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On the classification of non-integrable complex distributions

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1. INTRODUCTION

Given a point $p \in \mathbb{C}^n$ and a real number $r > 0$ we denote by $B^{2n}(p,r)$ the open ball of radius r centered at p in $\mathbb{C}^n$. The corresponding closed ball is denoted by $B^{2n}[p,r]$ and its boundary sphere by $S^{2n-1}(p,r) = \partial B^{2n}[p,r]$. We also write $B^{2n}(1) = B^{2n}(0,1)$, $B^{2n}[1] = B^{2n}[0,1]$ and $S^{2n-1}(1) = \partial B^{2n}[1]$. Let Ω be a germ of holomorphic one-form with an isolated singularity at the origin $0 \in \mathbb{C}^n$, $n \geqslant 3$. We address the problem of analytical classification of Ω in the *non-integrable* case. Motivated by the geometrical-analytical classification of singularities in dimension 2 we consider the case where the kernel of Ω generates a germ of distribution $\mathrm{Ker}(\Omega)$ transverse to small spheres $S^{2n-1}(0,\varepsilon)$. This is one, though not the only, central motivation for this work. The problem of existence of integral manifolds for germs of singularities of integrable one-forms is an ancient problem already considered in the work of Briot-Bouquet. The existence results in [3] (for dimension $n = 2$) and in [4] (for the non-dicritical case in dimension $n = 3$) motivate the very basic question below:

Question 1. *Is there a non-integrable germ of holomorphic one-form Ω with an isolated singularity at the origin $0 \in \mathbb{C}^n$ such that* $\mathrm{Ker}(\Omega)$ *is transverse to the spheres $S^{2n-1}(0,\varepsilon)$, for $\varepsilon > 0$ small enough and* $\mathrm{Ker}(\Omega)$ *admits no integral manifold through the origin?*

Theorem 2 gives a positive answer to this question. Other motivations are related to our previous work in [10] and [9] where we study the obstructions to the integrability of Ω. Our first main result reads as:

Theorem 1. *Let $n \geqslant 3$ and Ω be a holomorphic one-form defined in a neighborhood of $B^{2n}[1]$ and such that* $\operatorname{sing}(\Omega) \cap S^{2n-1}(1) = \emptyset$. *If there exists a holomorphic vector field ξ in a neighborhood of $B^{2n}[1]$, transverse to $S^{2n-1}(1)$, and such that $\Omega \cdot \xi = 0$, then Ω is not integrable.*

Let us give examples of distributions as in Theorem 1. Denote by $\mathcal{A}(2m)$ the set of all $2m \times 2m$ skew-symmetric complex matrices and by $\mathbb{A}(2m)$ the subset of nonsingular elements in $\mathcal{A}(2m)$. In [9] it is observed that if $A = (a_{ij})_{i,j=1}^{2m}$ belongs to $\mathbb{A}(2m)$ then, for $m \geqslant 2$, the one-form $\Omega_A = \sum_{i,j=1}^{2m} a_{ij} z_i \, dz_j$ defines a non-integrable holomorphic (linear) distribution transverse to the spheres $S^{4m-1}(0, r) \subset \mathbb{C}^{2m}$, $r > 0$. Such a one-form will be called *linear*. A particular case is the one-form $\Omega_{\mathbb{J}(2m)} = \sum_{j=1}^{m} (z_{2j-1}\, dz_{2j} - z_{2j}\, dz_{2j-1})$. One may ask for non-linear examples. Given $\ell = (\ell_1, \dots, \ell_m) \in \mathbb{N}^m$, we introduce the corresponding *non-integrable Poincaré–Dulac normal form* as $\Omega_{(\ell)} = \sum_{j=1}^{m} [z_{2j-1}\, dz_{2j} - (\ell_j z_{2j} + z_{2j-1}^{\ell_j})\, dz_{2j-1}]$ in coordinates $(z_1, z_2, \dots, z_{2m}) \in \mathbb{C}^{2m}$. We prove that also $\Omega_{(\ell)}$ is not integrable for $m \geqslant 2$, singular only at the origin, and $\operatorname{Ker}(\Omega_{(\ell)})$ is transverse to $S^{4m-1}(0, r)$, $\forall r > 0$ small enough (see Example 1). The one-forms Ω_A, $\Omega_{\mathbb{J}(2m)}$ and $\Omega_{(\ell)}$ are our basic models in the classification we pursue (see Section 4).

Let $\operatorname{Ker}(\Omega)$ be a codimension one holomorphic distribution on a complex manifold V^n. Let $p \in V^n$ be a *singularity* of $\operatorname{Ker}(\Omega)$, that is, of Ω. A germ of codimension one analytic subset Λ_p at p is an *integral manifold* of $\operatorname{Ker}(\Omega)$ through p if any vector $v_q \in T_q(V)$ which is tangent to Λ_p at a point q belongs to $\operatorname{Ker}(\Omega)(q)$. This means that if Λ is any representative of Λ_p in a neighborhood U of p in V^n and Λ^* denotes the smooth part of Λ then the tangent bundle $T\Lambda^*$ is a sub-bundle of $\operatorname{Ker}(\Omega)\big|_{\Lambda^*}$ (see Definition 1). We shall always assume Λ and Λ_p to be irreducible, nevertheless we do not require that $\Lambda \backslash \Lambda^* = \operatorname{sing}(\Lambda)$ is contained in $\operatorname{Sing}(\operatorname{Ker}(\Omega))$. Regarding the existence of integral manifolds for non-integrable distributions we have:

Theorem 2. *Let $m \geqslant 2$. Given $A \in \mathbb{A}(2m)$ and $\ell \in \mathbb{N}^m$ the distributions* $\operatorname{Ker}(\Omega_{\mathbb{J}(2m)})$, $\operatorname{Ker}(\Omega_A)$ *and* $\operatorname{Ker}(\Omega_{(\ell)})$ *admit no integral manifold through the origin.*

In the course of the proof of Theorem 2 we obtain the following Darboux's theorem type for (not necessarily integrable) polynomial distributions. This is actually a non-integrable version of the more precise Theorem 3.3 of [11, p. 102]:

Proposition 1. *Let Ω be a (not necessarily integrable) polynomial one-form on $\mathbb{C}^n$, $n \geqslant 2$, and assume that* $\operatorname{codsing}(\Omega) \geqslant 2$. *If* $\operatorname{Ker}(\Omega)$ *has infinitely many algebraic invariant hypersurfaces then Ω is integrable. Indeed $\Omega = P\,dQ - Q\,dP$ for some polynomials P, Q with no common factors and, in particular, the leaves of the* *foliation $\mathcal{F}_\Omega$ defined by Ω are contained in the algebraic subvarieties $\{\lambda P - \mu Q = 0\}$ where $(\lambda, \mu) \in \mathbb{C}^2 - \{(0,0)\}$.*

As already mentioned, the examples Ω_A, $\Omega_{\mathbb{J}(2m)}$ and $\Omega_{(\ell)}$ above constructed motivate the problem of analytical classification of germs of non-integrable one-forms defining distributions transverse to small spheres (see Questions 2 and 3 in Section 4). In this direction we prove:

Theorem 3. *Let Ω be a holomorphic one-form in a neighborhood U of the closed ball $B^{4m}[1] \subset \mathbb{C}^{2m}$ and such that* (1) $\Omega \cdot \vec{R} = 0$, *where $\vec{R}$ is the radial vector field in $\mathbb{C}^{2m}$ and* (2) $\mathrm{Sing}(\Omega) \cap S^{4m-1}(1) = \emptyset$. *Then* $\mathrm{Ker}(\Omega)$ *is homotopic to the linear distribution* $\mathrm{Ker}(\Omega_{\mathbb{J}(2m)})$ *by distributions* $\mathrm{Ker}(\Omega_s)$, $0 \leqslant s \leqslant 1$, *such that $\Omega_0 = \Omega$ and $\Omega_1 = \Omega_{\mathbb{J}_{2m}}$, where Ω_s is holomorphic and satisfies* (1) *and* (2) *above.*

2. INVARIANT MANIFOLDS

In this section we discuss Question 1 and prove Theorem 2 and Proposition 1. First we prove the examples mentioned in the introduction.

Example 1. Let $A = (a_{ij}) \in \mathbb{A}(2m)$ and $\Omega_A := \sum_{i,j=1}^{2m} a_{ij} z_i \, dz_j$ the corresponding linear one-form in $\mathbb{C}^{2m}$. Then $\mathrm{Sing}(\Omega_A) = \{0\} \subset \mathbb{C}^{2m}$ and $\Omega_A \cdot \vec{R} = 0$ for the radial vector field $\vec{R} = \sum_{j=1}^{2m} z_j \frac{\partial}{\partial z_j}$. This implies that $\mathrm{Ker}(\Omega_A)$ is transverse to every sphere $S^{4m-1}(0,r)$, $r > 0$. The non-integrability of Ω_A and $\Omega_{(\ell)}$ is a straightforward computation (cf. [10]). For the transversality of the non-integrable Poincaré–Dulac normal form $\Omega_{(\ell)}$ with small spheres $S^{4m-1}(0,\varepsilon)$ we observe that if $\xi_{(\ell)} = \sum_{j=1}^{m} [(\ell_j z_{2j} + z_{2j-1}^{\ell_j}) \frac{\partial}{\partial z_{2j}} + z_{2j-1} \frac{\partial}{\partial z_{2j-1}}]$ then $\Omega_{(\ell)} \cdot \xi_{(\ell)} = 0$ and, as it is well known, the vector field $\xi_{(\ell)}$ is transverse to the spheres $S^{4m-1}(0,\varepsilon)$ if $\varepsilon > 0$ is small enough.

Now we shall prove Theorem 2. Let us first precise some notions involved.

Definition 1. Let Ω be a holomorphic one-form on a complex manifold V. A *smooth* complex immersed submanifold $\Lambda^* \subset V$ is an *integral manifold* of Ω if $T\Lambda^* \subset \mathrm{Ker}(\Omega)|_{\Lambda^*}$. In other words, any tangent vector to Λ^* belongs to $\mathrm{Ker}(\Omega)$. If $\Lambda \subset V$ is a possibly singular complex analytic submanifold we say that Λ is an *integral manifold* of Ω if its regular part $\Lambda^* = \Lambda \setminus \mathrm{sing}(\Lambda)$ is an integral manifold of Ω.

The following lemma is found in the algebraic setting in [11, Section 3.1, p. 99].

Lemma 1. *Let* $\mathrm{Ker}(\Omega)$ *be given by a holomorphic one-form Ω in V with* $\mathrm{codsing}(\Omega) \geqslant 2$ *and let $\Lambda \subset V$ be a codimension one analytic subset given by a reduced equation Λ: $\{f = 0\}$ for some holomorphic $f: V \to \mathbb{C}$. The following conditions are equivalent:*

(1) Λ *is* $\mathrm{Ker}(\Omega)$*-invariant.*
(2) $\Omega \wedge \frac{df}{f}$ *is a holomorphic* 2*-form on* V.

Proof. Since all objects involved are analytic we may consider the local case also at a generic (and therefore non-singular) point $p \in \Lambda^*$. In suitable local coordinates $(z_1, \dots, z_n) = (z_1, \dots, z_{n-1}, f)$ we have $p = 0$ and Λ given by $\{f = z_n = 0\}$. Also we may write $\Omega = \sum_{j=1}^{n} a_j\, dz_j$. Suppose Λ is $\mathrm{Ker}(\Omega)$ invariant. Then, since Λ is given by $\{z_n = 0\}$ we have $\Omega \cdot \frac{\partial}{\partial z_j}|_{\{z_n=0\}} = 0$ for all $j \in \{1, \dots, n-1\}$. In other words z_n divides a_j in $\mathbb{C}\{z_1, \dots, z_n\}$ for every $j \in \{1, \dots, n-1\}$ and therefore $\Omega = \sum_{j=1}^{n-1} z_n \tilde{a}_j\, dz_j + a_n\, dz_n$ for some holomorphic $\tilde{a}_1, \dots, \tilde{a}_{n-1} \in \mathbb{C}\{z_1, \dots, z_n\}$. Thus $\Omega \wedge \frac{dz_n}{z_n} = \sum_{j=1}^{n-1} \tilde{a}_j\, dz_j \wedge dz_n$ is holomorphic. Conversely, if $\Omega \wedge \frac{dz_n}{z_n}$ is holomorphic then $\Omega = \sum_{j=1}^{n-1} z_n \tilde{a}_j\, dz_j + a_n\, dz_n$ as above therefore $\Omega \cdot \frac{\partial}{\partial z_j}$ vanishes on $\{z_n = 0\}$ for every $j = 1, \dots, n-1$. Since $\{\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_{n-1}}\}$ generate $T\Lambda$ in a neighborhood of 0 we obtain that $T\Lambda \subset \mathrm{Ker}(\Omega)$ in a neighborhood of p. Therefore Λ is $\mathrm{Ker}(\Omega)$-invariant. □

Proof of Proposition 1. Once we have Lemma 1, the proof is essentially the same given in [11], thus we will omit it and refer to [11]. □

2.1. Proof of Theorem 2

Let $\Omega_{\mathbb{J}(2m)} = \sum_{j=1}^{m} (z_{2j-1}\, dz_{2j} - z_{2j}\, dz_{2j-1})$ be given in $\mathbb{C}^{2m}$. Then $d\Omega_{\mathbb{J}(2m)} = 2\sum_{j=1}^{m} dz_{2j-1} \wedge dz_{2j}$ is a non-degenerate 2-form on $\mathbb{C}^{2m}$ and $(d\Omega_{\mathbb{J}(2m)})^m$ is a non-zero $2m$-form. Let L be a complex manifold and $\zeta : L \to \mathbb{C}^{2m}$ a smooth embedding.

Lemma 2. *The following statements are equivalent:*

(1) $T(\zeta(L)) \subset \mathrm{Ker}(\Omega_{\mathbb{J}(2m)})|_{\zeta(L)}$.
(2) $\zeta^*(\Omega_{\mathbb{J}(2m)}) = 0$.

Proof. To prove that (1) implies (2) we take any vector $v \in T(L)$, then $\zeta_*(v) \in T(\zeta(L)) \subset \mathrm{Ker}(\Omega_{\mathbb{J}(2m)})|_{\zeta(L)}$ means $\Omega_{\mathbb{J}(2m)}(\zeta_* v) = 0$. Therefore we get $\zeta^*(\Omega_{\mathbb{J}(2m)})(v) = \Omega_{\mathbb{J}(2m)}(\zeta_* v) = 0$. To prove that (2) implies (1) take any $\tilde{v} = \zeta_* v \in T(\zeta(L))$, $v \in T(L)$. Then $\zeta^*(\Omega_{\mathbb{J}(2m)}) = 0$ means that $\Omega(\tilde{v}) = \Omega_{\mathbb{J}(2m)}(\zeta_* v) = \zeta^*(\Omega_{\mathbb{J}(2m)})(v) = 0$. Thus $\tilde{v}$ belongs to $\mathrm{Ker}(\Omega_{\mathbb{J}(2m)})|_{\zeta(L)}$. □

Proposition 2. *Under the above notations, assume that*

$$T\big(\zeta(L)\big) \subset \mathrm{Ker}(\Omega_{\mathbb{J}(2m)})|_{\zeta(L)},$$

then the complex dimension of L *is less than* m *or equal to* m, *i.e.,* $\dim_{\mathbb{C}} L \leqslant m$.

Proof. First we observe that $\zeta^*(d\Omega_{\mathbb{J}(2m)}) = d(\zeta^* \Omega_{\mathbb{J}(2m)}) = 0$ on L. Assume that $\dim_{\mathbb{C}} L \geqslant m + 1$. Take $m + 1$ linearly independent vectors $v_1, \dots, v_{m+1}$

in $T(L)$. Moreover, we take $m-1$ vectors $u_1,\dots,u_{m-1}$ in $T\mathbb{C}^{2m}$ such that $\zeta_* v_1,\dots,\zeta_* v_{m+1},u_1,\dots,u_{m-1}$ are linearly independent in $T\mathbb{C}^{2m}$. Then we get $(d\Omega_{\mathbb{J}(2m)})^m(\zeta_* v_1,\dots,\zeta_* v_{m+1},u_1,\dots,u_{m-1})=0$ because $\zeta^*(d\Omega_{\mathbb{J}(2m)})=0$. It is contradictory with the fact that $(d\Omega_{\mathbb{J}(2m)})^m$ is a non-zero $2m$-form. □

Corollary 1. *Let $m\geqslant 2$. The one-form $\Omega_{\mathbb{J}(2m)}$ has no integral manifold through the origin.*

Proof. Assume that Λ is an integral manifold of $\Omega_{\mathbb{J}(2m)}$. Then $\Lambda^*=\Lambda-\{0\}$ is a complex hypersurface, i.e., $\dim_{\mathbb{C}}\Lambda^*=2m-1$. By Proposition 2, we get $\dim_{\mathbb{C}}\Lambda^*\leqslant m$. We have a contradiction with the hypothesis $m\geqslant 2$. □

This proves Theorem 2 for $\Omega_{\mathbb{J}(2m)}$. By the same argument, Ω_A and $\Omega_{(\ell)}$ have no integral manifold through the origin of $\mathbb{C}^{2m}$.

Regarding the smoothness of invariant hypersurfaces of holomorphic distributions with an isolated singularity we have:

Proposition 3. *Let Ω be a holomorphic one-form in a complex manifold M and $p\in M$ and $\Lambda\subset M$ an integral manifold of Ω point. Given a point $p\in M$ where Ω is nonzero then Λ is smooth at the point p.*

Proof. In suitable local coordinates $(z_1,\dots,z_n)$ centered at p the Pfaffian equation $\Omega=0$ is equivalent to an equation $dz_n=\sum_{j=1}^{n-1} g_j\,dz_j$ for some holomorphic functions g_j. Any integral manifold then writes as $z_n=f(z_1,\dots,z_{n-1})$ for a holomorphic function f satisfying $\partial f(z_1,\dots,z_{n-1})/\partial z_j=g_j(z_1,\dots,z_{n-1},f(z_1,\dots,z_{n-1}))$. This implies that f cannot develop any singularities. □

3. PROOF OF THEOREM 1

By hypothesis, ξ defines a one-dimensional holomorphic foliation in a neighborhood of the closed ball $B^{2n}[1]$, transverse to the sphere $S^{2n-1}(1)$. According to [8] ξ has a unique singularity p in the open ball $B^{2n}(1)$ and this singularity is in the Poincaré domain: the vector field ξ has a non-singular linear part at p having eigenvalues $\lambda_1,\dots,\lambda_n$ such that the origin does not belong to the convex hull of the set $\{\lambda_1,\dots,\lambda_n\}$ in $\mathbb{R}^2$. On the other hand, we have the following result from [9]:

Theorem 4. *Let ω be a holomorphic one-form in an open subset $U\subset\mathbb{C}^n$, $n\geqslant 2$. Suppose that the distribution* $\operatorname{Ker}(\omega)$ *is transverse to a sphere $S^{2n-1}(p,r)\subset U$, with $B^{2n}[p,r]\subset U$, $p\in U$, $r>0$, then n is even, ω has a single singular point in the ball $B^{2n}(p,r)$ and this is a simple singularity in the following sense: if we write $\omega=\sum_{j=1}^n f_j\,dz_j$ in local coordinates centered at the singularity then the matrix $(\partial f_j/\partial z_k)_{j,k=1}^n$ is non-singular at the singularity.*

By the above result Ω has a unique singularity q in $B^{2n}(1)$ and this is a simple singularity.

Claim 1. *We have $p = q$.*

Proof. Let $\Omega = \sum_{j=1}^{n} f_j\, dz_j$ in a neighborhood U of $B^{2n}[1]$ on $\mathbb{C}^n$. Write $\xi = \sum_{i=1}^{n} A_i \frac{\partial}{\partial z_i}$ then $0 = \Omega \cdot \xi = \sum_{j=1}^{n} f_j A_j$ and therefore $0 = \frac{\partial}{\partial z_k}(\sum_{j=1}^{n} f_j A_j) = \sum_{j=1}^{n}(\frac{\partial f_j}{\partial z_k} A_j + f_j \frac{\partial A_j}{\partial z_k})$. Thus $0 = \sum_{j=1}^{n} \frac{\partial f_j}{\partial z_k}(p) \cdot A_j(p)$, $\forall k$ and since the matrix $(\frac{\partial f_j}{\partial z_k}(p))_{j,k=1}^{n}$ is non-singular we have $A_j(p) = 0$, $j = 1, \dots, n$. By the uniqueness of the singularity of ξ we get $p = q$. This proves the claim. $\square$

Let us finish the proof. Suppose by contradiction that Ω is integrable. By a theorem of Malgrange [12], since $\operatorname{codimsing}(\Omega) = n \geqslant 3$ at p, the one-form Ω admits a holomorphic (Morse-type) first integral in a neighborhood of p, say $f\colon (W, p) \to (\mathbb{C}, 0)$. Then $\Omega \cdot \xi = 0$ implies that $\xi(f) = 0$. Because the germ of ξ at 0 is in the Poincaré domain, this implies that f is constant in a neighborhood of 0, contradiction with the fact that f is of Morse type at 0. This ends the proof of Theorem 1.

4. ON THE ANALYTICAL CLASSIFICATION

The problem of analytic classification of singularities of holomorphic one-forms in dimension two is a very well-developed topic. Recently (cf. [5]) the analytic classification was obtained for germs of reduced *integrable* one-forms at the origin $0 \in \mathbb{C}^3$. As far as we know, nothing is found regarding the non-integrable case. In the follow-up, "to classify" means to give a description in terms of objects which are completely understood. Obviously the class of germs of singular (non-integrable) holomorphic one-forms is too wide in order to be classified at a first moment and we also lack of geometric ingredients. This remark is one of the motivations for our approach in this section. Other motivation is given by the well-known results for holomorphic foliations with singularities in dimension two collected in the following omnibus theorem:

Theorem 5. [1,2,6,8] *Given a germ of singular holomorphic one-form Ω at $0 \in \mathbb{C}^2$ the following conditions are equivalent:*

(1) *Ω is in the Poincaré domain: $\Omega = A\, dy - B\, dx$ where the vector field $X = A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y}$ has a singularity at $(0, 0)$ whose linear part $DX(0, 0)$ has non-zero eigenvalues λ, μ with quotient $\lambda/\mu \in \mathbb{C}\backslash\mathbb{R}_-$.*
(2) $\operatorname{Ker}(\Omega)$ *is transverse to some (and therefore to every) sphere $S^3(0, \varepsilon)$, for $\varepsilon > 0$ small enough.*
(3) *There are local analytic coordinates $(x, y) \in (\mathbb{C}^2, 0)$ such that $\Omega(x, y)$ is either linear $\Omega = \lambda x\, dy - \mu y\, dx$ with $\lambda/\mu \in \mathbb{C}\backslash\mathbb{R}_-$, or it is of the form $\Omega = x\, dy - (ny + x^n)\, dx$ where $n \in \mathbb{N}$ (called Poincaré–Dulac normal form).*

Thus we have the following problem:

Problem 1. *To obtain the local analytical classification of germs of non-integrable holomorphic one-forms* Ω *at* $0 \in \mathbb{C}^n$ *under the hypothesis of transversality with small spheres* $S^{2n-1}(0,\varepsilon)$.

One may work with the following notions and models:

Definition 2. We shall say that a germ of singular non-integrable one-form Ω at the origin $0 \in \mathbb{C}^{2m}$ is *in the Poincaré domain* if $\mathrm{Ker}(\Omega)$ is transverse to small spheres $S^{4m-1}(0,\varepsilon)$, $\varepsilon > 0$. A germ Ω will be called *analytically linearizable* if $f^*\Omega = \Omega_A$ for some germ of biholomorphism $f \in \mathrm{Bih}(\mathbb{C}^{2m},0)$ fixing the origin and some $A \in \mathbb{A}(2m)$. Finally we shall say that Ω is (*analytically conjugate to*) *a non-integrable Poincaré–Dulac normal form* if $f^*\Omega = \Omega_{(\ell)}$ for some $f \in \mathrm{Bih}(\mathbb{C}^{2m},0)$ and some $\ell \in \mathbb{N}^m$.

Question 2. *Are conditions* (i), (ii) *and* (iii) *below equivalent*?

(i) Ω *is the Poincaré-domain.*
(ii) Ω *is analytically linearizable or conjugate to a non-integrable Poincaré–Dulac normal form.*
(iii) *There is a holomorphic vector field* ξ *transverse to the spheres* $S^{4m-1}(0,r)$ *for* $r > 0$ *small enough and such that* $\Omega \cdot \xi = 0$?

Remark 1. The non-integrable examples we have given, do admit holomorphic sections ξ transverse to small spheres $S^{4m-1}(0,r)$ (see Example 1). Let us observe that there are examples of non-integrable one-forms Ω with isolated singularity at the origin in $\mathbb{C}^n, n \geqslant 3$, which admit integral manifold. For example, take $f = \frac{1}{2}\sum_{j=1}^{2m} z_j^2$ and $\Omega = df + f\nu$ for some holomorphic one-form $\nu = \sum_{j=1}^{2m} \nu_j\, dz_j$ such that $d\nu(0)$ is nondegenerate. Then $\{f = 0\}$ is an integral manifold of Ω and also $\Omega(z) = 0$ if and only if $z_j + \sum_{k=1}^{2m} z_k^2 \nu_j(z) = 0$ for every j. This shows, because the Jacobian of the left hand side at $z = 0$ is the identity, that Ω has an isolated singularity at the origin $z = 0$. Finally, if $X_j \in \mathrm{Ker}(\Omega)$, then $(d\Omega)(X_1, X_2) = f(d\nu)(X_1, X_2)$, we obtain that near the origin Ω is non-integrable at every point where $f \neq 0$. This example suggests that a more interesting question might be:

Question 3. *Classify the non-integrable germs of holomorphic one-forms* Ω *with an isolated singularity at the origin and* $\mathrm{Ker}(\Omega)$ *transverse to all small spheres centered at the origin, which admit an integral manifold through the origin.*

The above construction suggests that if the integral manifold is taken in the form $\{f = 0\}$ for a holomorphic function f, then maybe we can write $\Omega = g(df + f\nu)$ for some function g and a one-form ν, and we have to study the kernels of $df + f\nu$.

4.1. Proof of Theorem 3

Now we shall prove Theorem 3. We need:

Lemma 3. *The open subset $\mathbb{A}(2m) \subset \mathcal{A}(2m)$ is arcwise-connected.*

Proof. First we recall that $\mathcal{A}(2m)$ is a complex vector space. Denote by $f : \mathcal{A}(2m) \to \mathbb{C}$ the determinant function, i.e., $f(A) = \mathrm{Det}(A)$, $\forall A \in \mathcal{A}(2m)$. Then f is holomorphic and $f \neq 0$ because if

$$\mathbb{J}(2m) = \begin{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & & \\ & \ddots & \\ & & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{bmatrix}$$

then $\mathbb{J}(2m) \in \mathcal{A}(2m)$ and $f(\mathbb{J}(2m)) = \mathrm{Det}(\mathbb{J}(2m)) = (-1)^m \neq 0$. Now $\mathbb{A}(2m) = \{A \in \mathcal{A}(2m)\colon\ f(A) \neq 0\} = \mathcal{A}(2m) \backslash f^{-1}(0)$. Since $\mathcal{A}(2m)$ is an affine space and $f^{-1}(0)$ is a thin subset of $\mathcal{A}(2m)$ it follows from [7] Corollary 4 page 20 that $\mathbb{A}(2m)$ is (arcwise) connected. □

We shall adopt the following natural definition:

Definition 3. Two codimension one holomorphic distributions Δ_1, Δ_2 on a complex manifold M^n are *homotopic* if there is a C^∞ family $\{P_s\}_{s \in [0,1]}$ of holomorphic distributions in M such that $P_0 = \Delta_1$ and $P_1 = \Delta_2$.

Lemma 4. *Let Ω_A be a linear one-form in $\mathbb{C}^{2m}$ with $A \in \mathbb{A}(2m)$. Then the distribution* $\mathrm{Ker}(\Omega_A)$ *is homotopic to the canonical distribution* $\Omega_{\mathbb{J}(2m)} = \sum_{j=1}^{2m}(z_{2j}\, dz_{2j-1} - z_{2j-1}\, dz_{2j})$.

Proof. This is an immediate consequence of Lemma 3. Given a smooth path $\alpha\colon\ [0,1] \to \mathbb{A}(2m)$ connecting $\alpha(0) = A$ to $\alpha(1) = \mathbb{J}(2m)$ we define a homotopy by setting $P_s = \mathrm{Ker}(\Omega_{\alpha(s)})$ (recall that $\alpha(s) \in \mathbb{A}(2m)$). □

Now we consider a holomorphic one-form Ω defined in a neighborhood U of the closed unit ball $B^{4m}[0,1] \subset \mathbb{C}^{2m}$ and such that $\Omega \cdot \vec{R} = 0$. Write $\Omega = \sum_{\nu=1}^{+\infty} \omega_\nu$ where ω_ν is a homogeneous one-form of degree $\nu \geqslant 1$ in U (assume U is a ball). Then $0 = \Omega \cdot \vec{R} = \sum_{\nu=1}^{+\infty} \omega_\nu \cdot \vec{R}$ and since $\omega_\nu \cdot \vec{R}$ is a homogeneous polynomial of degree $\nu + 1$ we conclude that $\omega_\nu \cdot \vec{R} = 0$, $\forall \nu \geqslant 1$. In particular, $\omega_1 \cdot \vec{R} = 0$ and therefore $\omega_1 = \Omega_A$ for some $A \in \mathcal{A}(2m)$. Assume now that $\mathrm{Ker}(\Omega)$ is transverse to $S^{4m-1}(1)$ then it follows from Theorem 4 that Ω has only one singularity which is simple. Because the group of holomorphic transformations of the unit ball acts transitively, we can assume by a holomorphic change of coordinates that the origin is the only singularity of Ω in $B^{4m}[0,1]$ and, since it is a simple singularity, A is non-singular, i.e., $A \in \mathbb{A}(2m)$.

Lemma 5. *Under the above hypothesis there is a real analytic deformation* $\{\mathrm{Ker}(\Omega^t)\}_{t \in [0,1]}$ *of* $\mathrm{Ker}(\Omega)$ *into* $\mathrm{Ker}(\Omega_A)$ *by holomorphic distributions* $\mathrm{Ker}(\Omega^t)$ *transverse to $S^{4m-1}(1)$ outside the intersection $S^{4m-1}(1) \cap \mathrm{Sing}(\Omega^t)$.*

Proof. We define Ω^t by $\Omega^t := t^{-1}\Omega(tz)$. Then Ω^t converges to a holomorphic one-form for each $t \in [0, 1]$ by canonical convergence criteria, indeed, Ω^t defines a holomorphic one-form in U for each t in the closed unit disc $\overline{\mathbb{D}} \subset \mathbb{C}$. We have $\Omega^1 = \Omega$ and an easy computation shows that Ω^t converges for $t \to 0$ to Ω_A where A is the Jacobi-matrix of Ω at the origin. Furthermore, because $\vec{R}(tz) = t\vec{R}(z)$, one has $\Omega^t(z) \cdot \vec{R}(z) = t^{-1}\Omega(tz) \cdot \vec{R}(z) = t^{-2}\Omega(tz) \cdot \vec{R}(tz) = 0$ for every $t \neq 0$. Thus $\mathrm{Ker}(\Omega^t)$ is transverse to $S^{4m-1}(1)$ outside $\mathrm{Sing}(\Omega^t) \cap S^{4m-1}(1)$. $\square$

Now we study the singular set $\mathrm{sing}(\Omega^t)$.

Lemma 6. *We have* $\mathrm{sing}(\Omega^t) = \{0\}$, $\forall t \in \overline{\mathbb{D}}$.

Proof. Take any $z \in S^{4m-1}(1)$. Consider the complex line $\ell(z) = \{tz\colon\ t \in \mathbb{C}\}$ through z. If $0 < |t| < 1$, then $tz \neq 0$ belongs to $B^{4m}[1]$. Therefore $\Omega^t(z) = \frac{1}{t}\Omega(tz)$ is non-zero. If $t = 0$, then $\Omega^0(z) = \Omega_A(z)$ is non-zero on $S^{4m-1}(1)$. This proves the lemma. $\square$

Proof of Theorem 3. For the proof we just have to apply Lemmas 4, 5 and 6 above. $\square$

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