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CLASSIFICATION OF THE AUSLANDER-REITEN QUIVERS OF LOCAL GORENSTEIN ORDERS AND A CHARACTERIZATION OF THE SIMPLE CURVE SINGULARITIES

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In this paper we give a complete list of all finite Auslander-Reiten quivers of local Gorenstein orders Λ over a complete Dedekind domain R of finite lattice type (i.e. Λ is an injective indecomposable left lattice over itself and has – up to isomorphism – only finitely many indecomposable left lattices) [7,19]. For each translation quiver Γ in this list, we indicate explicitly a Gorenstein order Λ with Γ as its Auslander-Reiten quiver. Moreover, in each case we describe the indecomposable Λ-lattices.

In particular, this list contains the Auslander-Reiten quivers of the plane simple curve singularities whose complete local rings can be viewed as Gorenstein orders over the power series ring in one variable over the complex numbers [9]. Briefly we recall a description of these singularities which turns out to be of interest in connection with the above translation quivers [1,6,21,22]:

Consider the ring of invariants of a finite nontrivial subgroup of $SL_2(\mathbb{C})$ acting linearly on the power series ring $\mathbb{C}[\![U, V]\!]$. It has three generators X, Y, Z satisfying one relation $f(X, Y)+Z^2=0$ which defines in the neighbourhood of the origin a surface with the origin as an isolated singularity. The singularities occuring in this way as quotient singularity of a finite group are usually known as rational double points or Kleinian singularities. It is well known that the resolution graph of these singularities are the Dynkin diagrams $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$ [6]. Then the intersection with the plane $Z=0$ is a reduced simple plane curve singularity [1] characterized by Greuel-Knörrer [14]:

The complete local ring Λ of a reduced plane curve singularity has finitely many nonisomorphic torsion free modules of rank 1 if and only if $\Lambda \cong \mathbb{C}[\![X, Y]\!]/f(X, Y)$ where $f(X, Y)+Z^2$ defines a Kleinian singularity.

Our characterization of the Auslander-Reiten quivers of the simple curve singularities uses one of the main results of [26] which we summarize as follows:

Let Λ be a basic R-order in the separable K-algebra A, where K is the quotient field of R, $A=K\Lambda$, and $\Lambda/\text{Rad}\,\Lambda$ is a product of skewfields. If $A=\prod_{i=1}^{s}(D_i)_{n_i}$, where $(D_i)_{n_i}$ is the $n_i \times n_i$-matrix ring over a finite-dimensional skewfield D_i over K, then both the number s of simple factors of A and all the numbers n_i, $i=1,\dots,s$ are determined by the Auslander-Reiten quiver of Λ.

This result together with the knowledge of all the examples we shall present, gives rise to the following:

Theorem. *Let Λ be any local not necessarily commutative Gorenstein R-order of finite lattice type in a product of skewfields such that the stable Auslander–Reiten quiver $\mathfrak{A}(\Lambda)_s$ has as tree class one of the Dynkin diagrams $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$. Then its Auslander–Reiten quiver $\mathfrak{A}(\Lambda)$ coincides with the Auslander–Reiten quiver of the category of lattices over the complete local ring of a simple curve singularity given by one of the equations $f(X,Y)=0$.*

In Riedtmann's notation [16], *$\mathfrak{A}(\Lambda)_s$ is one of the following:*

$\mathfrak{A}(\Lambda)_s$		type of the corresponding Kleinian singularity and defining polynomial $f(X,Y)$		
$\mathbb{Z}\mathbb{A}_1/\tau^{2\mathbb{Z}}$		$\mathbb{A}_1$,	X^2+Y^2	
$\mathbb{Z}\mathbb{A}_3/(\tau\varphi)^{\mathbb{Z}}$,	$\varphi^2=\mathrm{Id}$	$\mathbb{A}_3$,	X^2+Y^4	
$\mathbb{Z}\mathbb{D}_m/(\tau\varphi)^{\mathbb{Z}}$,	$m\geq 4$, $\varphi^2=\mathrm{Id}$	$\mathbb{A}_{2m-3}$,	X^2+Y^{2m-2}	*
$\mathbb{Z}\mathbb{A}_{2m}/\varrho^{\mathbb{Z}}$,	$m\geq 1$, $\varrho^2=\tau$	$\mathbb{A}_{2m}$,	X^2+Y^{2m+1}	
$\mathbb{Z}\mathbb{D}_n/\tau^{2\mathbb{Z}}$,	$n\geq 4$ and even	$\mathbb{D}_n$,	X^2Y+Y^{n-1}	
$\mathbb{Z}\mathbb{A}_{2n-3}/(\tau\varphi)^{\mathbb{Z}}$,	$n\geq 5$ and odd, $\varphi^2=\mathrm{Id}$	$\mathbb{D}_n$,	X^2Y+Y^{n-1}	*
$\mathbb{Z}\mathbb{E}_6/(\tau\varphi)^{\mathbb{Z}}$,	$\varphi^2=\mathrm{Id}$	$\mathbb{E}_6$,	X^3+Y^4	
$\mathbb{Z}\mathbb{E}_7/\tau^{2\mathbb{Z}}$		$\mathbb{E}_7$,	X^3+XY^3	
$\mathbb{Z}\mathbb{E}_8/\tau^{2\mathbb{Z}}$		$\mathbb{E}_8$,	X^3+Y^5	

(Here τ denotes the translation on the translation quiver $\mathbb{Z}\Delta$, Δ an oriented Dynkin diagram, φ and ϱ are automorphisms of $\mathbb{Z}\Delta$ satisfying the indicated relations induced by nontrivial automorphisms on Δ.)

* The discrepancy between the type of the Kleinian singularity and the tree class of the stable Auslander-Reiten quiver in these cases is explained in [9].

The paper is organized as follows:

In Section 1, using covering techniques, we derive necessary conditions relating the positions of projective and injective vertices for Auslander-Reiten quivers of arbitrary orders. In Section 2 we translate the results of Section 1 into concrete conditions for configurations of Gorenstein orders of finite type. Using these results, we derive in Section 3 a complete list of all possible finite Auslander-Reiten quivers of orders having exactly one projective vertex being simultaneously injective. In Section 4, we present for each translation quiver of Section 3 a Gorenstein order Λ having this translation quiver as Auslander-Reiten quiver.

Except for the Dynkin diagrams of type $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$ – where the reader should consult [13, 14] – we also indicate the whole Auslander-Reiten quiver of Λ and give a description of its indecomposable lattices. If Λ can be chosen to be commutative, we just take as Λ the local ring $\mathbb{C}[\![X, Y]\!]/f(X, Y)$ of a simple plane curve singularity.

The computations of many of these Auslander–Reiten quivers are already discussed elsewhere [8, 9, 20, 23, 24]. Our computations were rather technical and very often had to be worked out in many steps. So we have not included a detailed description of all these computations.

1. Relations between projective and injective vertices in $\mathfrak{A}(\Lambda)$

Let R be a complete Dedekind domain with quotient field K, residue class field $\mathfrak{k}$, and let Λ be an R-order in a separable K-algebra $A = K\Lambda$ of finite lattice type. We denote by $\mathfrak{A}(\Lambda) = \Gamma$ the Auslander–Reiten quiver of Λ, and we consider Γ as $\mathfrak{k}$-modulated translation quiver in the sense of [3, 25]. Moreover let $\tilde{\Gamma}$ be the universal cover of Γ, and let $F : \tilde{\Gamma} \to \Gamma$ be the covering morphism [5]. We recall the definition of the powers of the functorial radical $\mathbf{r}^l(M, N)$, $l \geq 0$, of the Λ-morphism space from M to N as the $\mathrm{End}_\Lambda(M)$-$\mathrm{End}_\Lambda(N)$-submodule of $\mathrm{Hom}_\Lambda(M, N)$ which is generated by those morphisms from M to N which are compositions of l irreducible maps. Recall also that to each vertex x and each arrow $x \to y$ in $\tilde{\Gamma}$ there is associated the finite-dimensional skewfield $f_x = \mathrm{End}_\Lambda(Fx)/\mathrm{Rad}\,\mathrm{End}_\Lambda(Fx)$ over $\mathfrak{k}$ and the finite-dimensional f_x-f_y-bimodule ${}_xB_y = \mathrm{Irr}(Fx, Fy) = \mathbf{r}(Fx, Fy)/\mathbf{r}^2(Fx, Fy)$ resp. For vertices x, y in $\tilde{\Gamma}$ let $H(x, y)$ be the morphisms from x to y in the mesh category $\mathfrak{k}(\tilde{\Gamma})$ of $\tilde{\Gamma}$ [5, 25]. By [25] there exists a covering functor for Λ: For x, y as above and Λ-lattices $M = Fx$, $N = Fy$ there exists a graded $\mathfrak{k}$-bilinear isomorphism

$$\mathbf{F} : \prod_{Fz = Fy} H(x, z) \to \prod_{l \geq 0} \mathbf{r}^l(M, N)/\mathbf{r}^{l+1}(M, N).$$

Our aim in this section is to find relations between the positions of projective and injective Λ-lattices in $\mathfrak{A}(\Lambda)$. First we consider an indecomposable Λ-lattice M and an indecomposable projective Λ-lattice Q. Then each morphism $\varphi : Q \to M$ factorizes over a projective cover $\psi : P_0(M) \to M$:

$$\begin{array}{ccc} & & P_0(M) \\ & \overset{\alpha}{\nearrow} & \downarrow{\scriptstyle\psi} \\ Q & \xrightarrow{\ \varphi\ } & M \end{array} \qquad \text{there exists an } \alpha \text{ such that } \alpha\psi = \varphi,$$

and φ can be extended to a projective cover of M if and only if α is a split monomorphism; otherwise $\alpha \in \mathbf{r}(Q, P_0(M))$.

This observation gives rise to the following definition for indecomposable Λ-lattices X and M, M nonprojective:

$$\mathbf{r}P(X,M)=\sum_{\substack{Q \text{ arbitrary}\\ \text{projective}}} \mathbf{r}(X,Q)\cdot\mathbf{r}(Q,M)$$

consists of all Λ-morphisms from X to M which factor nontrivially over a projective lattice.

If we abbreviate $\mathrm{End}_\Lambda(X)/\mathrm{Rad}\,\mathrm{End}_\Lambda(X)$ by $t(X)$ for X indecomposable, we have for Q indecomposable projective:

$$\begin{aligned}(1.1)\qquad \dim_{t(Q)}(\mathrm{Hom}_\Lambda(Q,M)/\mathbf{r}P(Q,M)) &= \mathrm{mult}_{P_0(M)}(Q)\\ &= \text{multiplicity of } Q - \text{up to isomorphism} - \text{as direct}\\ &\quad \text{summand in the projective cover of } M.\end{aligned}$$

We now want to make a similar construction in the mesh category $\mathfrak{k}(\tilde{\Gamma})$ and recover this multiplicity there:

For a nonprojective vertex z in $\tilde{\Gamma}$ and an arbitrary vertex x we define $H_\mathrm{p}(x,z)$ as quotient of $H(x,z)$ modulo the f_x-f_z-subspace $HP(x,z)$ generated by paths of the form

$$x\rightarrow\overset{\alpha}{\cdots}\rightarrow q\rightarrow\overset{\beta}{\cdots}\rightarrow z,$$

where q is projective, and α has length at least 1.

Note that for indecomposable Λ-lattices M,N the radical filtration

$$\mathrm{Hom}_\Lambda(M,N)\supseteq\mathbf{r}(M,N)\supseteq\mathbf{r}^2(M,N)\supseteq\cdots$$

induces a filtration on the quotient $\mathrm{Hom}_\Lambda(M,N)/\mathbf{r}P(M,N)$ with associated graded factors

$$\mathbf{r}^l(M,N)+\mathbf{r}P(M,N)/(\mathbf{r}^{l+1}(M,N)+\mathbf{r}P(M,N)).$$

In this situation we have the following:

Proposition 1. *Let $F:\tilde{\Gamma}\rightarrow\Gamma$ and $\mathbf{F}$ be as above, and let $M=Fx$, $N=Fy$. Then $\mathbf{F}$ induces a graded $\mathfrak{k}$-linear bijection*

$$\mathbf{F}_\mathrm{p}:\prod_{Fz=N}H_\mathrm{p}(x,z)\rightarrow\prod_{l\geq 0}\mathbf{r}^l(M,N)+\mathbf{r}P(M,N)/(\mathbf{r}^{l+1}(M,N)+\mathbf{r}P(M,N)).$$

Proof. Since F maps projective vertices of $\tilde{\Gamma}$ onto projective lattices, $\mathbf{F}_\mathrm{p}$ is well-defined; the surjectivity of $\mathbf{F}_\mathrm{p}$ is also clear.

Since Λ is of finite lattice type, there exists an $l_0\in\mathbb{N}$ such that $\mathbf{r}^l(M,N)\subseteq\mathbf{r}P(M,N)$ for all $l\geq l_0$. For vertices x,y in $\tilde{\Gamma}$ we denote by $l_{x,y}$ the length of any path from x to y in $\tilde{\Gamma}$. Then by the injectivity of $\mathbf{F}$ we conclude

$$\begin{aligned}\dim_{f_x}\Big(\prod_{Fz=N}H_\mathrm{p}(x,z)\Big) &= \sum_{\substack{Fz=N\\ l_{x,z}\leq l_0}}(\dim_{f_x}H(x,z)-\dim_{f_x}HP(x,z))\\ &= \sum_{\substack{Fz=N\\ l_{x,z}\leq l_0}}\dim_{f_x}H(x,z)-\sum_{\substack{Fz=N\\ l_{x,z}\leq l_0}}\dim_{f_x}HP(x,z)\end{aligned}$$

$$= \text{length}_{\text{End}_\Lambda(M)}(\text{Hom}_\Lambda(M,N)/\mathbf{r}^{l_0+1}(M,N))$$
$$- \text{length}_{\text{End}_\Lambda(M)}(\mathbf{r}P(M,N)/\mathbf{r}^{l_0+1}(M,N))$$
$$= \text{length}_{\text{End}_\Lambda(M)}(\text{Hom}_\Lambda(M,N)/\mathbf{r}P(M,N)).$$

Consequently, the injectivity of $\mathbf{F}_\text{p}$ follows from its surjectivity. □

We summarize the above considerations and Proposition 1 as follows:

Proposition 2. (i) *Let Q be an indecomposable projective Λ-lattice, N an arbitrary nonprojective indecomposable Λ-lattice. Then the following are equivalent:*

(a) *There exists a $\varrho \in \mathbf{r}^l(Q,N)\setminus \mathbf{r}^{l+1}(Q,N)$ which can be extended to a projective cover of N.*

(b) $\mathbf{r}^l(Q,N)\setminus \mathbf{r}^{l+1}(Q,N) \not\subseteq \mathbf{r}P(Q,N)$.

(c) *There exist vertices q, y in $\tilde{\Gamma}$ with $Fq=Q$, $Fy=N$, $l_{q,y}=l$ and $H_\text{p}(q,y)\neq 0$.*

(ii) *In the situation of* (i), *the multiplicity of Q in the projective cover $P_0(N)$ of N is given by the number $\sum_{Fq=Q}\dim_{f_q} H_\text{p}(q,y)$, and $P_0(N)$ decomposes into $\sum_{q \text{ projective in } \tilde{\Gamma}} \dim_{f_q} H_\text{p}(q,y)$ indecomposable direct summands.*

For a projective vertex q of $\tilde{\Gamma}$ we shall consider later in this section those vertices y such that $l_{q,y}$ is maximal with $H_p(q,y)\neq 0$.

We start with a fixed simple Λ-module S with projective cover P_S and denote by I_S that indecomposable injective Λ-lattice with minimal overlattice I_S^+ satisfying $I_S^+/I_S \cong S$.

Lemma 1. *If M is a Λ-lattice and $\varphi: M\to S$ an epimorphism, then φ factors over the projection $I_S^+ \to I_S^+/I_S \cong S$.*

Proof. Since I_S is an injective lattice, the following pullback via φ decomposes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I_S & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & 0 \\
 & & \Vert & & \downarrow & & \downarrow{\scriptstyle \varphi} & & \\
0 & \longrightarrow & I_S & \longrightarrow & I_S^+ & \longrightarrow & S & \longrightarrow & 0.
\end{array}$$ □

Since Λ is of finite lattice type there exists an $l_1 \in \mathbb{N}$ such that

$$\mathbf{r}^l(X, I_S^+)\cdot \text{Hom}_\Lambda(I_S^+, S)=0$$

for each Λ-lattice X and $l\geq l_1$. Moreover, since $\text{Hom}(P_S, S)\neq 0$, we can choose by Lemma 1 a nonzero morphism $\varrho \in \mathbf{r}^{l_0}(P_S, I_S^+)$ where l_0 is maximal with $\varrho\cdot \text{Hom}_\Lambda(I_S^+, S)\neq 0$.

Lemma 2. (i) *If P' is a projective Λ-lattice and $\tau\in \text{Hom}_\Lambda(P_S, P')$, $\varrho' \in \text{Hom}_\Lambda(P', I_S^+)$ such that $\varrho = \tau\varrho'$, then τ is a split monomorphism.*

(ii) *For an arbitrary Λ-lattice Y and each $\alpha \in \mathbf{r}(I_S^+, Y)$ there exists a projective Λ-*

lattice P', a nonsplit morphism $\tilde{\varrho}: P_S \to P'$ and a morphism $\psi: P' \to Y$ such that $\varrho\alpha = \tilde{\varrho}\psi$, i.e., $\varrho\alpha$ factors properly over another projective.

Proof. (i) Trivial.

(ii) Let $P' = P_0(Y) \xrightarrow{\psi} Y$ be a projective cover of Y. Then there exists a $\tilde{\varrho}$ with $\tilde{\varrho}\psi = \varrho\alpha$. If $\varrho\alpha \cdot \mathrm{Hom}_\Lambda(Y, S) = 0$, then $\mathrm{Im}\, \tilde{\varrho} \subset \mathrm{rad}_\Lambda P_0(Y)$ and $\tilde{\varrho}$ is not split mono. Otherwise suppose that there exists a nonzero β in $\mathrm{Hom}_\Lambda(Y, S)$ with $\varrho\alpha\beta \neq 0$. By Lemma 1 there exist morphisms β', σ' such that

$$\begin{array}{ccc} Y & \xrightarrow{\beta} & S \\ & {\scriptstyle\beta'}\searrow \quad \nearrow{\scriptstyle\sigma'} & \\ & I_S^+ & \end{array}$$

commutes. Therefore $\varrho\alpha\beta'\sigma' \neq 0$ and $\varrho\alpha\beta' \cdot \mathrm{Hom}(I_S^+, S) \neq 0$; moreover $\varrho\alpha\beta' \in \mathrm{r}^{l_0+1}(P_S, I_S^+)$: contradiction to the maximality of l_0. □

We summarize the above results in the following

Proposition 3. (i) *If X is an indecomposable Λ-lattice with a morphism $\varrho \in \mathrm{Hom}_\Lambda(P_S, X)$ satisfying $\varrho \cdot \mathrm{Hom}_\Lambda(X, S) \neq 0$ and $\varrho\alpha \cdot \mathrm{Hom}_\Lambda(Y, S) = 0$ for an arbitrary $\alpha \in \mathrm{r}(X, Y)$, then X is isomorphic to a direct summand of the unique minimal overlattice I_S^+ of I_S.*

(ii) *If q is a projective vertex in $\bar{\Gamma}$ and y is a vertex of $\bar{\Gamma}$ with $l_{q,y}$ maximal satisfying $H_p(q, y) \neq 0$, then y is a successor of an injective vertex.*

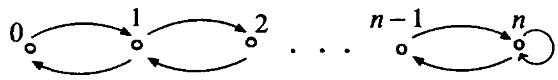
Proof. (i) follows immediately from the Lemmata above. (ii) is the direct translation of (i) using Proposition 1. □

2. Necessary conditions for the Auslander–Reiten quivers of Gorenstein orders

From now on we assume that Λ is a nonmaximal R-order and is an indecomposable injective lattice over itself, i.e. Λ is local but not necessarily commutative and Gorenstein in the terminology of [10].

If the Jacobson radical $\mathrm{Rad}\,\Lambda$ decomposes, then Λ is a Bäckström order with associated graph $\mathbb{A}_3$ or $\mathbb{C}_2$ [18], and its Auslander–Reiten quiver is described in [20]. Therefore we assume from now on that $\mathrm{Rad}\,\Lambda$ is indecomposable and Λ is of finite lattice type.

Then by [15] the stable Auslander-Reiten quiver $\mathfrak{A}(\Lambda)_s$ of Λ, i.e., the full subquiver of $\mathfrak{A}(\Lambda)$ of all nonprojective vertices has as tree class a Dynkin diagram Δ and is isomorphic to $\mathbb{Z}\Delta/G$ for G an admissible automorphism group of $\mathbb{Z}\Delta$ in the sense of Riedtmann [16] or is described in [23] in case $\mathfrak{A}(\Lambda)$ contains a loop $\circlearrowleft$.
In this last case $\mathfrak{A}(\Lambda)$ is of the form



, $n \geq 1$,

where the vertex 0 is projective-injective and the translation is the identity on the other vertices $1, \dots, n$. Obviously the stable Auslander–Reiten quiver is then isomorphic to the translation quiver

$$\mathbb{Z}\mathbb{A}_{2n}/\varrho^{\mathbb{Z}}, \quad \varrho^2 = \tau,$$

where ϱ is the automorphism of $\mathbb{Z}\mathbb{A}_{2n}$ induced by the nontrivial automorphism of $\mathbb{A}_{2n}$. Note that $\varrho^{r\mathbb{Z}}$ is admissible in the sense of Riedtmann for $r > 1$ only. So we call the automorphism group G of $\mathbb{Z}\Delta$ *l-admissible* (lattice-admissible) if G is admissible or Δ is of type $\mathbb{A}_{2n}$ and $G = \varrho^{\mathbb{Z}}$ as above.

With the notation of Section 1, we get $\tilde{\Gamma}$ by adding suitable projective-injective vertices to $\mathbb{Z}\Delta$. Since $\operatorname{Rad}\Lambda$ is indecomposable, a projective vertex q of $\tilde{\Gamma}$ has a unique predecessor q^- corresponding to $\operatorname{Rad}\Lambda$ and a unique successor q^+ corresponding to the unique minimal overlattice Λ^+ of Λ in the quiver $\tilde{\Gamma}$; moreover $q^- = \tau q^+$.

We call q^+ a *configuration vertex* and the set of vertices

$$C = \{q^+ \mid q^+ \text{ is a successor of a projective vertex } q\}$$

is called *configuration* of $\mathbb{Z}\Delta$ with respect to Λ.

If q is a projective-injective vertex of $\tilde{\Gamma}$ with successor q^+, then for each nonprojective vertex x in $\tilde{\Gamma}$ we have an isomorphism of f_q-vectorspaces

$$H_{\mathrm{p}}(q, x) \cong {}_qB_{q^+} \otimes_{f_{q^+}} H_{\mathrm{p}}(q^+, x).$$

If $f_q \not\cong f_{q^+}$, then Λ being of finite lattice type, the valuation $(\dim_{f_q} {}_qB_{q^+}, \dim_{f_{q^+}} {}_qB_{q^+})$ is of the form $(1, n)$ or $(n, 1)$ with $n = 2$ or $n = 3$ [2]. This implies either that Λ^+ decomposes – what we already excluded – or the middle term of the almost split sequence of Λ^+ contains n copies of Λ as direct summands. By rank arguments, we have $n = 2$ and an almost split sequence

$$0 \to \operatorname{Rad}\Lambda \to \Lambda^{(2)} \to \Lambda^+ \to 0.$$

Then Λ is a Bäckström order with associated graph $\mathbb{B}_2$; moreover $\operatorname{Rad}\Lambda \cong \Lambda^+$ and Λ has – up to isomorphism – exactly the two nonisomorphic indecomposable lattices Λ and $\operatorname{Rad}\Lambda$. Therefore we assume from now on that ${}_qB_{q^+} \cong f_q \cong f_{q^+}$ for each projective vertex q of $\tilde{\Gamma}$.

For q and x as above, we have under this hypothesis isomorphisms as f_q-vectorspaces

$$H_{\mathrm{p}}(q, x) \cong H_{\mathrm{p}}(q^+, x) \cong \mathfrak{k}(\mathbb{Z}\Delta)(q^+, x) \cong H_{\mathbb{Z}\Delta}(q^+, x),$$

where the last two terms stand for the morphisms from q^+ to x in the mesh category with respect to $\mathbb{Z}\Delta$. Altogether the computation of $H_p(q,x)$ is reduced to computations in the mesh category of $\mathbb{Z}\Delta$.

For each vertex z in $\mathbb{Z}\Delta$, we define the *cover vector* of z as the positive vector $(\dim_{f_z} H_{\mathbb{Z}\Delta}(z,y))_y$, where y runs over the vertices of $\mathbb{Z}\Delta$. (The cover vector consists essentially of the positive piece of the additive function starting at z in the terminology of Gabriel [11] and coincides with Bongartz's starting function of z [4, 5].) The *support* of the cover vector of z consists of those vertices y with $\dim_{f_z} H_{\mathbb{Z}\Delta}(z,y)>0$. Note that for a projective vertex q of $\tilde{\Gamma}$ a nonzero path from q to any vertex of $\mathbb{Z}\Delta$ contributes to a projective cover of Fx in the sense of Proposition 2 if and only if x belongs to the support of the cover vector of q^+.

We now make two important observations which also hold for arbitrary Gorenstein orders of finite lattice type:

First, if I is an injective indecomposable Λ-lattice with minimal overlattice I^+, then – I being projective and I^+/I being simple – the projective cover of I^+ decomposes exactly into two indecomposable nonzero direct summands.

Second, by Propositions 2 and 3 this implies: If $c=q^+$ is a configuration vertex in $\mathbb{Z}\Delta$, there exists a unique vertex c' of $\mathbb{Z}\Delta$ in the support of the cover vector of c such that $l_{c,c'}$ is maximal. Moreover, c' is also a configuration vertex, i.e., there exists a projective vertex q' in $\tilde{\Gamma}$ such that $(q')^+=c'$.

These observations imply immediately the following necessary conditions for a configuration of $\mathbb{Z}\Delta$. Similar conditions are given by Riedtmann for the algebra case in [17], cf. also [12].

Proposition 4. *Let C be a configuration of $\mathbb{Z}\Delta$ with respect to a Gorenstein order of finite lattice type. Then C satisfies the following conditions:*

(C_1) *For each $c\in C$ there exists a unique $c'\in C$ with $H_{\mathbb{Z}\Delta}(c,c')\neq 0$ and $H_{\mathbb{Z}\Delta}(c,d)=0$ for all successors d of c'.*

(C_2) *For each $x\in\mathbb{Z}\Delta$ there exists at least one $c\in C$ with $H_{\mathbb{Z}\Delta}(c,x)\neq 0$.*

3. The possible Auslander–Reiten quivers of local Gorenstein orders of finite lattice type

In this section we assume that Λ is a nonmaximal local Gorenstein order and that Λ is not a Bäckström order with associated graph $\mathbb{A}_3$, $\mathbb{B}_2$ or $\mathbb{C}_2$ (cf. Section 2).

We now discuss the various possibilities for the structure of $\Gamma=\mathfrak{A}(\Lambda)$ using the results of the previous section.

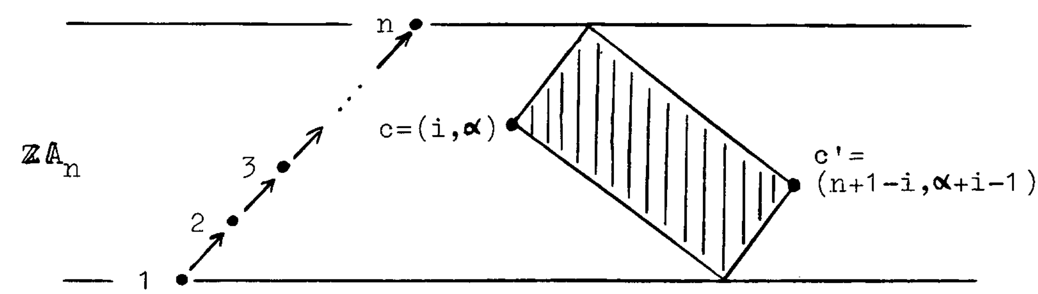
Let $\mathfrak{A}(\Lambda)_s=\mathbb{Z}\Delta/G$ for an oriented Dynkin diagram Δ and an l-admissible automorphism group G of $\mathbb{Z}\Delta$.

We label the vertices of Δ by integers $1,2,\dots,n$ and associate to the vertices of $\mathbb{Z}\Delta$ coordinates $(i,\alpha)\in\{1,\dots,n\}\times\mathbb{Z}$ such that $\tau(i,\alpha)=(i,\alpha-1)$, and there is an arrow from (i,α) to (j,β) in $\mathbb{Z}\Delta$ if and only if either $\alpha=\beta$ and $\overset{i}{\bullet} \longrightarrow \overset{j}{\bullet}$ in Δ or $\alpha=\beta-1$ and $\overset{j}{\bullet} \longrightarrow \overset{i}{\bullet}$ in Δ.

For the various Δ's we briefly sketch the support of the cover vector of a vertex c and indicate also the vertex c' as defined in Proposition 4; an explicit description of the cover vectors is e.g. given by Bongartz in [4].

Using Proposition 4 and – Λ being local – by the fact that there is exactly one G-orbit of configuration vertices in $\mathbb{Z}\Delta$, we determine the possible Auslander–Reiten quivers Γ.

$$\Delta=\mathbb{A}_n: \quad \overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longrightarrow \overset{3}{\bullet} \longrightarrow \cdots \longrightarrow \overset{n-1}{\bullet} \longrightarrow \overset{n}{\bullet}$$



Obviously the support of any cover vector hits the τ-orbit of the end vertices 1 and n together twice. By condition (C_2) this implies that $\tau^2 \in G$. Therefore by condition (C_1) for a configuration vertex $c=(i,\alpha)$ we must have $i=1,2,\ n-1$ or n. For simplicity we may assume that either $c=(1,0)$ or $c=(2,0)$ is a configuration vertex.

If n is even, $c=(1,0)$ is a possible configuration vertex and

(1) $\qquad \Gamma_s \cong \mathbb{Z}\mathbb{A}_n/\varrho^{\mathbb{Z}}, \quad \varrho^2=\tau.$

For $n\geq 4$, $c=(2,0)$ as configuration vertex is excluded by the following argument: G has to be of the form $(\varrho\tau^s)^{\mathbb{Z}}$ with $\varrho^2=\tau$, $s\geq 0$. By condition (C_1) $\tau\notin G$, and we have $s\geq 1$; moreover $\tau^2\in G$. On the other hand the elements of G are of the form

$$(\varrho\tau^s)^m=\varrho^m\tau^{sm}=\begin{cases}\tau^{(s+1/2)m} & \text{for even } m,\\ \varrho\tau^{sm+(m-1)/2} & \text{for odd } m.\end{cases}$$

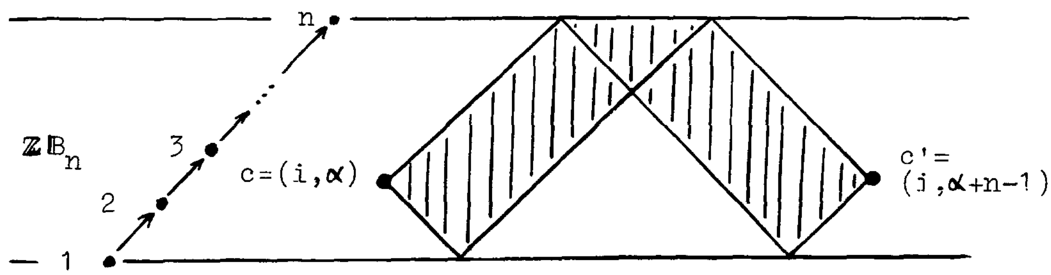
But $(s+1/2)m=2$ is not possible for $s\geq 1$ and $m\in\mathbb{Z}$.

If n is odd and $n>3$, G is of the form $(\varphi\tau^r)^{\mathbb{Z}}$, $\varphi^2=\mathrm{Id}$ and $r\geq 1$. This immediately excludes $c=(1,0)$ because this would force $\tau\in G$. For $c=(2,0)$ the G-orbit of c must contain $c'=(n-1,1)$, and this forces $n\equiv 3 \pmod 4$. This gives the following possibilities:

(2) $\qquad \Gamma_s \cong \mathbb{Z}\mathbb{A}_n/(\varphi\tau)^{\mathbb{Z}}, \quad \varphi^2=\mathrm{Id}, \quad n\equiv 3 \pmod 4,$

(3) $\qquad \Gamma_s \cong \mathbb{Z}\mathbb{A}_3/\tau^{\mathbb{Z}}.$

$$\Delta=\mathbb{B}_n:\quad \overset{1}{\bullet}\longrightarrow\overset{2}{\bullet}\longrightarrow\overset{3}{\bullet}\longrightarrow\cdots\longrightarrow\overset{n-1}{\bullet}\xrightarrow[(2,1)]{}\overset{n}{\bullet},\quad n\geq 2.$$



The possible automorphism groups clearly are of the form $G=\tau^{s\mathbb{Z}}$, $s\geq 1$. Each support of a cover vector hits the τ-orbit of vertex 1 at most twice, therefore $\tau^2\in G$. Possible configuration vertices are up to translation (1,0) or (2,0).

If (1,0) is a configuration vertex, then

(4) $\Gamma_s\cong\mathbb{Z}\mathbb{B}_n/\tau^{\mathbb{Z}}$.

If (2,0) is a configuration vertex then n has to be odd and

(5) $\Gamma_s\cong\mathbb{Z}\mathbb{B}_n/\tau^{2\mathbb{Z}}$,

or for $n=2$, we get the same as for $\Delta=\mathbb{C}_2$ in case (6).

$$\Delta=\mathbb{C}_n:\quad \overset{1}{\bullet}\longrightarrow\overset{2}{\bullet}\longrightarrow\overset{3}{\bullet}\longrightarrow\cdots\longrightarrow\overset{n-1}{\bullet}\xrightarrow[(1,2)]{}\overset{n}{\bullet},\quad n\geq 2.$$

We have the same pattern of the supports as in case $\Delta=\mathbb{B}_n$ and get therefore:

If (1,0) is a configuration vertex, then

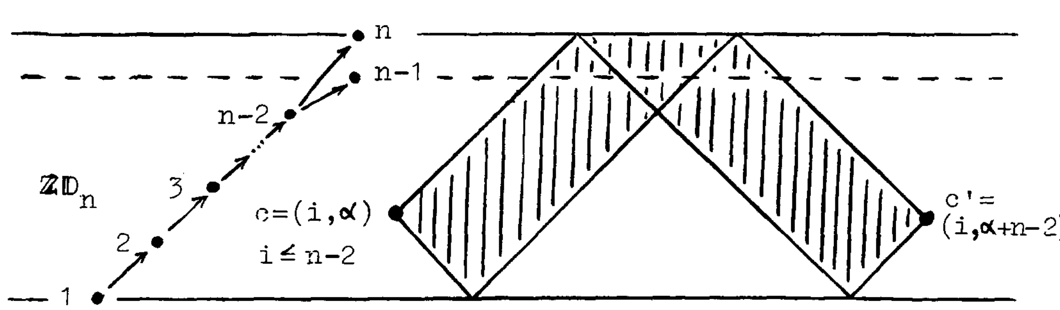
(6) $\Gamma_s\cong\mathbb{Z}\mathbb{C}_n/\tau^{\mathbb{Z}}$.

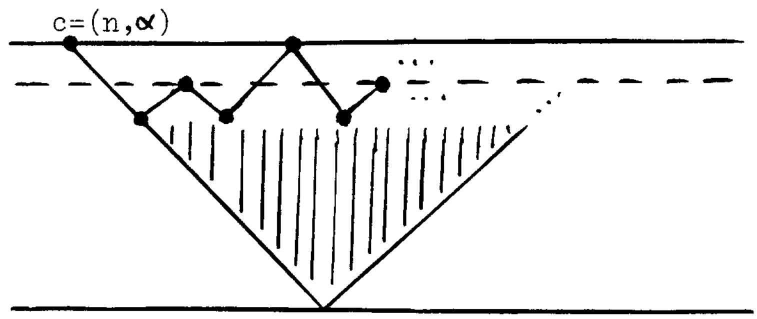
If (2,0) is a configuration vertex, then n is odd and

(7) $\Gamma_s\cong\mathbb{Z}\mathbb{C}_n/\tau^{2\mathbb{Z}}$,

or for $n=2$, we get the same as for $\Delta=\mathbb{B}_2$ in case (4).

$$\Delta=\mathbb{D}_n:\quad \overset{1}{\bullet}\longrightarrow\overset{2}{\bullet}\longrightarrow\overset{3}{\bullet}\longrightarrow\cdots\longrightarrow\overset{n-2}{\bullet}\begin{matrix}\nearrow\overset{n-1}{\bullet}\\ \searrow\underset{n}{\bullet}\end{matrix},\quad n\geq 4.$$





The admissible automorphism groups are of the form $G=\tau^{s\mathbb{Z}}$, $G=(\varrho\tau^s)^{\mathbb{Z}}$ or $G=(\sigma\tau^s)^{\mathbb{Z}}$, where ϱ and σ for $n=4$ are induced by nontrivial automorphisms of $\mathbb{D}_n$ and satisfy $\varrho^2=\mathrm{Id}$ and $\sigma^3=\mathrm{Id}$. Each support of a cover vector hits the τ-orbit of vertex 1 at most twice. Therefore G contains an element of the form $\varphi\tau^2$ for an automorphism φ of finite order of $\mathbb{Z}\mathbb{D}_n$. Possible configuration vertices are up to translation $(1,0)$ and $(2,0)$, and we get the following:

If $(1,0)$ is a configuration vertex, then

(8) $\Gamma_s=\mathbb{Z}\mathbb{D}_n/(\varrho\tau)^{\mathbb{Z}}$, $\varrho^2=\mathrm{Id}$ or

(9) $\Gamma_s=\mathbb{Z}\mathbb{D}_n/\varrho^{\mathbb{Z}}$.

If $(2,0)$ is a configuration vertex, then n has to be even and either

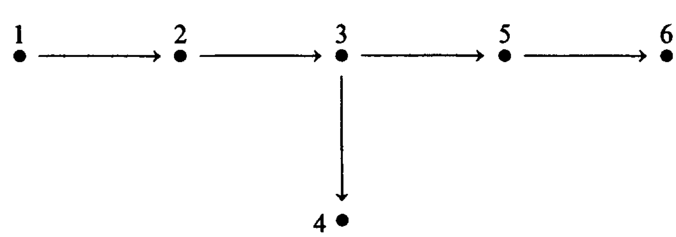
(10) $\Gamma_s=\mathbb{Z}\mathbb{D}_n/\tau^{2\mathbb{Z}}$, or

(11) $\Gamma_s=\mathbb{Z}\mathbb{D}_n/(\varrho\tau^2)^{\mathbb{Z}}$, $\varrho^2=\mathrm{Id}$ or

(12) $\Gamma_s=\mathbb{Z}\mathbb{D}_4/(\sigma\tau^2)^{\mathbb{Z}}$, $\sigma^3=\mathrm{Id}$.

Similar arguments and the structure of the supports of the cover vectors (cf. [4]) give the following possibilities for the remaining Dynkin diagrams:

$\Delta=\mathbb{E}_6$:

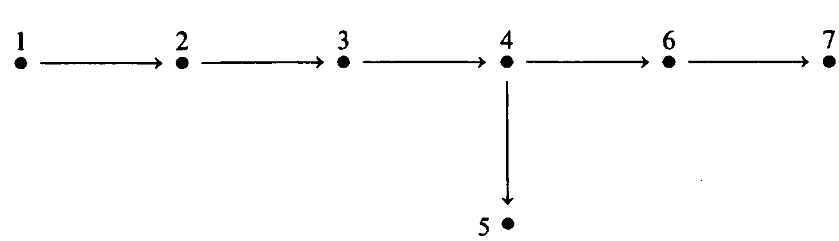


$(1,0)$ is a configuration vertex, and

(13) $\Gamma_s=\mathbb{Z}\mathbb{E}_6/(\varrho\tau)^{\mathbb{Z}}$, $\varrho^2=\mathrm{Id}$,

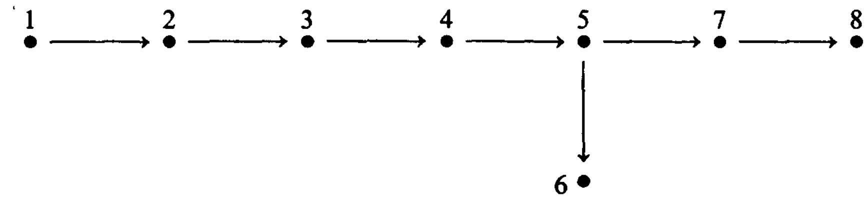
and ϱ is induced by the nontrivial automorphism of $\mathbb{E}_6$.

$\Delta=\mathbb{E}_7$:



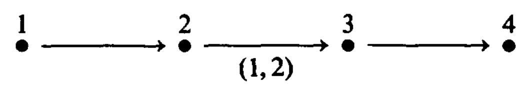
$(7,0)$ is a configuration vertex, and

(14) $\Gamma_s = \mathbb{Z}\mathbb{E}_7/\tau^{2\mathbb{Z}}$.

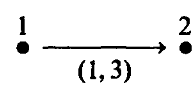
$\Delta = E_8$: 

$(1,0)$ is a configuration vertex, and

(15) $\Gamma_s = \mathbb{Z}\mathbb{E}_8/\tau^{2\mathbb{Z}}$.

$\Delta = \mathbb{F}_4$: 

is not possible.

$\Delta = \mathbb{G}_2$: 

$(1,0)$ is a configuration vertex, and

(16) $\Gamma_s = \mathbb{Z}\mathbb{G}_2/\tau^{2\mathbb{Z}}$

or $(2,0)$ is a configuration vertex, and

(17) $\Gamma_s = \mathbb{Z}\mathbb{G}_2/\tau^{2\mathbb{Z}}$.

4. The realization of the possible Auslander–Reiten quivers of local Gorenstein orders of finite lattice type

For all the possible Auslander–Reiten quivers of local Gorenstein orders we listed in the previous section we now give concrete examples.

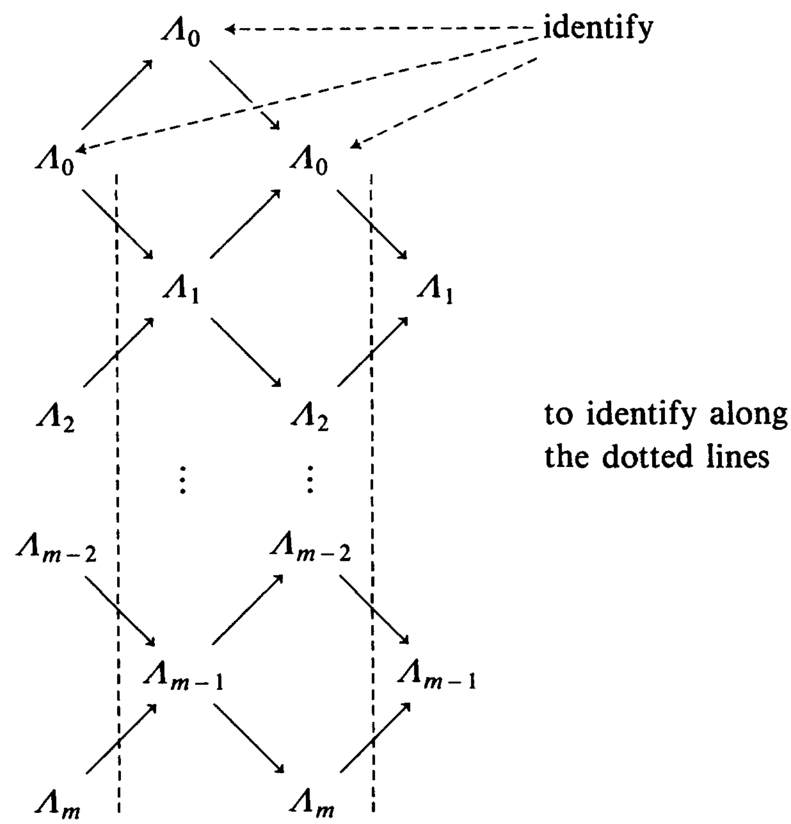
We work over power series rings in one variable over a field in order to avoid arithmetic difficulties in the ground ring R. In all cases, except for $\Delta = \mathbb{G}_3$ we work with the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$ or the Hamiltonian quaternions $\mathbb{H}$. Mostly one can take arbitrary fields, however in the cases $\Delta = \mathbb{B}_n$ or $\mathbb{C}_n$ one has to be careful.

Besides the ground ring R we indicate a maximal R-order in $A = K\Lambda$ containing Λ and then describe the R-order Λ such that $\mathfrak{A}(\Lambda)_s = \Gamma_s$ occurs in the list of Section 3. Then we describe the indecomposable Λ-lattices and write down $\mathfrak{A}(\Lambda)$ explicitly.

Case (1). $R = \mathbb{C}[\![t^2]\!]$, maximal order $\Omega = \mathbb{C}[\![t]\!]$, $\Lambda = R + t^{2m}\Omega$, $m \geq 1$.

Let $\Lambda_i = R + t^{2i}\Omega$ for $i = 0, \dots, m$; in particular, $\Lambda_0 = \Omega$. $\mathfrak{A}(\Lambda)$ is shown in Diagram 1.

$\mathfrak{A}(\Lambda)$:



$\mathfrak{A}(\Lambda)_s \cong \mathbb{Z}\mathbb{A}_{2m}/\varrho^{\mathbb{Z}}, \quad \varrho^2 = \tau.$

Diagram 1.

For $m=0$, Λ is a maximal order.

For $m \geq 1$, Λ is the local ring of the singularity associated to the Kleinian singularity of type $\mathbb{A}_{2m}$. (Shortly we write from now on "local ring of the singularity of type Δ".)

Case (2). $R = \mathbb{C}[\![t^2]\!]$, $\Omega = \mathbb{C}[\![t]\!]$, maximal order $\Omega \prod \Omega$.

Put $n = 2m+3$, with $m \geq 1$. Λ is as ring generated by the elements $(1,1)$, (t, t^{n-2}) and $(0, t^2)$ in $\Omega \prod \Omega$. Let

$$U_l = \{0\} \oplus (R + t^{2l+1}\Omega), \quad 0 \leq l \leq m,$$

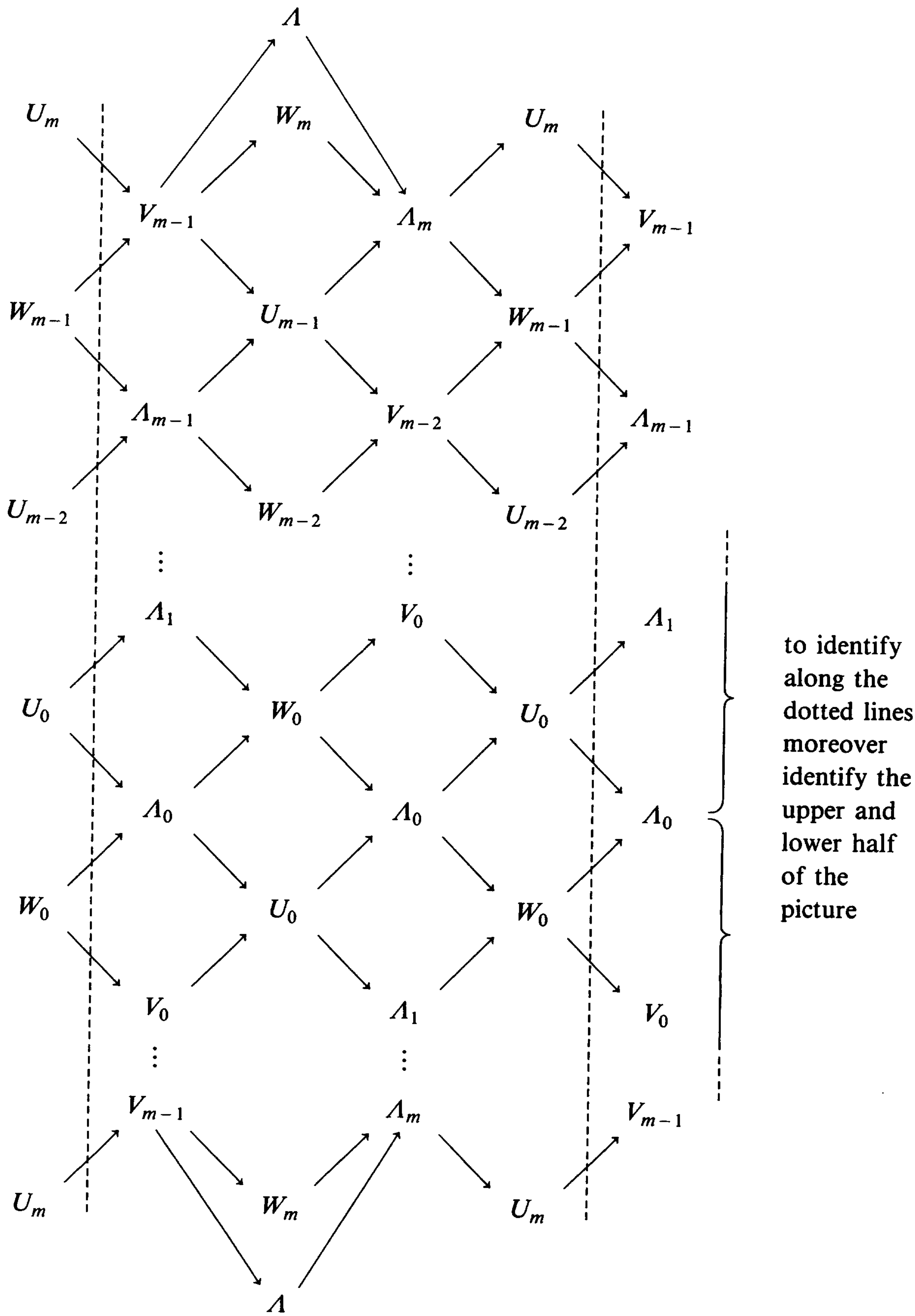
$$\Lambda_l = \{(f,g) \in \Omega^{(2)} \mid f - g \in t\Omega \text{ and } g \in U_l\}, \quad 0 \leq l \leq m-1,$$

$$V_l = \{(f,g) \in \Lambda \mid g \in t^{2(m-l)}\Omega\}, \quad 0 \leq l \leq m-1,$$

$$W_l = \tau U_l, \quad 0 \leq l \leq m.$$

$\mathfrak{A}(\Lambda)$ is shown in Diagram 2.

$\mathfrak{A}(\Lambda)$:



$\mathfrak{A}(\Lambda)_s = \mathbb{Z}\mathbb{A}_{4m+3}/(\varphi\tau)^{\mathbb{Z}}, \quad \varphi^2 = \mathrm{Id}.$

Diagram 2.

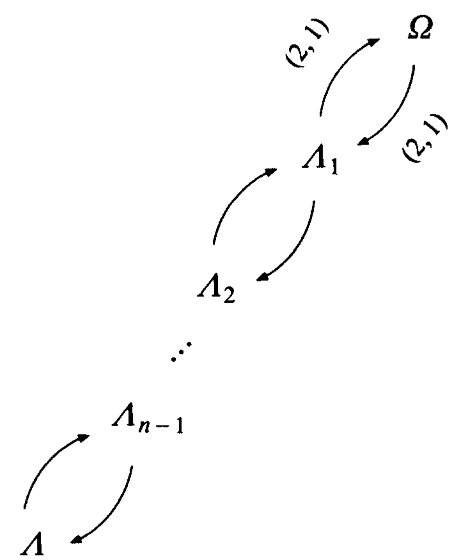
Λ is the local ring of the singularity of type $\mathbb{D}_n$. $m=0$ will be handled together with case (8).

Case (3). This will be a special case of the orders we consider in case (9) and will be handled there.

Case (4). $R=\mathbb{R}[\![t]\!]$, maximal order $\Omega=\mathbb{C}[\![t]\!]$, $\Lambda=\Lambda_n=R+t^n\Omega$, $n\geq 1$.

Let $\Lambda_i=R+t^i\Omega$, $1\leq i\leq n$. $\mathfrak{A}(\Lambda)$ is shown in Diagram 3.

$\mathfrak{A}(\Lambda)$:



$\mathfrak{A}(\Lambda)_s=\mathbb{Z}\mathbb{B}_n/\tau^{\mathbb{Z}}$ for $n\geq 2$.

Diagram 3.

Λ is a Bäckström order with associated graph $\mathbb{B}_2$ in case $n=1$.

Case (5). $R=\mathbb{R}[\![t^2]\!]$, let $\Omega_1=\mathbb{R}[\![t]\!]$, $\Omega_2=\mathbb{C}[\![t]\!]$. As usual let $\mathbb{C}=\mathbb{R}(\mathrm{i})$, $\mathrm{i}^2=-1$. Maximal order $\Omega_1\prod\Omega_2$,

$$\Lambda=\{(a_0+a_1t+\Omega_1t^2,a_0+b_1t+\cdots+(b_{n-1}+\mathrm{i}a_1)t^{n-1}+t^n\Omega_2)$$
$$\in\Omega_1\prod\Omega_2\,|\,a_0,a_1,b_1,\ldots,b_{n-1}\in\mathbb{R}\},\quad n\geq 2.$$

In a more suggestive way we write Λ as

$$\begin{bmatrix} \mathbb{R} = & \mathbb{R} \\ \mathbb{R}t & \mathbb{R}t \\ & \vdots \\ \Omega_1t^2 & (\mathbb{R}+\mathrm{i}\mathbb{R})t^{n-1} \\ ; & \Omega_2t^n \end{bmatrix}.$$

Similarly we define the following:

$$U_0=\{0\}\oplus\Omega_2,$$
$$U_l=\{0\}\oplus(\mathbb{R}[\![t]\!]+\Omega_2t^l),\quad 1\leq l\leq n-1,$$

$$V_l=\begin{bmatrix} \mathbb{R} & \mathbb{R}t^{n-1-l} \\ & \vdots \\ \Omega_1 t & (\mathbb{R}+\mathrm{i}\mathbb{R})t^{n-1} \\ ; & \Omega_2 t^n \end{bmatrix}, \quad 0\le l\le n-2,$$

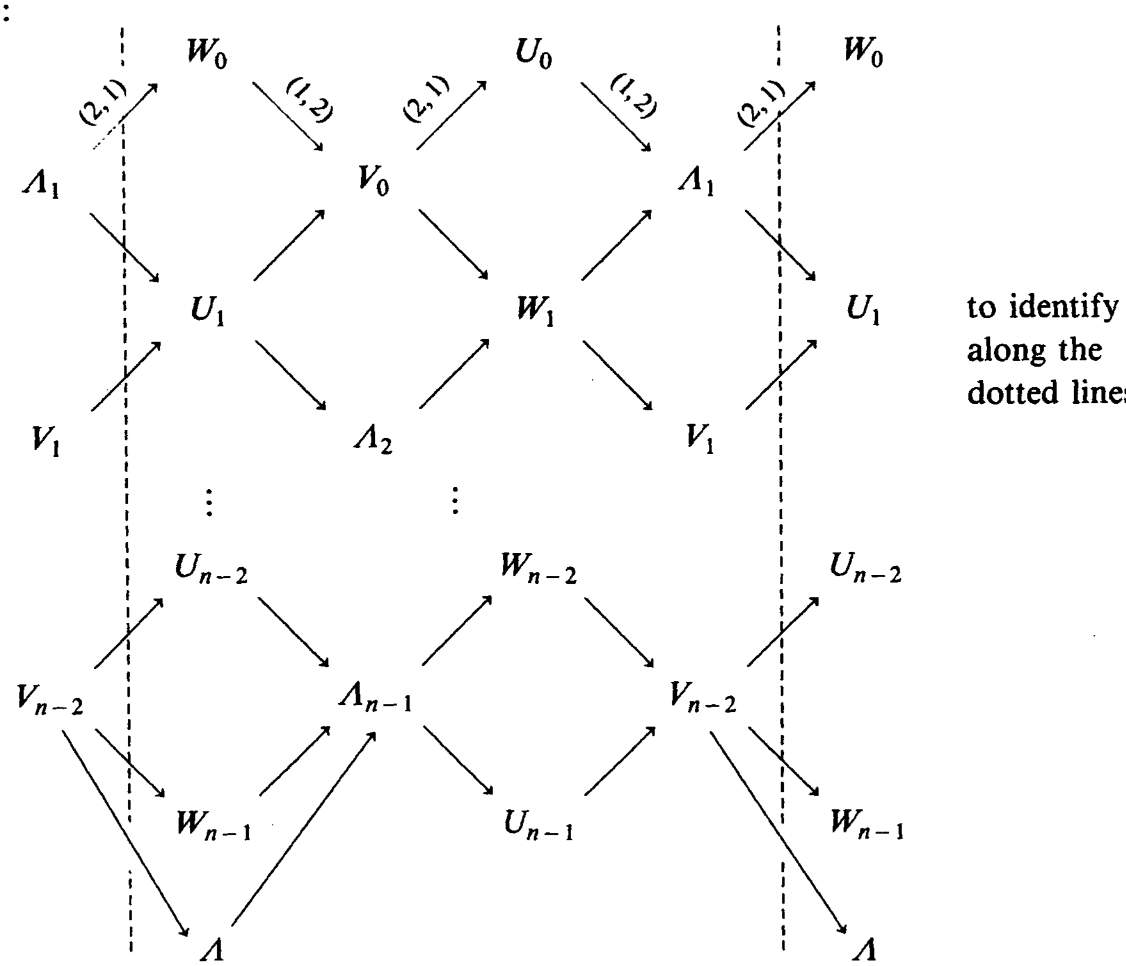
$$\Lambda_l=\begin{bmatrix} \mathbb{R}=\!=\mathbb{R} & \\ \Omega_1 t & \mathbb{R}t \\ & \vdots \\ ; & \Omega_2 t^l \end{bmatrix}, \quad 1\le l\le n-1,$$

$$W_{n-1}=\Omega_1\oplus\{0\},$$

$$W_l=\begin{bmatrix} \mathbb{R} & =\!=\!=\!= & \mathbb{R} \\ & \mathbb{R} & \mathbb{R}t \\ \Omega_1 t & & \vdots \\ & ,\ \Omega_1 t & (\mathbb{R}+\mathrm{i}\mathbb{R})t^l \\ & ; & \Omega_2 t^{l+1} \end{bmatrix}, \quad 0\le l\le n-2.$$

$\mathfrak{A}(\Lambda)$ is shown in Diagram 4.

$\mathfrak{A}(\Lambda)$:



$\mathfrak{A}(\Lambda)_s\cong\mathbb{Z}\mathbb{B}_{2n-1}/\tau^{2\mathbb{Z}}$.

Diagram 4.

Case (6). We view $\mathbb{C}$ as subring of $(\mathbb{R})_2$ by

$$\alpha + i\beta \mapsto \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \quad \text{for } \alpha, \beta \in \mathbb{R}.$$

$R = \mathbb{R}[\![t]\!]$, maximal order $\Omega = (R)_2$, $\Lambda = \mathbb{C}[\![t]\!] + \Omega t^n$, $n \geq 1$. Let $\Lambda_i = \mathbb{C}[\![t]\!] + \Omega t^i$. $\mathfrak{A}(\Lambda)$ is shown in Diagram 5.

$\mathfrak{A}(\Lambda)$:

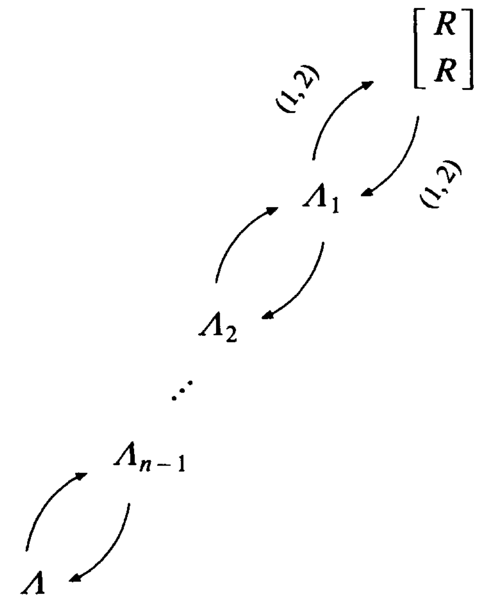


Diagram 5.

For $n = 1$, Λ is a Bäckström oder with associated graph $\mathbb{C}_2$,

$$\mathfrak{A}(\Lambda)_s \cong \mathbb{Z}\mathbb{C}_n / \tau^{\mathbb{Z}} \quad \text{for } n \geq 2.$$

Case (7). For the quaternions $\mathbb{H}$ we do the same construction as in case (5) for the reals $\mathbb{R}$: $R = \mathbb{R}[\![t^2]\!]$, let $\tilde{\Omega}_1 = \mathbb{H}[\![t]\!]$, $\tilde{\Omega}_2 = (\mathbb{C}[\![t]\!])_2$.

We view $\mathbb{H}$ as subring of $(\mathbb{C})_2$ of the form

$$\left\{ \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix} \middle| \alpha, \beta \in \mathbb{C}, \bar{x} = \text{complex conjugate of } x \right\}.$$

If $\tilde{i} = \left[\begin{smallmatrix} i & 0 \\ 0 & i \end{smallmatrix}\right]$, then $\mathbb{H}(\tilde{i}) = (\mathbb{C})_2$. Maximal order: $\tilde{\Omega}_1 \coprod \tilde{\Omega}_2$,

$$\Lambda = \{(a_0 + a_1 t + \tilde{\Omega}_1 t^2, a_0 + b_1 t + \cdots + (b_{n-1} + \tilde{i} a_1) t^{n-1} + \tilde{\Omega}_2 t^n)$$
$$\in \Omega_1 \coprod \Omega_2 \mid a_0, a_1, b_1, \ldots, b_{n-1} \in \mathbb{H}\}, \quad n \geq 2.$$

The lattices are similar to the lattices in case (5). However the lattices U_0 and W_0 yield under this translation lattices which decompose into two isomorphic indecomposables. This causes the reversion of the valuations. The Auslander–Reiten quiver of Λ is similar to the Auslander–Reiten quiver in case (5), and

$$\mathfrak{A}(\Lambda)_s \cong \mathbb{Z}\mathbb{C}_{2n-1} / \tau^{\mathbb{Z}}.$$

Case (8). $R = \mathbb{C}[\![t]\!]$, maximal order $R^{(2)}$. Put $n = 2m$ for $m \geq 1$.

$$\Lambda = R \overset{m}{\equiv\equiv} R = \{(f, g) \in R^{(2)} \mid f - g \in Rt^m\}.$$

$\mathfrak{A}(\Lambda)$ is shown in Diagram 6.

$\mathfrak{A}(\Lambda)$:

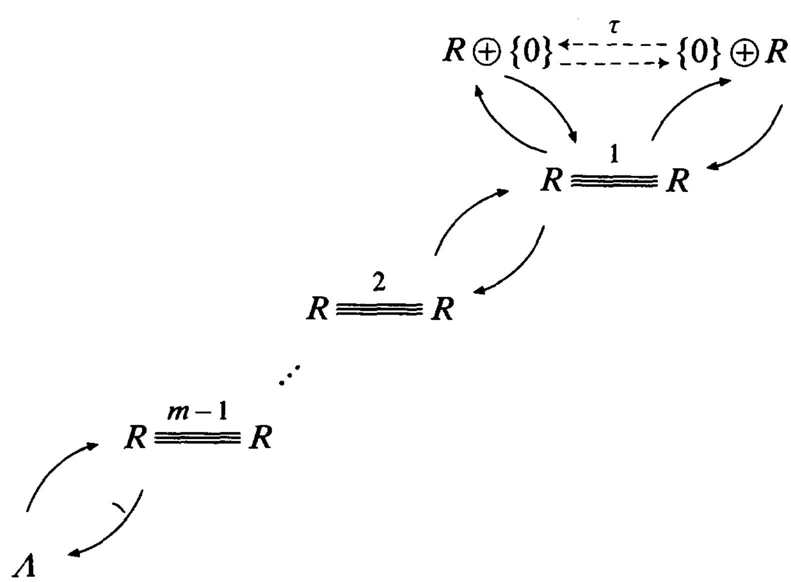


Diagram 6.

$$\mathfrak{A}(\Lambda)_s \cong \begin{cases} \mathbb{Z}\mathbb{A}_1/\tau^{2\mathbb{Z}} & \text{for } m = 1, \\ \mathbb{Z}\mathbb{A}_3/(\varphi\tau)^{\mathbb{Z}}, \varphi^2 = \mathrm{Id} & \text{for } m = 2, \\ \mathbb{Z}\mathbb{D}_{m+1}/(\varrho\tau)^{\mathbb{Z}}, \varrho^2 = \mathrm{Id} & \text{for } m \geq 3. \end{cases}$$

Λ is the local ring of the singularity of type $\mathbb{A}_{n-1}$. Moreover Λ is a Bäckström order with associated graph $\mathbb{A}_3$ for $m = 1$, and $m = 2$ covers the remaining case of case (2).

Case (9). $R = \mathbb{C}[\![t]\!]$, maximal order: $(R)_2$.

For $m, l \geq 0$, we put

$$\Lambda_{m,l} = \left\{ \begin{bmatrix} f & g \\ ht & k \end{bmatrix} \in (R)_2 \,\middle|\, f - k \in Rt^m, g - h \in Rt^l \right\}.$$

Then $\Lambda_{m,l}$ is a ring if and only if $l = m$ or $l = m - 1$. Moreover $\Lambda_{m,m}$ and $\Lambda_{m,m-1}$ are Bass orders [10, 19] with minimal overorder $\Lambda_{m,m-1}$ and $\Lambda_{m-1,m-1}$ respectively. $\mathfrak{A}(\Lambda_{m,m})$ and $\mathfrak{A}(\Lambda_{m,m-1})$ are shown in Diagram 7.

$\mathfrak{A}(\Lambda_{m,m})$:

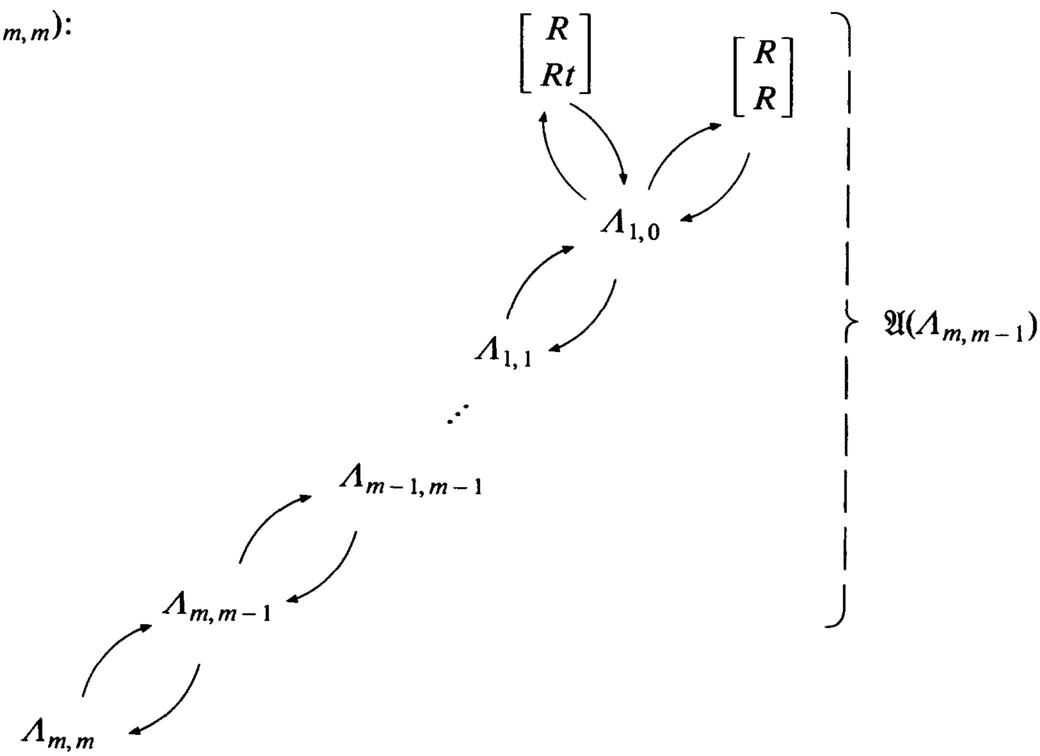


Diagram 7.

Note. $\Lambda_{0,0}$ is hereditary, $\Lambda_{1,0}$ is a Bäckström order with associated graph $\mathbb{A}_3$, and $\Lambda_{1,1}$ covers case (3).

For $m \geq 2$,

$$\mathfrak{A}(\Lambda_{m,m})_s \cong \mathbb{Z}\mathbb{D}_{2m+1}/\tau^{\mathbb{Z}} \quad \text{and} \quad \mathfrak{A}(\Lambda_{m,m-1})_s \cong \mathbb{Z}\mathbb{D}_{2m}/\tau^{\mathbb{Z}}.$$

Case (10). $R = \mathbb{C}[\![t]\!]$, maximal order $R^{(3)}$.

Let $m \geq 1$ and $\Lambda \subset R^{(3)}$ be generated as ring by the elements $(1,1,1)$, $(t,0,t^m)$ and $(0,t,t)$ in $R^{(3)}$.

We define the following:

$$U_l = \{(0,f,g) \in R^{(3)} \mid f-g \in Rt^l\}, \quad 1 \leq l \leq m,$$

$$W_l = \tau U_l, \quad 1 \leq l \leq m,$$

$$V_l = \{(f,g,h) \in \Lambda \mid g \in Rt^{m-l}\}, \quad 0 \leq l \leq m-1,$$

$$\Lambda_l = \{(f,g,h) \in R^{(3)} \mid f-g \in Rt \text{ and } g-h \in Rt^l\}, \quad 1 \leq l \leq m,$$

$$X_1 = \{(f,g,0) \in R^{(3)} \mid f-g \in Rt\},$$

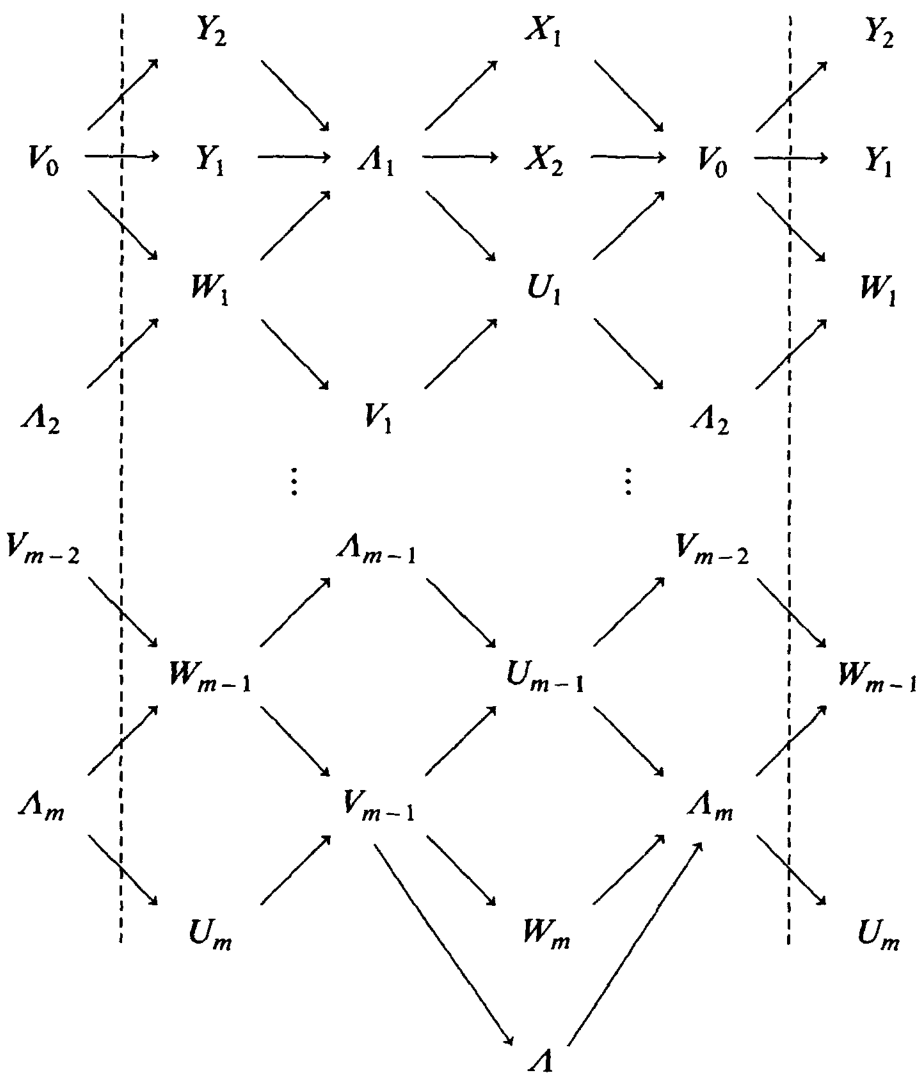
$$X_2 = \{(f,0,g) \in R^{(3)} \mid f-g \in Rt\},$$

$$Y_1 = \{0\} \oplus R \oplus \{0\},$$

$$Y_2 = \{(0,0)\} \oplus R.$$

$\mathfrak{A}(\Lambda)$ is shown in Diagram 8.

$\mathfrak{A}(\Lambda)$:



$\mathfrak{A}(\Lambda)_s \cong \mathbb{Z}\mathbb{D}_{2m+2}/\tau^{2\mathbb{Z}}$

Diagram 8.

Λ is the local ring of the singularity of type $\mathbb{D}_{2m+2}$. Note the similarity between Λ and the group ring $\mathbb{Z}_p C_{p^2}$ of the cyclic group of order p^2 over the p-adics [24].

Case (11). $R = \mathbb{C}[\![t^2]\!]$, $\Omega = \mathbb{C}[\![t]\!]$, maximal order $\Omega \prod (\Omega)_2$. Let $m \geq 1$ and let

$$\Lambda = \left\{ \left(a_0 + a_1 t + \Omega t^2, \right.\right.$$

$$\left.\left. \begin{bmatrix} a_0 + b_1 t + \cdots + b_{m-1} t^{m-1} + \Omega t^m & c_0 + c_1 t + \cdots + c_{m-1} t^{m-1} + \Omega t^m \\ t(c_0 + c_1 t + \cdots + (a_1 + c_{m-1}) t^{m-1}) + \Omega t^{m+1} & a_0 + b_1 t + \cdots + b_{m-1} t^{m-1} + \Omega t^m \end{bmatrix} \right) \right.$$

$$\left. \middle| \, a_0, a_1, b_1, \ldots, b_{m-1}, c_0, \ldots, c_{m-1} \in \mathbb{C} \right\}$$

In a more suggestive way we write

$$\Lambda=\begin{bmatrix} \mathbb{C} & \left[\begin{matrix} \Omega & \overset{m}{\times} & \Omega \\ & & \\ \Omega t & & \Omega \end{matrix}\right] \\ \mathbb{C}t,\ \overset{m}{\cdots} & \\ \Omega t^2; & \end{bmatrix}.$$

Similarly we describe the following indecomposable lattices for $m \geq 1$ and $l=m$ or $l=m-1$.

$$\Lambda_{m,l}=\begin{bmatrix} \Omega & \overset{m}{\times} & \Omega \\ & {\scriptstyle l} & \\ \Omega t & & \Omega \end{bmatrix},$$

$$W_{m,l}=\begin{bmatrix} \mathbb{C} & \left[\begin{matrix} \Omega & \overset{m}{\times} & \Omega \\ & {\scriptstyle l} & \\ \Omega t & & \Omega \end{matrix}\right] \\ \Omega t,\ \mathbb{C}\overset{l}{\cdots} & \\ \Omega t; & \end{bmatrix},$$

$$U_{l,m}=\begin{bmatrix} \mathbb{C}\overset{m}{\cdots} & \left[\begin{matrix} t\Omega & \overset{l}{\times} & \Omega \\ & {\scriptstyle m} & \\ t\Omega & & \Omega \end{matrix}\right] \\ \Omega t; & \end{bmatrix},$$

$$\Lambda'_{m,l}=\begin{bmatrix} \mathbb{C} & \left[\begin{matrix} \Omega & \overset{m}{\times} & \Omega \\ & {\scriptstyle l} & \\ \Omega t & & \Omega \end{matrix}\right] \\ \Omega t; & \end{bmatrix}.$$

$\mathfrak{A}(\Lambda)$ is shown in Diagram 9.

$$\mathfrak{A}(\Lambda)_s \cong \mathbb{Z}\mathbb{D}_{4m}/(\varrho\tau^2)^{\mathbb{Z}}, \quad \varrho^2=\mathrm{Id}.$$

If we take for $m \geq 2$

$$\Lambda=\begin{bmatrix} \mathbb{C} & \left[\begin{matrix} \Omega & \overset{m}{\times} & \Omega \\ & {\scriptstyle m-1} & \\ \Omega t & & \Omega \end{matrix}\right] \\ \mathbb{C}t\overset{m-1}{\cdots} & \\ \Omega t^2; & \end{bmatrix}$$

we similarly get

$$\mathfrak{A}(\Lambda)_s \cong \mathbb{Z}\mathbb{D}_{4m-2}/(\varrho\tau^2)^{\mathbb{Z}}, \quad \varrho^2=\mathrm{Id}.$$

Case (12). $R=\mathbb{C}[\![t]\!]$, maximal order $\Omega=(R)_3$,

$$\Lambda=\left\{\begin{bmatrix} a+Rt & b+Rt & R \\ Rt & a+Rt & c+Rt \\ -(b+c)t+Rt^2 & Rt & a+Rt \end{bmatrix} \;\middle|\; a,b,c \in \mathbb{C}\right\}$$

$$=\begin{bmatrix} R & R & R \\ Rt & R & R \\ Rt & Rt & R \end{bmatrix}.$$

$\mathfrak{A}(\Lambda)$ is shown in Diagram 10.

$\mathfrak{A}(\Lambda)$:

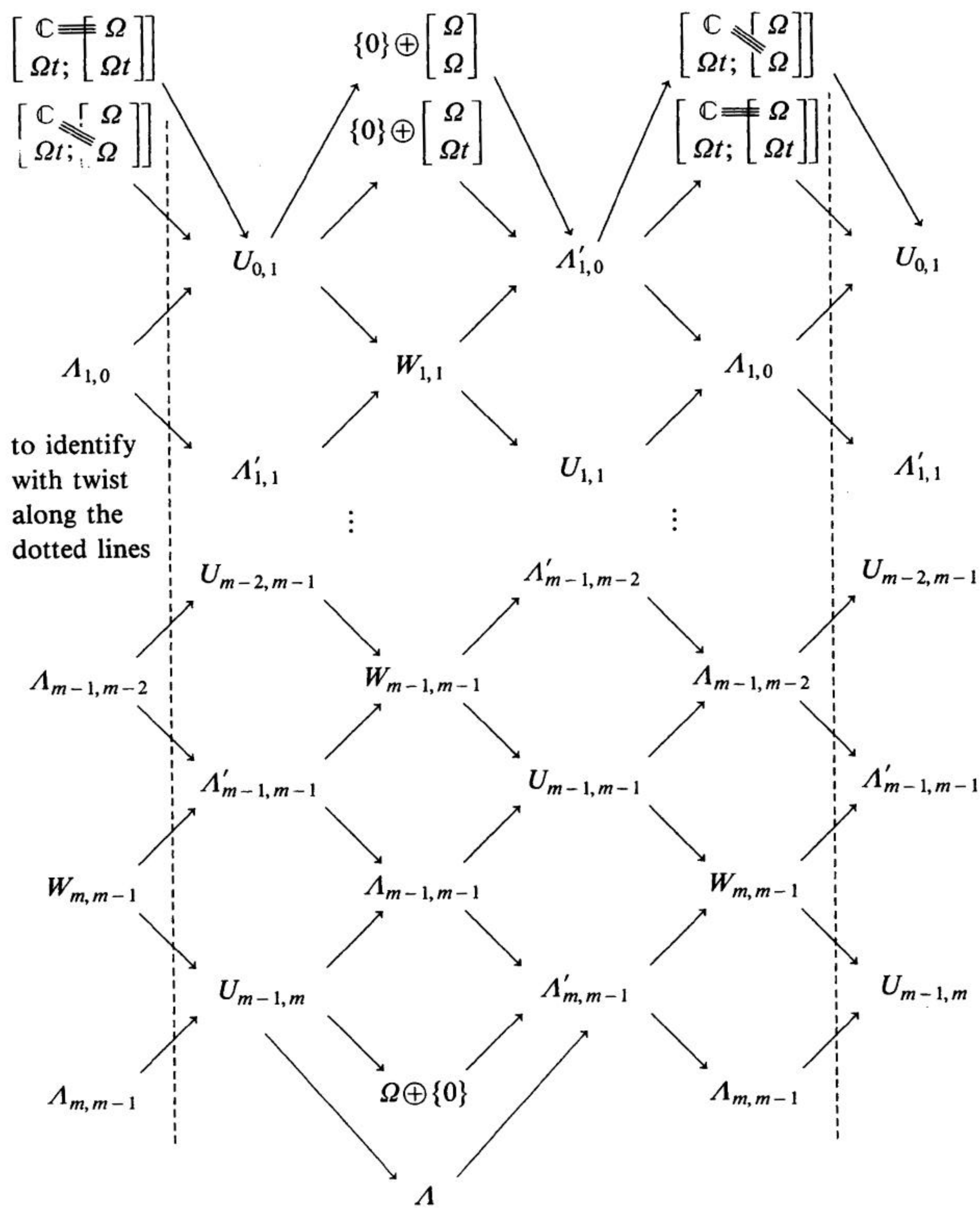
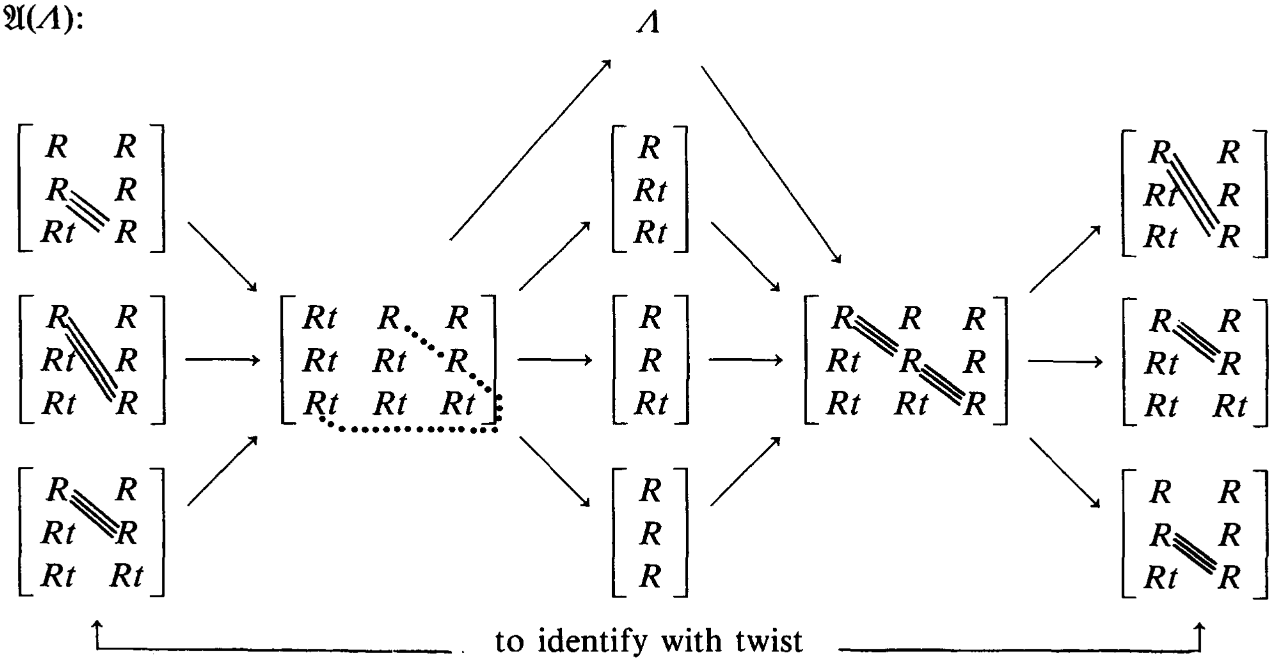


Diagram 9.

$\mathfrak{A}(\Lambda)$:



$\mathfrak{A}(\Lambda)_s \cong \mathbb{Z}\mathbb{D}_4/(\sigma\tau^2)^{\mathbb{Z}}, \quad \sigma^3 = \mathrm{Id}.$

Diagram 10.

In the cases (13), (14) and (15) one can take as Λ the local ring of the singularity of type $\mathbb{E}_6$, $\mathbb{E}_7$ and $\mathbb{E}_8$ respectively. The Auslander-Reiten quivers are given in [9] explicitly. Moreover the indecomposable lattices are described in [13, 14], and it is left to the reader to find their positions in the Auslander-Reiten quiver.

For the last two cases (16) and (17) let $\mathfrak{k}$ be a field with an extension field $\mathfrak{f}$ of degree $|\mathfrak{f}:\mathfrak{k}| = 3$.

Case (16). $R = \mathfrak{k}[\![t]\!]$, maximal order $\Omega = (R)_3$.

The isomorphism $\mathfrak{f} \cong \mathfrak{k}^{(3)}$ as $\mathfrak{k}$-spaces induces a representation of $\mathfrak{f}$ on $\mathfrak{k}^{(3)}$ and therefore an inclusion $\mathfrak{f} \hookrightarrow (\mathfrak{k})_3$. Thus we view $\mathfrak{f}$ as subring of $(\mathfrak{k})_3$. Moreover let $x \in (\mathfrak{k})_3 \setminus \mathfrak{f}$. We define

$$\Lambda = \mathfrak{f} + (\mathfrak{f} + x\mathfrak{f})t + \Omega t^2.$$

$\mathfrak{A}(\Lambda)$ is shown in Diagram 11.

$\mathfrak{A}(\Lambda)$:

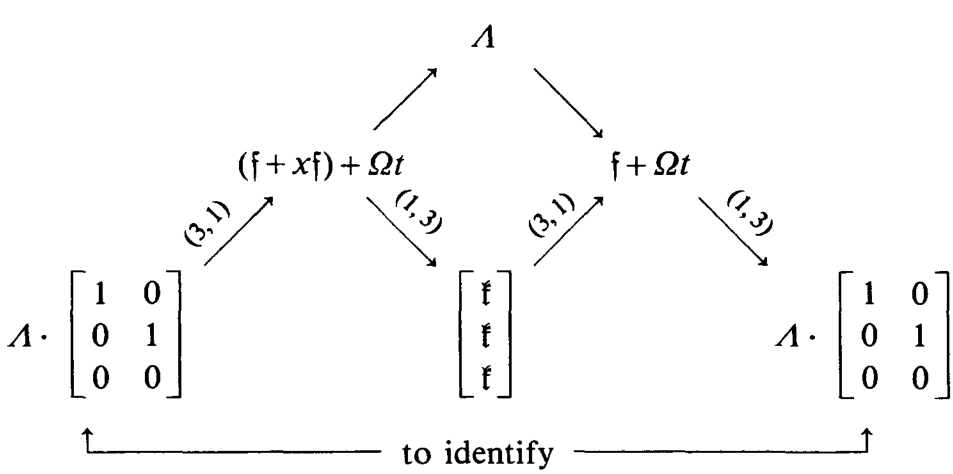


Diagram 11.

Case (17). $R=\mathfrak{k}[\![t]\!]$, maximal order $\Omega=\mathfrak{f}[\![t]\!]$. Let $y\in\mathfrak{f}\setminus\mathfrak{k}$, and let

$$\Lambda=\mathfrak{k}+(\mathfrak{k}+y\mathfrak{k})t+\Omega t^2.$$

$\mathfrak{A}(\Lambda)$ is shown in Diagram 12.

$\mathfrak{A}(\Lambda)$:

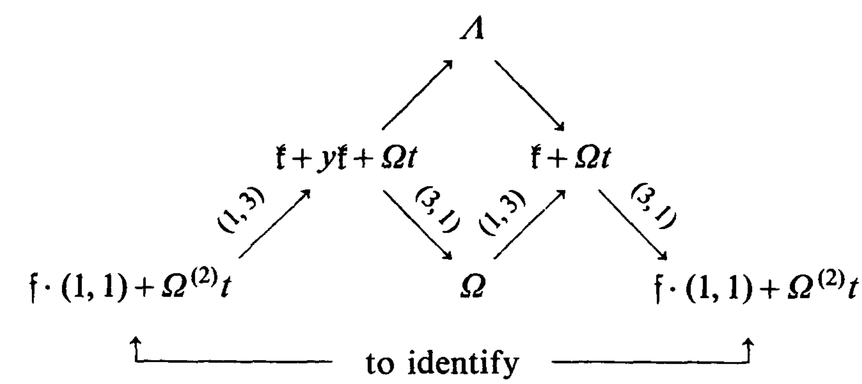


Diagram 12.

In the cases (16) and (17)

$$\mathfrak{A}(\Lambda)_s\cong\mathbb{Z}\mathbb{G}_2/\tau^{2\mathbb{Z}}.$$

Finally let us mention that, if $K\Lambda=\prod_{i=1}^{s}(D_i)_{n_i}$ as in the introduction, then if all n_i are 1 and $\mathfrak{A}(\Lambda)$ has only trivial valuations, $\mathfrak{A}(\Lambda)$ occurs as Auslander-Reiten quiver of a simple curve singularity.

This shows the theorem in the introduction.

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