatorname{dist}(x, K).$$

Since $x \notin K$ and $y(x) \in K$, we have $h(x) > 0$ and $h(y(x)) \leqslant 0$. Thus we have

$$h(x) \leqslant h(x) - h(y(x)) = |h(x) - h(y(x))| \leqslant \theta \operatorname{dist}(x, K),$$

and hence the conclusion follows. □

Lemma 2.4. *Let X be a nonempty closed convex set,* $\bar{x} = \Pi_X(x)$ *and* $x^* \in X$. *Then*

$$\|\bar{x} - x^*\|^2 \leqslant \|x - x^*\|^2 - \|x - \bar{x}\|^2.$$

Proof. Since $\|\bar{x} - x^*\|^2 = \|x - x^*\|^2 + \|x - \bar{x}\|^2 + 2\langle \bar{x} - x, x - x^*\rangle$ and since $\langle \bar{x} - x, x^* - \bar{x}\rangle \geqslant 0$, the conclusion follows immediately. □

3. Convergence and convergence rate

Theorem 3.1. If F is continuous on C and condition (2) holds, then either Algorithm 2.1 terminates in a finite number of iterations or generates an infinite sequence $\{x_i\}$ converging to a solution of (1).

Proof. Let x^* be a solution of the variational inequality problem. We assume that Algorithm 2.1 generates an infinite sequence $\{x_i\}$. In particular, $r_\mu(x_i) \neq 0$ for every i. Since $x_{i+1} = \Pi_{C_i}(x_i)$, it follows from Lemma 2.4 that

$$\|x_{i+1} - x^*\|^2 \leqslant \|x_i - x^*\|^2 - \|x_{i+1} - x_i\|^2 = \|x_i - x^*\|^2 - \text{dist}^2(x_i, C_i). \tag{9}$$

It follows that the sequence $\{\|x_{i+1} - x^*\|^2\}$ is nonincreasing, and hence is a convergent sequence. Therefore, $\{x_i\}$ is bounded and

$$\lim_{i\to\infty} \text{dist}(x_i, C_i) = 0. \tag{10}$$

Since $F(x)$ and hence $p(x, \mu)$ are continuous, we have the sequence $\{p(x_i, \mu)\}$ and hence the sequence $\{z_i\}$ is bounded. Thus the continuity of F implies that $\{F(z_i)\}$ is a bounded sequence, that is, for some $M > 0$,

$$\|\eta_i r_\mu(x_i) + F(z_i)\| \leqslant M, \quad \text{for all } i. \tag{11}$$

Clearly each function h_i is Lipschitz continuous on C with modulus M. Applying Proposition 2.3 and noting that $x_i \notin C_i$, we obtain that

$$\text{dist}(x_i, C_i) \geqslant M^{-1} h_i(x_i), \quad \text{for all } i. \tag{12}$$

It follows from (9), (12) and Lemma 2.2 that

$$\text{dist}(x_i, C_i) \geqslant M^{-1} h_i(x_i) \geqslant M^{-1}(\mu^{-1} - \sigma)\eta_i \|r_\mu(x_i)\|^2.$$

Thus (10) implies that

$$\lim_{i\to\infty} \eta_i \|r_\mu(x_i)\|^2 = 0. \tag{13}$$

If $\limsup_{i\to\infty} \eta_i > 0$, then we must have $\liminf_{i\to\infty} \|r_\mu(x_i)\| = 0$. Since $r_\mu(x)$ is continuous and $\{x_i\}$ is a bounded sequence, there exists an accumulation point $\bar{x}$ of $\{x_i\}$ such that $r_\mu(\bar{x}) = 0$. This implies that $\bar{x}$ solves the variational inequality (1). Replacing x^* by $\bar{x}$ in the preceding argument, we obtain that the sequence $\{\|x_i - \bar{x}\|\}$ is nonincreasing and hence converges. Since $\bar{x}$ is an accumulation point of $\{x_i\}$, some subsequence of $\{\|x_i - \bar{x}\|\}$ converges to zero. This shows that the whole sequence $\{\|x_i - \bar{x}\|\}$ converges to zero, and hence $\lim_{i\to\infty} x_i = \bar{x}$.

Suppose now that $\lim_{i\to\infty} \eta_i = 0$. Let $\bar{x}$ be any accumulation point of $\{x_i\}$: there exists some subsequence $\{x_{i_j}\}$ converging to $\bar{x}$. By the choice of η_i, (3) implies that

$$\begin{aligned}\sigma\|r_\mu(x_{i_j})\|^2 &< \langle F(x_{i_j}) - F(x_{i_j} - \gamma^{k_{i_j}-1} r_\mu(x_{i_j})), r_\mu(x_{i_j})\rangle \\ &= \langle F(x_{i_j}) - F(x_{i_j} - \gamma^{-1}\eta_{i_j} r_\mu(x_{i_j})), r_\mu(x_{i_j})\rangle \\ &\leqslant \|F(x_{i_j}) - F(x_{i_j} - \gamma^{-1}\eta_{i_j} r_\mu(x_{i_j}))\|\|r_\mu(x_{i_j})\|, \quad \text{for all } j,\end{aligned}$$

Since $\{r_\mu(x_i)\}$ is bounded and F is continuous, we obtain by letting $j \to \infty$ that $r_\mu(\bar{x}) = 0$. Applying the similar argument in the previous case, we get that $\lim_{i\to\infty} x_i = \bar{x}$. □

Before ending this section, we provide a result on the convergence rate of the iterative sequence generated by Algorithm 2.1. To establish this result, we need a certain error bound to hold locally (see (14) below). The research on error bound is a large topic in mathematical programming. One can refer to the survey [6] for some sufficient conditions ensuring the existence of error bounds and for the roles played by error bounds in the convergence analysis of iterative algorithms; more recent developments on this topic are included in Chapter 6 in the excellent book [1]. A condition similar to (14) has also been used in [9] (see expression (5) therein) to analyze the convergence rate in very general framework.

Theorem 3.2. *In addition to the assumptions in the above theorem, if F is Lipschitz continuous with modulus $L > 0$ and if there exist positive constants c and δ such that*

$$\mathrm{dist}(x, S) \leqslant c\|r_\mu(x)\|, \quad \textit{for all } x \textit{ satisfying } \|r_\mu(x)\| \leqslant \delta; \tag{14}$$

then there is a constant $\alpha > 0$ such that for sufficiently large i,

$$\mathrm{dist}(x_i, S) \leqslant \frac{1}{\sqrt{\alpha i + \mathrm{dist}^{-2}(x_0, S)}}.$$

Proof. Put $\eta := \min\{1/2, L^{-1}\gamma\sigma\}$. We first prove that $\eta_i > \eta$ for all i. By the construction of η_i, we have $\eta_i \in (0, 1]$. If $\eta_i = 1$, then clearly $\eta_i > 1/2 \geqslant \eta$. Now we assume that $\eta_i < 1$. Since $\eta_i = \gamma^{k_i}$, it follows that the nonnegative integer $k_i \geqslant 1$. Thus the construction of k_i implies that

$$\langle F(x_i) - F(x_i - \gamma^{-1}\eta_i r_\mu(x_i)), r_\mu(x_i)\rangle > \sigma\|r_\mu(x_i)\|^2. \tag{15}$$

It follows from the Lipschitz continuity of F that

$$\begin{aligned}\sigma\|r_\mu(x_i)\|^2 &< \langle F(x_i) - F(x_i - \gamma^{-1}\eta_i r_\mu(x_i)), r_\mu(x_i)\rangle \\ &\leqslant L\gamma^{-1}\eta_i\|r_\mu(x_i)\|^2.\end{aligned}$$

Therefore $\eta_i > L^{-1}\gamma\sigma \geqslant \eta$.

Let $x^* \in \Pi_S(x_i)$. By the proof of the above theorem and (14), we obtain that for sufficiently large i,

$$\begin{aligned}\mathrm{dist}^2(x_{i+1}, S) \leqslant \|x_{i+1} - x^*\|^2 &\leqslant \|x_i - x^*\|^2 - M^{-2}\eta_i^2(\mu^{-1} - \sigma)^2\|r_\mu(x_i)\|^4 \\ &\leqslant \|x_i - x^*\|^2 - M^{-2}\eta^2(\mu^{-1} - \sigma)^2\|r_\mu(x_i)\|^4 \\ &\leqslant \mathrm{dist}^2(x_i, S) - M^{-2}\eta^2(\mu^{-1} - \sigma)^2 c^{-4}\mathrm{dist}(x_i, S)^4.\end{aligned}$$

Write α for $M^{-2}\eta^2(\mu^{-1}-\sigma)^2c^{-4}$. Applying Lemma 6 in [8, Chapter 2], we have

$$\operatorname{dist}(x_i, S) \leqslant \operatorname{dist}(x_0, S)/\sqrt{\alpha i \operatorname{dist}^2(x_0, S)+1} = 1/\sqrt{\alpha i + \operatorname{dist}^{-2}(x_0, S)}.$$

This completes the proof. □

4. Numerical experiments

In this section, we present some numerical experiments for the proposed algorithm. The MATLAB codes are run on a PC (with CPU Intel P4) under MATLAB Version 6.5.1.199709 (R13) Service Pack 1 which contains Optimization Toolbox Version 2.3. We compare the performance of our algorithm [10, Algorithm 2.2] and [12, Algorithm NVE-2]. We take $\|r(x)\| \leqslant 10^{-4}$ as the termination criterion. We choose $\gamma = 0.5$, $\sigma = 4$ and $\mu = 0.2$ for our algorithm; $\sigma = 0.3$ and $\gamma = 0.5$ for Algorithm 2.2 in [10] and $\sigma = 0.4$ and $\gamma = 0.8$ for Algorithm NVE-2 in [12]. The choices of the parameters for the latter two algorithms are what the corresponding references proposed. Example 1 is tested in [11]. Example 2 contains test results for several nonlinear variational inequality problems. We thank an anonymous referee for pointing out some problems in the original numerical test results which helps us to correct some bugs in the original MATLAB code and for suggesting us to test more nonlinear problems to compare our algorithm with some known algorithms in the literature.

Example 1. Consider the affine variational inequality (1) with $C = [0, 1]^n$ and $F(x) = Mx + d$ where

$$M = \begin{pmatrix} 4 & -2 & & & \\ 1 & 4 & -2 & & \\ & 1 & 4 & -2 & \\ & & \cdot & \cdot & \cdot \\ & & & 1 & 4 \end{pmatrix} \quad \text{and} \quad d = \begin{pmatrix} -1 \\ -1 \\ \cdots \\ -1 \end{pmatrix}.$$

The initial point x_0 is chosen to be the origin. We use *nf* to denote the total number of times that F is evaluated (Table 1).

Example 2. Nonlinear variational inequality problems. Mathiesen's test problem is tested in [5,7,10]. *PMnash5* and *PMnash10* are called Nash–Cournot NCP (with $n = 5$ and $n = 10$, respectively) and tested

Table 1
Example 1

	Algorithm 2.1		[10, Algorithm 2.2]		[12, Algorithm NVE-2]	
	iter.(*nf*)	CPU	iter.(*nf*)	CPU	iter.(*nf*)	CPU
$n = 10$	22 (33)	0.141	21 (65)	0.151	30 (331)	0.221
$n = 50$	23 (32)	0.251	23 (71)	0.261	34 (375)	0.34
$n = 100$	24 (32)	0.551	23 (71)	0.541	36 (397)	0.781
$n = 200$	24 (31)	2.304	25 (77)	2.423	37 (408)	3.305
$n = 500$	25 (32)	24.9	25 (77)	26.1	38 (419)	38.605

Table 2
Example 2

	Algorithm 2.1		[10, Algorithm 2.2]		[12, Algorithm NVE-2]	
	Iter.(*nf*)	CPU	Iter.(*nf*)	CPU	Iter.(*nf*)	CPU
Mathiesen	19 (51)	0.14	12 (35)	0.15	204 (2522)	1.472
PMnash5	11 (67)	0.14	21 (67)	0.18	31 (807)	0.34
PMnash10	10 (31)	0.12	17 (54)	0.16	33 (595)	0.351
Harnash5	9 (55)	0.1	18 (58)	0.181	25 (651)	0.271
Harnash10	34 (169)	0.271	39 (122)	0.33	71 (1412)	0.811

in [7,10]. Harker [2] defined and tested *Harnash5* and *Harnash10* with $n = 5$ and $n = 10$, respectively. For Mathiesen's test problem, we use $x_0 = (0.3, 0.4, 0.3)$ as the initial point, while the initial point of other test problems is $x_0 = (1, \dots, 1)$ (Table 2).

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