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Xor-Implications and E-Implications: Classes of Fuzzy Implications Based on Fuzzy Xor

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Abstract

The main contribution of this paper is to introduce an autonomous definition of the connective "fuzzy exclusive or" (fuzzy Xor, for short), which is independent from others connectives. Also, two canonical definitions of the connective Xor are obtained from the composition of fuzzy connectives, and based on the commutative and associative properties related to the notions of triangular norms, triangular conorms and fuzzy negations. We show that the main properties of the classical connective Xor are preserved by the connective fuzzy Xor, and, therefore, this new definition of the connective fuzzy Xor extends the related classical approach. The definitions of fuzzy Xor-implications and fuzzy E-implications, induced by the fuzzy Xor connective, are also studied, and their main properties are analyzed. The relationships between the fuzzy Xor-implications and the fuzzy E-implications with automorphisms are explored.

Keywords: Fuzzy Logic, Fuzzy Xor, Xor-implication, E-implication, Automorphism

1 Introduction

The connective *exclusive or* (Xor, for short) plays an important role in computer programming. For example, it is used as a primitive operation in many encryption algorithm (e.g, DES, blowfish, RC5, CAST, RIJNDAEL) [29,47]. The *one-time pad* [43] is an encryption algorithm where the plaintext is combined with a random

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key (called pad) by a modular addition, or the operation XOR, when binary data are considered.

Several other applications of the Xor connective can be found, such as in simple threshold activated neural networks [39], in the identification of elemental emission spectra [15], in algorithms to eliminate cache conflict misses [50], in the construction of conflict-free hash functions [49], in techniques to exploit the parallelism in IP routers [11], etc. Also, the boolean web search logic capability may be improved with the Xor operator, in order to consider the search of mutually exclusive sites. Moreover, due to its non-linearity, the connective Xor is frequently used as a problem, e.g., in Neural Networks [24], in support vector machines (SVM) [25] and Quantum Computing [31,35].

Different versions of the *fuzzy Xor connective* have been used in the literature. In [34], a fuzzy Xor operation $\oplus$, defined as $x \oplus y = x + y - 2xy$, is used to identify preference rules from interactions in the linear model. In [39], a generalized Xor operation is given as a family of fuzzy Xor operations, based on a composition of the fuzzy negation, and triangular conorms and norms (t-conorms and t-norms, for short). In [6,28], three distinct definitions of the fuzzy Xor are considered, in order to introduce a semantics of interval fuzzy logics related to the checklist paradigm:

$$x \oplus_{\perp} y = max(x - y, y - x), \tag{1}$$

$$x \oplus_{\top} y = min(2 - x - y, x + y), \tag{2}$$

$$x \oplus_{mid} y = (1 - x)y + x(1 - y). \tag{3}$$

In the literature, special attention has been given to the research on the validity of many classical logic tautologies in fuzzy logic, especially those that are related to *fuzzy implications*. Fuzzy implications have been widely studied, playing important roles in different domains [18,19,46,48,51,52,54,55]. Recent papers studied different classes of fuzzy implications [2,4,5,12,18,44,53].

Several properties of classical implications can be generalized from the multivalued implications: [4]

(i) R-implication, related to a residuation concept from the intuitionistic logic [21,22];

(ii) S-implication, arising from the notion of disjunction and negation of classical logic, and generated by a t-conorm and a fuzzy negation [4];

(iii) QL-implication, which may be generated from a left-continuous t-norm, a t-conorm, joint with a strong fuzzy negation [44];

(iii) D-implication, whose definition is obtained as a contraposition concerning a fuzzy negation of a QL-implication [32,33];

(iv) Łukasiewicz implication, whose definition is based on the Łukasiewics t-norm [1].

In this paper, we introduce an autonomous definition of the fuzzy Xor connective, which is independent of the other connectives, generalizing the previous definitions

[4] See also [3,5,53], for other different definitions of fuzzy implications.

referred above (equations (1), (2) and (3)).

We also provide two canonical constructions based on the composition of other fuzzy connectives. In particular, one of them constitutes a generalization of the fuzzy Xor connective introduced in [39], by considering arbitrary fuzzy negations.

Since the main properties of the classical Xor connective are preserved, this new definition of the fuzzy Xor connective extends the related classical approach.

We use this definition of the fuzzy Xor connective to construct two new classes of fuzzy implications, namely *E-implications* and *Xor-implications*, analyzing their main properties and their relationship with automorphisms.

The results can be applied in soft computing, which deals with the design of flexible information processing systems [36], with applications in control systems [14], decision making [13], expert systems [45], pattern recognition [7,36], etc.

This paper is organized as follows. In Sect. 2, we review the main concepts related to ordinary fuzzy connectives. Fuzzy t-conorms (and t-norms), negations and implications are presented in the subsections 2.1, 2.2 and 2.3, respectively. The fuzzy Xor connective and corresponding properties are considered in Sect. 3. In Subsect. 3.1, the canonical definition of the fuzzy Xor operator is introduced. The fuzzy Xor implications and E-implications are defined in Sect. 4, where we also show how to construct an E-implication as a composition of a t-norm, a t-conorm, a negation and an Xor operator. Automorphisms are presented in Sect. 5, where it is shown that the action of the generalization of an automorphism introduced in [20,21] preserves the generalization of an E-implication. Section 6 is the Conclusion.

2 Usual Fuzzy Connectives

In this section, basic definitions related to the fuzzy connectives t-norm, t-conorm and fuzzy negation are considered.

2.1 T-norms and T-conorms

Let $U = [0, 1]$ be the unitary interval. A *t-norm* is a function $T : U^2 \to U$ satisfying, for all $x, y, z \in U$, the following properties:

- T1: $T(x, y) = T(y, x)$ (commutativity);
- T2: $T(x, T(y, z)) = T(T(x, y), z)$ (associativity);
- T3: If $y \leq z$ then $T(x, y) \leq T(x, z))$ (monotonicity);
- T4: $T(x, 1) = x$ (boundary condition).

A *t-conorm* is a function $S : U^2 \to U$ satisfying, for all $x, y, z \in U$, the following properties:

- S1: $S(x, y) = S(y, x)$ (commutativity);
- S2: $S(x, S(y, z)) = S(S(x, y), z)$ (associativity);
- S3: If $y \leq z$ then $S(x, y) \leq S(x, z)$ (monotonicity);
- S4: $S(x, 0) = 0$ (boundary condition).

2.2 Fuzzy Negation

A function $N : U \to U$ is a *fuzzy negation* if

- N1: $N(0) = 1$ and $N(1) = 0$;
- N2: If $x \geq y$ then $N(x) \leq N(y)$, $\forall x, y \in U$.

Fuzzy negations satisfying the involutive property are called *strong fuzzy negations* [12,27]:

- N3: $N(N(x)) = x$, $\forall x \in U$.

2.3 Fuzzy Implications

Several definitions for fuzzy implications together with related properties have been given in the literature (see, e.g., [2,5,12,17,18,23,30,42,51,52,53]). The unique consensus in these definitions is that the fuzzy implication should have the same behavior as the classical implication for the crisp case. Thus, a binary function $I : U^2 \to U$ is a *fuzzy implication* if I satisfies the minimal boundary conditions:

$$I(1,1) = I(0,1) = I(0,0) = 1 \text{ and } I(1,0) = 0. \tag{4}$$

Several reasonable properties may be required for fuzzy implications. The properties considered in this paper are listed below:

I1: If $x \leq z$ then $I(x,y) \geq I(z,y)$ (first place antitonicity);

I2: If $y \leq z$ then $I(x,y) \leq I(x,z)$ (second place isotonicity);

I3: $I(1,x) = x$ (left neutrality principle);

I4: $I(x, I(y,z)) = I(y, I(x,z))$ (exchange principle);

I5: $I(x,y) = I(x, I(x,y))$ (iterative boolean-like law);

I6: $I(x, N(x)) = N(x)$, and N is a strong fuzzy negation;

I7: $N(x) = I(x,0)$ is a strong fuzzy negation;

I8: $I(x,1) = 1$;

I9: $I(x,y) \geq y$;

I10: $I(x,y) = I(N(y), N(x))$, and N is a strong fuzzy negation (contra-positive);

I11: $I(0,x) = 1$, dominance falsity.

3 The Connective Fuzzy Xor

In fuzzy logic, one can use the Xor connective in order to evaluate the degree with which one and only one of its immediate antecedents is true. Unfortunately, in the literature, there is no autonomous definitions for that fuzzy connective, in the sense that it would be independent from the other connectives. In the following, we introduce a definition for the fuzzy Xor connective satisfying this condition of independence.

Definition 3.1 A function $E : U^2 \rightarrow U$ is a *fuzzy Xor* if it satisfies the properties:

- E1: $E(x, y) = E(y, x)$ (symmetry);
- E2: $E(x, E(y, z)) = E(E(x, y), z)$ (associativity);
- E3: $E(0, x) = x$ (0-Identity);
- E4: $E(1, 1) = 0$ (boundary condition).

Example 3.2 The fuzzy Xor connective introduced in [34], defined by $x \oplus y = x + y - 2xy$, trivially satisfies the properties E1, E3 and E4. It also satisfies the property E2, since:

$$\begin{aligned} x \oplus (y \oplus z) &= x + (y \oplus z) - 2x(y \oplus z) \\ &= x + (y + z - 2yz) - 2x(y + z - 2yz) \\ &= x + y + z - 2yz - 2xy - 2xz + 4xyz \\ &= x + y - 2xy + z - 2xz - 2yz + 4xyz \\ &= x + y - 2xy + z - 2z(x + y - 2xy) \\ &= (x \oplus y) \oplus z. \end{aligned}$$

It follows from properties E3 and E4 that the fuzzy Xor connective generalizes the classical Xor connective.

Based on Definition 3.1, the following extra reasonable properties can be considered for the fuzzy Xor connective:

- E5: $E(x, x) = 0$
- E6: $E(E(x, y), x) = y$
- E7: If $x \leq y \leq z$ then $E(x, y) \leq E(x, z)$ and $E(y, z) \geq E(x, z)$
- E8: $N_E(x) = E(x, 1)$ is a strong fuzzy negation.
- E9: If $E(x, y) = 0$ then $x = y$.
- E10: If $E(x, y) = 1$ then $|x - y| = 1$.
- E11: $E(N_E(x), x) = 1$.
- E12: E is continuous.

These properties are not necessarily primitive, that is, some of them can be obtained from some other properties.

Proposition 3.3 *Let $E : U^2 \rightarrow U$ be a fuzzy Xor connective. Then, the following relations hold:*

E5 $\Rightarrow$ E6: If E satisfies E5 then E satisfies E6.

E7 $\Rightarrow$ E8: If E satisfies E7 then E satisfies E8.

E6 $\Rightarrow$ E11: If E satisfy E6 then E satisfies E11.

Proof. Let $E : U^2 \rightarrow U$ be a fuzzy Xor connective. Then it follows that:

E5 $\Rightarrow$ E6: $E(E(x, y), x) = E(x, E(x, y)) = E(E(x, x), y)) = E(0, y) = y$.

E7 $\Rightarrow$ E8: The function $N_E : U \to U$, given by $N_E(x) = E(x, 1)$, satisfies the properties:

N1: $N_E(0) = E(0, 1) = 1$ and $N_E(1) = E(1, 1) = 0$.

N2: If $x \leq y$, then, by the property E7, $N_E(x) = E(x, 1) \geq E(y, 1) = N_E(y)$.

N3: One has that $N_E(N_E(x)) = N_E(E(x, 1)) = E(E(x, 1), 1)$. Based on the associativity (E2) and boundary condition (E4) properties in Definition 3.1, it holds that $E(E(x, 1), 1) = E(x, E(1, 1)) = E(x, 0) = x$, and, thus, $N_E(N_E(x)) = x$.

Therefore, $N_E(x) = E(x, 1)$ is a strong fuzzy negation.

E6 $\Rightarrow$ E11: If $E(E(x, y), x) = y$ then $E(N_E(x), x) = E(E(x, 1), x) = 1$.

□

3.1 Obtaining the Xor Connective from Other Connectives

Proposition 3.4 *Let T, S and N be a t-norm, a t-conorm and a fuzzy negation, respectively. A fuzzy Xor connective can be given by the function $E_T : U^2 \to U$, defined by:*

$$E_T(x, y) = T(S(x, y), N(T(x, y))). \tag{5}$$

Proof. The function $E_T : U^2 \to U$, given by $E_T(x, y) = T(S(x, y), N(T(x, y)))$, satisfies the properties in Definition 3.1:

E1: Based on the commutativity of T and S, it follows that:

$$E_T(x, y) = T(S(x, y), N(T(x, y))) = T(S(y, x), N(T(y, x))) = E_T(y, x).$$

E2: Based on the associativity of T and S, it follows that:

$$\begin{aligned}
&E_T(x, E_T(y, z))\\
&= E_T(x, T(S(y, z), N(T(y, z))))\\
&= T(S(x, T(S(y, z), N(T(y, z)))), N(T(x, T(S(y, z), N(T(y, z))))))\\
&= T(S(T(S(x, y), N(T(x, y), z))), N(T(S(x, y), N(T(x, y))), z))\\
&= E_T(T(S(x, y), N(T(x, y))), z)\\
&= E_T(E_T(x, y), z).
\end{aligned}$$

E3: Considering the boundary conditions in the definitions of T, N and S, it follows that:

$$E_T(0, x) = T(S(0, x), N(T(0, x))) = T(x, N(0)) = T(x, 1) = x.$$

E4: The same conditions in the definitions of T, N and S assure that

$$E_T(1, 1) = T(S(1, 1), N(T(1, 1))) = T(1, N(1)) = T(1, 0) = 0.$$

□

Example 3.5 Consider the Łukasiewicz t-norm, defined by $T_L(x, y) = max(x + y - 1, 0)$, the t-conorm (or bounded sum), defined by $S_L(x, y) = min(x + y, 1)$, and the fuzzy negation, given by $N(x) = 1 - x$ (see [44]). The fuzzy Xor operator E_{T_L}, canonically obtained as in Eq. (5), can be expressed as:

$$E_{T_L}(x,y) = max(min(x+y,1) + (1 - max(x+y-1,0)) - 1, 0)$$
$$= \begin{cases} x+y & \text{if } x+y \leq 1 \\ 2-(x+y) & \text{if } x+y > 1 \end{cases}$$

Considering the properties discussed in this paper, this operator only verifies two of them:

E8: It is straightforward, following from Proposition 3.8.

E11: $E_{T_L}(N_{E_{T_L}}(x), x) = E_{T_L}(1-x, x) = 1$.

Proposition 3.6 *Let T, S and N be a t-norm, a t-conorm and a fuzzy negation, respectively. A fuzzy Xor connective can be given by the function $E_S : U^2 \to U$, defined by:*

$$E_S(x,y) = S(T(N(x),y), T(x,N(y))). \quad (6)$$

Proof. The function $E_S : U^2 \to U$, given by $S(T(N(x),y), T(x,N(y)))$, satisfies the properties in Definition 3.1:

E1: E_S satisfies the commutativity property, that is,

$$E_S(x,y) = S(T(N(x),y), T(x,N(y))) = S(T(N(y),x), T(y,N(x))) = E_S(y,x)$$

which is a consequence of the commutative properties of T and S.

E2: Based on the associativity of T and S, it follows that

$$\begin{aligned} &E_S(E_S(x,y),z) \\ &= E_S(S(T(N(x),y),T(x,N(y))),z) \\ &= S(T(N(S(T(N(x),y),T(x,N(y)))),z),T(S(T(N(x),y),T(x,N(y))),N(z))) \\ &= S(T(N(x)),S(T(N(y),z),T(y,N(z))),T(x,N(S(T(N(y),z),T(y,N(z)))))) \\ &= E_S(x,S(T(N(y),z),T(y,N(z)))) \\ &= E_S(x,E_S(y,z)). \end{aligned}$$

E3: Considering the boundary conditions T4 and S4 in the definitions of T and S, respectively, it follows that

$$E_S(0,x) = S(T(N(0),x), T(0,N(x))) = S(T(1,x),0) = S(x,0) = x.$$

E4: Considering the boundary conditions T4 and S4 in definitions of T and S, respectively, one has that

$$E_S(1,1) = S(T(N(1),1), T(1,N(1))) = S(N(1),N(1)) = S(0,0) = 0.$$

□

Example 3.7 Consider the Łukasiewicz t-norm, defined by $T_L(x,y) = max(x+y-1,0)$, the t-conorm (or bounded sum), defined by $S_L(x,y) = min(x+y,1)$, and the fuzzy negation, given by $N(x) = 1 - x$ (see [44]). The fuzzy Xor operator E_{S_L}, canonically obtained as in Eq. (6), can be expressed as:

$$E_{S_L}(x, y) = |x - y|.$$

This operator satisfies all the properties from E5 to E12. The fuzzy Xor operator E_{S_L} coincides with the one introduced in [28] (Table 3), which was presented in the Introduction (see Eq. (1)).

Proposition 3.8 *Let T, S and N be a t-norm, a t-conorm and a fuzzy negation, respectively. Then it holds that $N_{E_T} = N_{E_S} = N$.*

Proof. It follows that: $N_{E_T}(x) = E_T(x, 1) = T(S(x, 1), N(T(x, 1))) = T(1, N(x)) = N(x)$ and $N_{E_S}(x) = S(T(N(x), 1), T(x, N(1))) = S(N(x), T(x, 0)) = S(N(x), 0) = N(x)$. □

4 Fuzzy Implications Induced by the Fuzzy Xor Connective

The fuzzy Xor connective allows us to define two new fuzzy implications, which are presented in the next subsections.

4.1 Xor-implications

Proposition 4.1 *Let S, N and E be a t-conorm, a fuzzy negation and a fuzzy Xor connective, respectively. Then, the function $I_{E,S,N} : U^2 \to U$, defined by*

$$I_{E,S,N}(x, y) = E(x, S(N(x), N(y))). \tag{7}$$

is a fuzzy implication, called a fuzzy Xor-implication.

Proof. It follows that $I_{E,S,N}$ satisfies the following properties:

$I_{E,S,N}(0, 0) = E(0, S(N(0), N(0))) = E(0, S(1, 1)) = E(0, 1) = 1;$
$I_{E,S,N}(0, 1) = E(0, S(N(0), N(1))) = E(0, S(1, 0)) = E(0, 1) = 1;$
$I_{E,S,N}(1, 1) = E(1, S(N(1), N(1))) = E(1, S(0, 0)) = E(1, 0) = 1;$
$I_{E,S,N}(1, 0) = E(1, S(N(1), N(0))) = E(1, S(0, 1)) = E(1, 1) = 0.$

Therefore, $I_{E,S,N}$ is a fuzzy implication. □

Proposition 4.2 *Let S be a t-conorm and E be a fuzzy Xor connective satisfying the property E6. Then the fuzzy implication I_{E,S,N_E} satisfies the properties I2 and I5.*

Proof. It follows that

I2: $I_{E,S,N_E}(1, x) = E(1, S(N_E(1), N_E(x))) = E(1, S(0, N_E(x))) = E(1, N_E(x)) = N_E(N_E(x)) = x.$

I5: $I_{E,S,N_E}(x,0) = E(x, S(N_E(x), N_E(0))) = E(x, S(N_E(x), 1)) = E(x,1)$, which, by property E6, is a strong fuzzy negation.

□

4.2 E-Implications

Proposition 4.3 *Let S, N and E be a t-conorm, a fuzzy negation and a fuzzy Xor connective, respectively. Then the function $I_{S,N,E} : U^2 \to U$, defined by*

$$I_{S,N,E}(x,y) = S(N(x), E(N(x),y)). \tag{8}$$

is a fuzzy implication, called a fuzzy E-implication.

Proof. It follows that $I_{S,N,E}$ satisfies the following properties:

$I_{S,N,E}(0,0) = S(N(0), E(N(0),0)) = S(1, E(1,0)) = S(1,1) = 1;$
$I_{S,N,E}(0,1) = S(N(0), E(N(0),1)) = S(1, E(1,1)) = S(1,0) = 1;$
$I_{S,N,E}(1,1) = S(N(1), E(N(1),1)) = S(0, E(0,1)) = S(0,1) = 1;$
$I_{S,N,E}(1,0) = S(N(1), E(N(1),0)) = S(0, E(0,0)) = S(0,0) = 0.$

Therefore, $I_{S,N,E}$ is a fuzzy implication. □

Proposition 4.4 *Let S be a t-conorm and E be a fuzzy Xor connective satisfying the property E6. Then the fuzzy implication $I_{S,N_E,E}$ satisfies the properties I2 and I6.*

Proof. It follows that:

I2: $I_{S,N_E,E}(1,x) = S(N_E(1), E(N_E(1),x)) = S(0, E(0,x)) = E(0,x) = x$.

I5: $I_{S,N_E,E}(x,0) = I_{S,N_E,E}(0,x) = S(N_E(0), E(N_E(0),x)) = S(1, E(1,x)) = E(x,1)$, which, by Property E6, is a strong fuzzy negation.

□

5 Automorphism

Definition 5.1 A function $\rho : U \to U$ is an *automorphism* if it is bijective and monotonic, that is: [26,37]

$$x \leq y \;\Rightarrow\; \rho(x) \leq \rho(y).$$

An equivalent definition is given in [12], where $\rho : U \to U$ is an automorphism if it is a continuous and strictly increasing function such that $\rho(0) = 0$ and $\rho(1) = 1$.

Denote by $Aut(U)$ the set of all automorphisms on U.

Automorphisms are closed under composition, that is, if ρ and ρ' are automorphisms then $\rho \circ \rho'(x) = \rho(\rho'(x))$ is also an automorphism.

The inverse of an automorphism is also an automorphism.

The *action* of ρ on a function $F : U^n \rightarrow U$, denoted by F^ρ, is defined as follows:

$$F^\rho(x_1, \ldots, x_n) = \rho^{-1}(F(\rho(x_1), \ldots, \rho(x_n))). \tag{9}$$

As it is well known (see, e.g., [12,38]), the action of ρ preserves the fuzzy connectives, that is, S^ρ,T^ρ, N^ρ and I^ρ are a fuzzy t-conorm, a t-norm, a (strong) negation and an implication, respectively.

Proposition 5.2 *If E is a fuzzy Xor connective, then E^ρ is also a fuzzy Xor connective.*

Proof. Considering that E is a fuzzy Xor connective, one has that E^ρ satisfies the following properties:

E1: $E^\rho(x, y) = \rho^{-1}(E(\rho(x), \rho(y))) = \rho^{-1}(E(\rho(y), \rho(x))) = E^\rho(y, x)$, based on the symmetry of E;

E2: One has that

$$\begin{aligned} E^\rho(x, E^\rho(y, z)) &= \rho^{-1}(E(\rho(x), \rho \circ \rho^{-1}(E(\rho(y), \rho(z))))) \\ &= \rho^{-1}(E(\rho(x), E(\rho(y), \rho(z))) = \rho^{-1}(E(E(\rho(x)\rho(y)), \rho(z))), \end{aligned}$$

considering that E satisfies the associativity property, and then it follows that

$$E^\rho(x, E^\rho(y, z)) = \rho^{-1}(E(\rho \circ \rho^{-1}(E(\rho(x)\rho(y)), \rho(z)))) = E^\rho(E^\rho(x, y), z);$$

E3: $E^\rho(0, x) = \rho^{-1}(E(\rho(0), \rho(x))) = \rho^{-1}(E(0, \rho(x))) = \rho^{-1}(\rho(x))) = x$, since E satisfies the 0-Identity property;

E4: $E^\rho(1, 1) = \rho^{-1}(E(\rho(1), \rho(1))) = \rho^{-1}(E(1, 1)) = \rho^{-1}(0) = 0$, based on the boundary condition related to E.

Therefore, E^ρ is a fuzzy Xor connective, whenever E is a fuzzy Xor connective. □

Proposition 5.3 *Let S, N and E be a t-conorm, a fuzzy negation and a fuzzy Xor connective, respectively. Then it holds that $I_{S^\rho,N^\rho,E^\rho}(x, y) = (I_{S,N,E})^\rho(x, y)$.*

Proof. Considering $x, y \in U$, one has that:

$$\begin{aligned} I_{S^\rho,N^\rho,E^\rho}(x, y) &= S^\rho(N^\rho(x), E^\rho(N^\rho(x), y)) \text{ by Eq. (8)} \\ &= S^\rho(\rho^{-1}N(\rho(x)), E^\rho(\rho^{-1}N(\rho(x), y))) \text{ by Eq. (9)} \\ &= S^\rho(\rho^{-1}N(\rho(x)), \rho^{-1}E(N(\rho(x), \rho(y)))) \text{ by Eq. (9)} \\ &= \rho^{-1}S(N(\rho(x)), E(N(\rho(x), \rho(y)))) \text{ by Def. 5.1} \\ &= \rho^{-1}S(N(\rho(x)), E(N(\rho(x), \rho(y)))) \text{ by Eq. (8)} \\ &= \rho^{-1}I_{S,N,E}(\rho(x), \rho(y)) = (I_{S,N,E})^\rho(x, y) \text{ by Def. 9} \end{aligned}$$

□

Corollary 5.4 *If I is a fuzzy E-implication then I^ρ is also a fuzzy E-implication.*

Proof. It follows from the definition of E-implication and the Property 5.3. □

Proposition 5.5 *Let S be a t-conorm, E be a Xor and N be a fuzzy negation. Then it holds that $I_{E^\rho,S^\rho,N^\rho}(x, y) = (I_{E,S,N})^\rho(x, y)$.*

Proof. Considering $x, y \in U$, one has that

$$
\begin{aligned}
I_{E^\rho,S^\rho,N^\rho}(x,y) &= E^\rho(x, S^\rho(N^\rho(x), N^\rho(y))) \text{ by Eq.(8)}\\
&= E^\rho(x, S^\rho(\rho^{-1}N(\rho(x), \rho(y)))) \text{ by Eq.(9)}\\
&= E^\rho(x, \rho^{-1}(S(N(\rho(x)), N(\rho(y))))) \text{ by Eq.(9)}\\
&= \rho^{-1}E(\rho(x), S(N(\rho(x)), N(\rho(y)))) \text{ by Def. 5.1}\\
&= \rho^{-1}E(\rho(x), \rho(S(N(x), N(y)))) \text{ by Eq.(7)}\\
&= \rho^{-1}S(N(\rho(x)), E(N(\rho(x), \rho(y)))) = (I_{E,S,N})^\rho(x,y) \text{ by Def. 9}
\end{aligned}
$$

□

Corollary 5.6 *Let I be a fuzzy E-implication then I^ρ is also a fuzzy E-implication.*

Proof. It follows from the definition of E-implication and Proposition 5.5. □

6 Conclusion and Further Work

Fuzzy implications play an important role in fuzzy logic, both in a broad sense (heavily applied to fuzzy control, analysis of vagueness in natural language and techniques of soft-computing) and in a narrow sense (developed as a branch of many-valued logic which are able to investigate deep logical questions).

One of the main contributions of this paper is the introduction of an autonomous definition of the fuzzy Xor connective, independently of the other fuzzy connectives. Also, two canonical constructions of the fuzzy Xor connective, denoted by E_T and E_S, were obtained by the composition of t-conorms, t-norms and fuzzy negations.

Based on this definition of the fuzzy Xor connective, this paper introduced two new fuzzy implications called Xor-implication and E-implication, denoted by $I_{S,N,E}$ and $I_{E,S,N}$, respectively.

Moreover, considering an automorphism ρ, we showed that the action of ρ on the fuzzy implications $I_{S,N,E}$ and $I_{E,S,N}$ preserves these connectives, that is, $I^\rho_{S,N,E}$ and $I^\rho_{E,S,N}$ are both fuzzy implications.

Important properties, such as commutative and associative properties of Xor-implications and E-implications were considered in this paper, which resulted in a canonical definition concerned with the notions of triangular norms and triangular conorms. Although these implications are still in an early analysis phase, new properties can be explored and general comparisons concerning their relationships with the other interesting implications (R-implications, S-implications, QL-implications, D-implications) can be developed.

The definition of *interval-valued* Xor-implications and E-implications is an ongoing work, following our previous works [8,9,10,16,40,41] on the study of the the various interval-valued implication functions derived from interval t-norms and interval t-conorms.

Also, additive and multiplicative generators to obtain the fuzzy Xor connective and the new implications can be considered in further work. So, the investigation of the most important properties preserved under the action of automorphisms and the related interval extension can be carried out.

Finally, since bounded lattices may be considered from the point of view of fuzzy logic, it seems interesting to extend the notions of E-implications and Xor-implications from the unit interval to an arbitrary bounded lattice. The interval extensions of such implications also motivate the study of their general properties on bounded lattices.

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References

[1] Aguzzoli1, S. and A. Ciabattoni, *Finiteness in infinite-valued Lukasiewicz logic*, Journal of Logic, Language and Information **9** (2000), pp. 5–29.

[2] Baczynski, M., *Residual implications revisited. Notes on the Smets-Magrez*, Fuzzy Sets and Systems **145** (2004), pp. 267–277.

[3] Baczynski, M. and B. Jayaram, *(s,n)- and r-implications: A state-of-the-art survey*, Fuzzy Sets and Systems (2007), (accepted to journal).

[4] Baczynski, M. and B. Jayaran, *On the characterization of (S,N)-implications generated from continuous negations*, in: *Proc. of the 11th Conference on Information Processing and Management of Uncertainty in Knowledge-based Systems*, Paris, 2006, pp. 436–443.

[5] Balasubramaniam, J., *Yager's new class of implications J_f and some classical tautologies*, Information Sciences (2006).

[6] Bandler, W. and L. J. Kohout, *Approximate reasoning in expert systems*, in: M. M. Gupta, A. Kandel, W. Bandler and J. B. Kiszka, editors, *The Interrelations of the Principal Fuzzy Logical Operators*, North-Holland, Amsterdam, 1985 pp. 767–780.

[7] Bedregal, B. C., A. C. R. Costa and G. P. Dimuro, *Fuzzy rule-based hand gesture recognition*, in: M. Bramer, editor, *Artificial Intelligence in Theory And Practice*, number 271 in IFIP Series, Springer, Boston, 2006 pp. 285–294.

[8] Bedregal, B. C., R. H. N. Santiago, G. P. Dimuro and R. H. S. Reiser, *Interval valued R-implications and automorphisms*, in: *Pre-Proceedings of the 2nd Work. on Logical and Semantic Frameworks, with Applications*, Ouro Preto, 2007, pp. 82–97.

[9] Bedregal, B. C., R. H. N. Santiago, R. H. S. Reiser and G. P. Dimuro, *The best interval representation of fuzzy S-implications and automorphisms*, in: *Proc. of the IEEE International Conference on Fuzzy Systems, Londres, 2007* (2007), pp. 3220–3230.

[10] Bedregal, B. C., R. H. N. Santiago, R. H. S. Reiser and G. P. Dimuro, *Properties of fuzzy implications obtained via the interval constructor*, TEMA **8** (2007), pp. 33–42, (available at http://www.sbmac.org.br/tema).

[11] Bongiovanni, G. and P. Penna, *XOR-based schemes for fast parallel ip lookups*, Theoretical Computer Systems **38** (2005), pp. 481–501.

[12] Bustince, H., P. Burilo and F. Soria, *Automorphism, negations and implication operators*, Fuzzy Sets and Systems **134** (2003), pp. 209–229.

[13] Carlsson, C. and R. Fuller, "Fuzzy Reasoning in Decision Making and Optimization," Physica-Verlag Springer, Heidelberg, 2002.

[14] Chen, G. and T. T. Pham, "Fuzzy Sets, Fuzzy Logic, and Fuzzy Control Systems," CRC Press, Boca Raton, 2001.

[15] Codding, E. G. and G. Horlick, *Application of AND and exclusive-Or (xor) logic operations to the identification of elemental emission spectra measured using a photodiode array direct reading spectrometer*, Applied Spectroscopy **27** (1973), pp. 390–394.

[16] Dimuro, G. P., B. Bedregal and R. H. S. Reiser, *Interval additive generators of interval t-norms*, in: *Proc. of Logic, Language, Information and Computation15th International Workshop, WoLLIC 2008, Edinburgh, UK, July 1-4, 2008*, LNCS **5110**, Springer, Berlin, 2008 pp. 123–135.

[17] Fodor, J. and M. Roubens, "Fuzzy Preference Modelling and Multicriteria Decision Support," Kluwer Academic Publisher, Dordrecht, 1994.

[18] Fodor, J. C., *On fuzzy implication operators*, Fuzzy Sets and Systems **42** (1991), pp. 293–300.

[19] Fodor, J. C., *Contrapositive symmetry of fuzzy implications*, Fuzzy Sets and Systems **69** (1995), pp. 141–156.

[20] Gehrke, M., C. Walker and E. Walker, *Some comments on interval valued fuzzy sets*, International Journal of Intelligent Systems **11** (1996), pp. 751–759.

[21] Gehrke, M., C. Walker and E. Walker, *Algebraic aspects of fuzzy sets and fuzzy logics*, in: *Proc. of the Workshop on Current Trends and Development in Fuzzy Logic*, Thessaloniki, 1999, pp. 101–170.

[22] Gottwald, S., "A Treatise on Many-Valued Logics," Taylor & Francis Group, London, 2001.

[23] Horcik, R. and M. Navara, *Validation sets in fuzzy logics*, Kybernetika **38** (2002), pp. 319–326.

[24] Horikawa, Y., *Landscapes of basins of local minima in the XOR problem*, in: *Proc. IEEE Intl. Conf. on Neural Networks* (1993), pp. 1677–1680.

[25] Kecman, V., *Support vector machines - an introduction*, in: L. Wang, editor, *Support Vector Machines: Theory and Applications*, Springer, Berlin, 2005 pp. 1–48.

[26] Klement, E. and M. Navara, *A survey on different triangular norm-based fuzzy logics*, Fuzzy Sets and Systems **101** (1999), pp. 241–251.

[27] Klement, E. P., R. Mesiar and E. Pap, "Triangular Norms," Kluwer Academic Publisher, Dordrecht, 2000.

[28] Kohout, L. and E. Kim, *Characterization of interval fuzzy logic systems of connectives by group transformations*, Reliable Computing **10** (2004), pp. 299–334.

[29] Konheim, A. G., "Computer Security And Cryptography," John Wiley & Sons, Hoboken, 2007.

[30] Leski, J., *ε-insensitive learning techniques for approximate reasoning system*, Int. Jour. of Computational Cognition **1** (2003), pp. 21–77.

[31] Maeda, M., M. Suenaga and H. Miyajima, *Qubit neuron according to quantum circuit for XOR problem*, Applied Mathematics and Computation **185** (2007), pp. 1015–1025.

[32] Mas, M., M. Monserrat and J. Torrens, *Ql-implications versus d-implications*, Kybernetika **42** (2006), pp. 351–366.

[33] Mas, M., M. Monserrat and J. Torrens, *Two types of implications derived from uninorms*, Fuzzy Sets and Systems **158** (2007), pp. 2612–2626.

[34] Mela, C. F. and D. R. Lehmann, *Using fuzzy set theoretic techniques to identify preference rules from interactions in the linear model: an empirical study*, Fuzzy Sets and Systems **71** (1995), pp. 165–181.

[35] Mermin, N. D., "Quantum Computer Science - An Introduction," Cambridge University Press, Cambridge, 2007.

[36] Mitra, S. and S. K. Pal, *Fuzzy sets in pattern recognition and machine intelligence*, Fuzzy Sets and Systems **156** (2005), pp. 381–386.

[37] Navara, M., *Characterization of measures based on strict triangular norms*, Mathematical Analysis and Applications **236** (1999), pp. 370–383.

[38] Nguyen, H. and E. Walker, "A First Course in Fuzzy Logic," Chapman & Hall/CRC, Boca Raton, 1999.

[39] Pedrycz, W. and G. Succi, *fXOR fuzzy logic networks*, Soft Computing **7** (2002), pp. 115–120.

[40] Reiser, R. H. S., G. P. Dimuro, B. Bedregal and R. Santiago, *Interval valued QL-implications*, in: D. Leivant and R. Queiroz, editors, *Logic, Language, Information and Computation, Proc. of 14th International Workshop, WoLLIC 2007, Rio de Janeiro, 2007*, LNCS **4576**, Springer, Berlin, 2007 pp. 307–321.

[41] Reiser, R. H. S., G. P. Dimuro, B. R. C. Bedregal, H. Santos and R. C. Bedregal, *S-implications on bounded lattices and the interval constructor*, in: *Proc. XXX Cong. Nac. Mat. Aplicada e Computacional* (2007), pp. 1–7.

[42] Ruan, D. and E. Kerre, *Fuzzy implication operators and generalized fuzzy methods of cases*, Fuzzy Sets and Systems **54** (1993), pp. 23–37.

[43] Shannon, C., *Communication theory of secrecy systems*, Bell System Technical Journal **28** (1949), pp. 656–715.

[44] Shi, Y., D. Ruan and E. E. Kerre, *On the characterizations of fuzzy implications satisfying* $I(x, y) = I(x, I(x, y))$, Information Sciences **177** (2007), pp. 2954–2970.

[45] Siler, W. and J. J. Buckley, "Fuzzy Expert Systems and Fuzzy Reasoning," John Wiley, NY, 2004.

[46] Smets, P. and P. Magretz, *Implication in fuzzy logic*, International Journal of Approximate Reasoning **1** (1987), pp. 327–347.

[47] Stallings, W., "Cryptography and network security: Principles and practice," Prentice Hall, Upper saddle River, 1999.

[48] Türksen, I., V. Kreinovich and R. Yager, *A new class of fuzzy implications*, Axioms of fuzzy implication revisited, Fuzzy Sets and Systems **100** (1998), pp. 267–272.

[49] Vandierendonck, H. and K. Bosschere, *XOR-based hash functions*, IEEE Transactions on Computers **54** (2005), pp. 800–812.

[50] Vandierendonck, H., P. Manet and J. D. Legat, *Application-specific reconfigurable XOR-indexing to eliminate cache conflict misses*, in: *Proc. of the conference on Design, automation and test in Europe* (2006), pp. 357–362.

[51] Yager, R., *On the implication operator in fuzzy logic*, Information Sciences **31** (1983), pp. 141–164.

[52] Yager, R., *On global requirements for implication operators in fuzzy modus ponens*, Fuzzy Sets and Systems **106** (1999), pp. 3–10.

[53] Yager, R., *On some new classes of implication operators and their role in approximate reasoning*, Information Sciences **167** (2004), pp. 193–216.

[54] Zadeh, L. A., *Fuzzy sets*, Information and Control (1965), pp. 338–353.

[55] Zadeh, L. A., *The concept of a linguistic variable and its application to approximate reasoning - I*, Information Sciences **6** (1975), pp. 199–249.