An Explicit Extension Formula of Bounded Holomorphic Functions from Analytic Varieties to Strictly Convex Domains

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We consider a strictly convex domain $D \subset \mathbb{C}^n$ and $m$ holomorphic functions, $\phi_1, \ldots, \phi_m$, in a domain $\Omega \supset D$. We set $V = \{z \in \Omega: \phi_1(z) = \cdots = \phi_m(z) = 0\}$, $M = V \cap D$ and $\partial M = V \cap \partial D$. Under the assumptions that the variety $V$ has no singular point on $\partial M$ and that $V$ meets $\partial D$ transversally we construct an explicit kernel $K(\zeta, z)$ defined for $\zeta \in \partial M$ and $z \in D$ so that the integral operator $Ef(z) = \int_{\zeta \in \partial M} f(\zeta) K(\zeta, z) \, (z \in D)$, defined for $f \in H^\infty(M)$ (using the boundary values $f(\zeta)$ for a.e. $\zeta \in \partial M$), is an extension operator, i.e., $Ef(z) = f(z)$ for $f \in H^\infty(M)$ and $z \in M$ and furthermore $E$ is a bounded operator from $H^\infty(M)$ to $H^\infty(D)$.

INTRODUCTION

A classical theorem of Cartan states that if $\Omega \subset \mathbb{C}^n$ is a domain of holomorphy and $V \subset \Omega$ is an analytic subvariety of $\Omega$ then every holomorphic function on $V$ extends to a holomorphic function on $\Omega$. An analytic version of this theorem was proved by Henkin [7]. He proved that if $D$ is a strictly pseudoconvex domain and $M \subset D$ is a submanifold of $D$ which meets $\partial D$ transversally then there exists a bounded operator $E$ from $H^\infty(M)$ to $H^\infty(D)$ so that $Ef(z) = f(z)$ for $f \in H^\infty(M)$ and $z \in M$.

Henkin's proof of the existence of the extension operator split into two parts: in the first part the proof was carried out in the case $D$ is strictly convex and $M$ is a plane of the form $\{z_{s+1} = \cdots = z_n = 0\}$; in that case an explicit extension formula was given; in the second part, the proof combined a version of the Oka-Cartan theory together with the special case considered in the first part to obtain the existence of the operator $E$ in general.

In this paper we use an integral formula developed in [6] to write down explicitly an extension operator in the case $D$ is a strictly convex domain and $M$ is an analytic subvariety of $D$ with no singular points on $\partial D$ and
which meets $\partial D$ transversally (in particular we allow $M$ to have, finitely many, singular points inside $D$). Then we use some of the ideas of the first part of Henkin's proof to show that our extension operator is a bounded operator from $H^\infty(M)$ to $H^\infty(D)$.

The case codim $M = 1$ was studied by Adachi [1] who used the integral formula of Stout [10] to obtain an extension operator (the integral formula in [6] is a generalization of the integral formula in [10]). The extension to our more general setting is by no means immediate. Moreover our extension operator is, in a sense, more explicit.

A similar problem was also studied by Fornaess [4]. He obtained an extension operator in the case the variety $M$ is biholomorphically equivalent to a domain in $\mathbb{C}^N$, more precisely in the case that $M$ is the image of a strictly pseudoconvex domain under an embedding of it into a strictly convex domain. He used this embedding to obtain an extension operator. Since such an embedding is not explicit enough the corresponding extension operator is not quite explicit either. Our work is, in a sense, simpler and gives an explicit extension operator.

Although there are certain similarities of the local analysis of the extension operator of our paper with the papers by Adachi [1], Fornaess [4], and Henkin [7], there are some further technical difficulties to deal with in our case and it is a purpose of this paper to give the necessary modifications of the arguments to obtain the extension operator in our setting and to study it.

As far as notation is concerned, we will use the standard notation for differential forms (see Rudin [8, Chap. 16]). $H(X)$ denotes the set of holomorphic functions on $X$ and $H^\infty(X)$ the set of bounded holomorphic functions on $X$, equipped, as usual, with the sup-norm: $\|f\|_\infty = \sup_{z \in X} |f(z)|$. Also $A(X)$ is the set of holomorphic functions on $X$ which are continuous on $X$.

**Description of the Setting**

Let $D$ be a bounded strictly convex domain with $C^\infty$-boundary and $\rho$ a defining function for $D$, i.e., $\rho \in C^\infty(\overline{D})$, $D = \{\rho < 0\}$, $\partial D = \{\rho = 0\}$, $d\rho \neq 0$ on $\partial D$ and $\rho$ is a strictly convex function, i.e., the real Hessian of $\rho$ is strictly positive definite. Let $\Omega$ be a convex domain with $\Omega \supset \overline{D}$ and let $\phi_1, \ldots, \phi_m \in H(\Omega)$ $(m < n)$. Define

$$V = \{z \in \Omega: \phi_1(z) = \cdots = \phi_m(z) = 0\}$$

and set

$$M = V \cap D \quad \text{and} \quad \partial M = V \cap (\partial D).$$
Let us introduce the following quantities:

\[ |\nabla (\phi_1, \ldots, \phi_m)(\zeta)|^2 = \sum_{1 \leq j_1 < \cdots < j_m \leq n} \left| \frac{\partial (\phi_1, \ldots, \phi_m)}{\partial (\zeta_{j_1}, \ldots, \zeta_{j_m})} (\zeta) \right|^2 \]

and

\[ |\nabla (\rho, \phi_1, \ldots, \phi_m)(\zeta)|^2 = \sum_{1 \leq j_0 < \cdots < j_n < n} \left| \frac{\partial (\rho, \phi_1, \ldots, \phi_m)}{\partial (\zeta_{j_0}, \zeta_{j_1}, \ldots, \zeta_{j_m})} (\zeta) \right|^2. \]

Our assumptions are:

(i) \(|\nabla (\phi_1, \ldots, \phi_m)| \neq 0\) on \(\partial M\), i.e., \(V\) has no singular point on \(\partial M\), and

(ii) \(V\) meets \(\partial D\) transversally.

Conditions (i) and (ii) are equivalent to the following condition:

(iii) \(|\nabla (\rho, \phi_1, \ldots, \phi_m)| \neq 0\) on \(\partial M\) (see Proposition I.9 of [5]).

(In particular condition (iii) is a geometric condition, i.e., it is independent of the defining function \(\rho\) and of the functions \(\phi_1, \ldots, \phi_m\) which define the variety; of course it depends on the geometry.)

Thus \(V\) is a complex manifold close to \(\partial D\), \(\partial M\) is a smooth \((2n - 2m - 1)\)-dimensional manifold and \(V\) has finitely many singular points inside \(D\).

### The Extension Operator

Let \(\phi_i(\zeta, z) \in H(\Omega \times \Omega)\) so that

\[ \phi_i(\zeta) - \phi_i(z) = \sum_{j=1}^{n} \phi_{ij}(\zeta, z)(\zeta_j - z_j), \quad i = 1, \ldots, m. \]

(Since \(\Omega\) is convex we can write down explicitly a choice of functions \(\phi_{ij}\).)

Set \(\gamma_j(\zeta) = (\partial \rho / \partial \zeta_j)(\zeta), \quad j = 1, \ldots, n\). Then, as is well known (since \(D\) is strictly convex), we have

\[ F(\zeta, z) = \sum_{j=1}^{n} (\zeta_j - z_j) \frac{\partial \rho}{\partial \zeta_j} (\zeta) \neq 0 \quad \text{for} \quad z \in D, \quad \zeta \in \partial D. \]

Define

\[ \alpha(\zeta, z) = \sum_{1 \leq j_0 < \cdots < j_m < n} (-1)^{j_0 + \cdots + j_m} \begin{vmatrix} \gamma_{j_0} & \cdots & \gamma_{j_m} \\ \phi_{j_0} & \cdots & \phi_{j_m} \\ \vdots & \cdots & \vdots \\ \phi_{m_{j_0}} & \cdots & \phi_{m_{j_m}} \end{vmatrix} \wedge \bar{\gamma}_k(\zeta), \]

where

\[ \bar{\gamma}_k(\zeta) = \begin{cases} 1 & \text{if } \gamma_k(\zeta) > 0 \\ 0 & \text{if } \gamma_k(\zeta) < 0 \\ \frac{\gamma_k(\zeta)}{\gamma_k(\zeta) + 1} & \text{otherwise} \end{cases} \]
\[ \tilde{\beta}(\zeta) =: \sum_{1 \leq j_1 < \cdots < j_m \leq n} (-1)^{h + \cdots + j_m} \left( \sum_{k \neq j_1, \ldots, j_m} \frac{\partial \phi_1, \ldots, \phi_m}{\partial (\zeta_{j_1}, \ldots, \zeta_{j_m})} \right) d\zeta_k, \]
\[ \beta(\zeta) =: |\nabla (\phi_1, \ldots, \phi_m)(\zeta)|^{-2} \cdot \tilde{\beta}(\zeta), \]

and
\[ K(\zeta, z) =: c(n, m) \frac{\alpha(z, \zeta) \wedge \beta(\zeta)}{[F(z, \zeta)]^{n-m}}, \]

where \( c(n, m) =: (-1)^{((n-m)(n-m-1)/2) + 1} \cdot ((n-m-1)!/(2\pi i)^{n-m}) \). Let \( f \in H^\infty(M) \). Then, as is well known, the boundary values \( f(\zeta) \), a.e. \( \zeta \in \partial M \) exist for a.e. \( \zeta \in \partial M \) and the function \( \{ f(\zeta): \text{a.e. } \zeta \in \partial M \} \) is in \( L^\infty(\partial M) \). (see Stein [9]). We use these boundary values to define our extension operator as follows:
\[ Ef(z) =: \int_{\zeta \in \partial M} f(\zeta) K(\zeta, z) \quad \text{for } z \in D. \]

Our result is:

**Theorem 1.** The operator \( E \) is a bounded operator from \( H^\infty(M) \) to \( H^\infty(D) \). Moreover \( E \) is an extension operator, i.e.,
\[ Ef(z) = f(z) \quad \text{for } f \in H^\infty(M) \text{ and } z \in M. \]

Obviously if \( f \in H^\infty(M) \) then \( Ef \in H(D) \) and it follows from the integral formula of theorem I.1 of [6] that \( Ef(z) = f(z) \) for \( f \in H^\infty(M) \) and \( z \in M \). It remains to prove that \( E \) is a bounded operator from \( H^\infty(M) \) to \( H^\infty(D) \). We will prove it by showing that
\[ \sup_{z \in D} |Ef(z)| \leq C \sup_{\zeta \in M} |f(\zeta)| \quad \text{for } f \in A(M) \quad (1) \]

for some constant \( C \) which is stable under small perturbations of the boundary of \( D \) and this will complete the proof of theorem 1. As we pointed out in the introduction we will prove (1) using ideas from the first part of Henkin [7]. In fact we show that the technique used by Henkin in the special case \( M \) is a plane works in general, in our setting. This is by no means immediate and we will give the necessary modifications.

**Notation and Comments**

Throughout this paper \( s =: n-m \). In proving (1) we will work close to a fixed point \( p \in \partial M \) and we will always assume that we are working close
enough to $p$ so that the various constructions are possible. Thus fix a point $p \in \partial M$. We may assume that
\[
\frac{\partial (\rho, \phi_1, ..., \phi_m)}{\partial (\zeta_s, \zeta_{s+1}, ..., \zeta_n)} \neq 0 \quad \text{and} \quad \frac{\partial (\phi_1, ..., \phi_m)}{\partial (\zeta_{s+1}, ..., \zeta_n)} \neq 0
\]
in a neighborhood of $p$.

Then it follows, by the implicit function theorem, that the equations
\[\phi_1(\zeta) = \cdots = \phi_m(\zeta) = 0\]
can be solved for $\zeta_{s+1}, ..., \zeta_n$, locally at $p$, giving, say, $\zeta_{s+1} = g_1(\zeta'), ..., \zeta_n = g_m(\zeta')$ for some holomorphic functions $g_1, ..., g_m$ of $\zeta'$ ($\zeta' = (\zeta_1, ..., \zeta_s)$ if $\zeta = (\zeta_1, ..., \zeta_n)$).

It will be convenient to use the following notation:
\[G =: (g_1, ..., g_m) \quad \text{and} \quad h_j(\zeta) =: \zeta_{s+j} - g_j(\zeta'), \quad j = 1, ..., m.\]

Let us point out that (2) implies
\[
\frac{\partial (\rho, h_1, ..., h_m)}{\partial (\zeta_s, \zeta_{s+1}, ..., \zeta_n)} \neq 0
\]
or
\[
\frac{\partial \rho}{\partial \zeta_s} + \sum_{j=1}^{m} \frac{\partial \rho}{\partial \zeta_{s+j}} \frac{\partial g_j}{\partial \zeta_s} \neq 0 \quad (2')
\]
(this follows from the chain rule; see Proposition I.9 of [5]).

We will also use factorizations of $g_j(\zeta')$,
\[g_j(\zeta') - g_j(\zeta) = \sum_{k=1}^{s} g_{jk}(\zeta, \zeta')(\zeta_k - \zeta_k) \quad (3)
\]
for some holomorphic functions $g_{jk}$.

Note that
\[g_{jk}(p', z') = \frac{\partial g_j}{\partial \zeta_k} (p').\]

Sometimes it will be convenient to write $D_k$ for $\partial / \partial \zeta_k$ (e.g., $D_k \rho = \partial \rho / \partial \zeta_k$) and $z''$ for $(z_1, ..., z_{s-1})$ (if $z = (z_1, ..., z_n)$). Finally the symbols $\leq$ and $\approx$ will mean the following:

\[A \leq B \iff \text{there exists a finite positive constant } c_0, \text{ depending only on } M, \text{ so that } A \leq c_0 B.\]

and
\[A \approx B \iff A \leq B \quad \text{and} \quad B \leq A.\]
For the proof of Theorem 1 we need some preparation. The following lemma is of the implicit function theorem type and its proof is based on a fixed point argument.

**Lemma 1.** If \( z \) is sufficiently close to \( p \) then there exists a unique point \( w = w(z) = (w_1(z), \ldots, w_n(z)) \) so that

\[
\begin{align*}
  w_k(z) &= z_k & \text{for } k = 1, \ldots, s - 1, \\
  F(w(z), z) &= 0, \\
  h_j(w(z)) &= 0 & \text{for } j = 1, \ldots, m,
\end{align*}
\]

i.e., \( w(z) \) has the same first \((s - 1)\) coordinates with \( z \), it lies on the variety \( V \) and \( z \) lies on the complex tangent space to the hypersurface \( \rho = \rho(w(z)) \) at the point \( w(z) \). Recall

\[
F(\xi, z) = \sum_{j=1}^{n} \frac{\partial \rho}{\partial \xi_j} (\xi_j - z_j).
\]

**Proof.** To solve (4) for \( w \) it suffices to solve the following system for \((w_1, \ldots, w_n)\):

\[
\begin{align*}
  D_s \rho(z'', w_s, \ldots, w_n) \cdot (w_s - z_s) + \sum_{j=1}^{m} D_{s+j} \rho(z'', w_s, \ldots, w_n) \cdot (w_{s+j} - z_{s+j}) &= 0, \\
  w_{s+j} &= g_j(z'', w_s), & j = 1, \ldots, m
\end{align*}
\]

By substituting the last \( m \) equations of (5) into the first one and using (3) we see that it suffices to find \( w_s \) which satisfies the following equation:

\[
w_s = z_s + \frac{\sum_{i=1}^{m} D_{s+i} \rho(\bar{G}(z'', w_s))(z_{s+i} - g_j(z'', z_s))}{D_s \rho(\bar{G}(z'', w_s)) + \sum_{j=1}^{m} D_{s+j} \rho(\bar{G}(z'', w_s)) g_j((z'', z_s), (z'', w_s))},
\]

where we have set

\[
\bar{G}(z'', w_s) =: (z'', w_s, \bar{G}(z'', w_s)).
\]

Note that the denominator in the ratio of (6) is different from 0 by (2)' (since \( g_{js}(z', z') = (\partial g_j / \partial \xi_s)(\bar{G}(z')) \)); recall that we always assume that we are working sufficiently close to the point \( p \).

We will show that (6) has a solution for \( w_s \) by showing that the following function has a fixed point,

\[
\eta(x) =: z_s + \frac{\sum_{i=1}^{m} D_{s+i} \rho(\bar{G}(z'', x))(z_{s+i} - g_j(z'', z_s))}{D_s \rho(\bar{G}(z'', x)) + \sum_{j=1}^{m} D_{s+j} \rho(\bar{G}(z'', x)) g_j((z'', z_s), (z'', x))},
\]
i.e., we will show that there exists $w_s$ so that

$$\eta(w_s) = w_s$$

(the function $\eta(x)$ is defined on $\{x \in \mathbb{C}: |x - z_s| \leq \delta\}$ for some small $\delta > 0$). Clearly

$$|\eta(x) - z_s| = O \left( \sum_{j=1}^{m} |z_{s+j} - g_j(z')| \right) = O(|z - p|)$$

and therefore if $z$ is sufficiently close to $p$ and $\delta$ sufficiently small then

$$|x - z_s| \leq \delta \Rightarrow |\eta(x) - z_s| \leq \delta.$$  

Also

$$|\eta(x) - \eta(y)| = O \left( |x - y| \sum_{j=1}^{m} |z_{s+j} - g_j(z')| \right) = O(|x - y| \cdot |z - p|)$$

and thus if $z$ is sufficiently close to $p$ then

$$|\eta(x) - \eta(y)| \leq \frac{1}{2} |x - y|$$

(it is easy to introduce appropriate constants and make the above arguments more precise; see also Henkin [7]).

Thus $\eta$ is a contraction of $\{x \in \mathbb{C}: |x - z_s| \leq \delta\}$ to itself and as such it has a unique fixed point $w_s = w_s(z)$. This completes the proof of the lemma.

**Lemma 2.** The point $w = w(z)$ constructed in lemma 1 satisfies the following:

(i) $|z - w(z)|^2 \approx |\rho(z) - \rho(w(z))|,$

(ii) $|z - w(z)| \approx \sum_{j=1}^{m} |h_j(z) - h_j(w(z))| = \sum_{j=1}^{m} |h_j(z)|,$

(iii) $w(z) \in M$ provided that $z \in \bar{D}$.

**Proof.** Since $\rho$ is strictly convex we have (as is well known):

$$2 \Re F(\zeta, z) \geq \rho(\zeta) - \rho(z) + c_1 |\zeta - z|^2$$  

for some constant $c_1 > 0$. But

$$F(w(z), z) = 0$$

and hence we obtain

$$\rho(z) - \rho(w(z)) \geq |z - w(z)|^2.$$  

In particular (iii) holds.
On the other hand (by Taylor’s expansion) we have

$$\rho(z) = \rho(w(z)) - 2 \text{Re} F(w(z), z) + O(|z - w(z)|^2)$$

and again by (8)

$$|\rho(z) - \rho(w(z))| \lesssim |z - w(z)|^2$$

which together with (9) proves (i). Next we prove (ii). Obviously

$$\sum_{j=1}^m |h_j(z) - h_j(w(z))| \lesssim |z - w(z)|.$$

For the other direction let us consider the matrix

$$T = \begin{bmatrix} D_s \rho & D_{s+1} \rho & \cdots & D_n \rho \\ D_s h_1 & D_{s+1} h_1 & \cdots & D_n h_1 \\ \vdots & \vdots & \ddots & \vdots \\ D_s h_m & D_{s+1} h_m & \cdots & D_n h_m \end{bmatrix}.$$ 

Since det($T$) $\neq 0$ (by (2)') we may assume (without loss of generality) that:

either

(*) \{D_n h_m\} $\neq 0,$

or

(**) $D_n \rho \neq 0.$

(\{D_n h_m\} denotes the minor determinant of $T$ which corresponds to the term $D_n h_m$, i.e., the determinant of the $m \times m$ matrix obtained from $T$ by eliminating the column and the row to which the term $D_n h_m$ belongs.) Suppose (*) holds. Then

$$|h_m(z) - h_m(w)| = |(z_n - w_n) - (g_m(z') - g_m(w'))|$$

$$- |(z_n - w_n) - g_m(z_s - w_s)| \quad \text{by (3).} \quad (10)$$

($g_{\rho} = g_{\rho}(z', w')$). But (4) gives

$$D_s \rho(w)(z_s - w_s) = - \sum_{j=1}^m D_{s+j} \rho(w)(z_{s+j} - w_{s+j})$$

$$= - \sum_{j=1}^{m-1} D_{s+j} \rho(w)(z_{s+j} - w_{s+j}) - D_n \rho(w)(z_n - w_n)$$

$$= - \sum_{j=1}^{m-1} D_{s+j} \rho(w)[g_j(z') - g_j(w')]$$

$$+ \sum_{j=1}^{m-1} D_{s+j} \rho(w)[g_j(z') - z_{s+j}] - D_n \rho(w)(z_n - w_n)$$
or, by (3),

$$\left[ D_s \rho(w) + \sum_{j=1}^{m-1} D_{s+j} \rho(w) g_{js} \right] (z_s - w_s)$$

$$= -D_n \rho(w) (z_n - w_n) + \sum_{j=1}^{m-1} D_{s+j} \rho(w) [g_j(z') - z_{s+j}]. \quad (11)$$

Let

$$P = D_s \rho(w) + \sum_{j=1}^{m-1} D_{s+j} \rho(w) g_{js}.$$

By (\*), \(P \neq 0\) and thus by (11)

$$z_s - w_s = -\frac{D_n \rho(w)}{P} (z_n - w_n) + \sum_{j=1}^{m-1} \frac{D_{s+j} \rho(w)}{P} [g_j(z') - z_{s+j}]. \quad (12)$$

Substituting (12) in (10) we obtain

$$|h_m(z) - h_m(w)| \approx \left| \det(T)(z_n - w_n) - g_{ms} \sum_{j=1}^{m-1} D_{s+j} \rho(w) [g_j(z') - z_{s+j}] \right|. \quad (13)$$

Now (13) gives

$$|h_m(z) - h_m(w)| \geq |z_n - w_n| - \sum_{j=1}^{m-1} |g_j(z') - z_{s+j}|$$

or

$$\sum_{j=1}^{m} |h_j(z) - h_j(w)| \geq |z_n - w_n|. \quad (14)$$

But

$$|z - w| \leq |z_n - w_n| + \sum_{j=1}^{m-1} |z_{s+j} - w_{s+j}| + |z_s - w_s|$$

$$\leq |z_n - w_n| + \sum_{j=1}^{m-1} |h_j(z) - h_j(w)| + |z_s - w_s| \quad \text{(by (3))}$$

$$\leq |z_n - w_n| + \sum_{j=1}^{m-1} |h_j(z) - h_j(w)| \quad \text{(by (12))}$$

$$\leq \sum_{j=1}^{m} |h_j(z) - h_j(w)| \quad \text{(by (14))}$$
which proves (ii) in the case (*) holds. Now suppose (**) holds. Then
\[ z_n - w_n = - \sum_{j=0}^{m-1} \frac{D_{s+j} \rho(w)}{D_n \rho(w)} (z_{s+j} - w_{s+j}). \] (15)

Substitute (15) in (10) to obtain
\[
|h_m(z) - h_m(w)| = \left| \sum_{j=1}^{m-1} \frac{D_{s+j} \rho(w)}{D_n \rho(w)} (z_{s+j} - w_{s+j}) + g_{mS}(z_s - w_s) \right|
\]
\[= \left| \sum_{j=1}^{m-1} \frac{D_{s+j} \rho(w)}{D_n \rho(w)} [(z_{s+j} - g_j(z')) + (g_j(z') - g_j(w'))] \right|
\]
\[+ \frac{D_s \rho(w)}{D_n \rho(w)} (z_s - w_s) + g_{mS}(z_s - w_s) \right| \quad \text{(by (3))}
\]
\[= \left| \sum_{j=1}^{m-1} \frac{D_{s+j} \rho(w)}{D_n \rho(w)} [(z_{s+j} - g_j(z')) + g_j(z_s - w_s)] \right|
\]
\[+ \frac{D_s \rho(w)}{D_n \rho(w)} (z_s - w_s) + g_{mS}(z_s - w_s) \right|
\]
\[\approx \left| \det(T)(z_s - w_s) + \sum_{j=1}^{m-1} D_{s+j} \rho(w)(z_{s,j} - g_j(z')) \right|
\]
\[\geq |z_s - w_s| - \sum_{j=1}^{m-1} |z_{s+j} - g_j(z')|\]
or
\[\sum_{j=1}^{m} |h_j(z) - h_j(w)| \geq |z_s - w_s|. \] (16)

But
\[ |z - w| \leq \sum_{j=s+1}^{n} |z_j - w_j| + |z_s - w_s| \]
\[\leq |z_s - w_s| + \sum_{j=1}^{m} |h_j(z) - h_j(w)| \quad \text{(by (3))}
\]
\[\leq \sum_{j=1}^{m} |h_j(z) - h_j(w)| \quad \text{(by (16))},
\]
which proves (ii) holds in the case (**). This concludes the proof of the lemma.
LEMMA 3. Fix a \( z \) close to \( p \). Then there exist real coordinates \( t_1, t_2, \ldots, t_{2n} \) for points \( \zeta \) close to \( z \) so that

\[
\begin{align*}
    t_1 + it_2 &= (\rho(\zeta) - \rho(z)) + i \text{Im} F(\zeta, z), \\
    t_3 + it_4 &=, \\
    &\vdots \\
    t_{2s-1} + it_{2s} &=, \\
    t_{2s+1} + it_{2s+2} &= h_1(\zeta) - h_1(z), \\
    &\vdots \\
    t_{2n-1} + it_{2n} &= h_m(\zeta) - h_m(z).
\end{align*}
\]

Proof. Let \( u_0 = :\rho(\zeta) - \rho(z), \quad v_0 = :\text{Im} F(\zeta, z), \quad u_j = \text{Re}[h_j(\zeta) - h_j(z)] \) and \( v_j = \text{Im}[h_j(\zeta) - h_j(z)], \quad j = 1, \ldots, m \). Write also

\[\zeta_j = x_j + iy_j \quad (x_j, y_j \in \mathbb{R}).\]

To prove the lemma it suffices to show that the \( 2m + 2 \) vectors (of \( \mathbb{R}^{2n} \))

\[\{ \nabla u_k, \nabla v_k | k = 0, 1, \ldots, m \} \]

are linearly independent (over \( \mathbb{R} \)). \( (17) \)

Here \( \nabla u_0 = \nabla \zeta u_0 \rvert_{\zeta = z} = (\partial u_0/\partial x_1, \partial u_0/\partial y_1, \ldots, \partial u_0/\partial x_n, \partial u_0/\partial y_n) \rvert_{\zeta = z} \) and similarly for the others. But it is easy to see that

\[
\begin{align*}
    \frac{\partial v_0}{\partial x_j} \rvert_{\zeta = z} &= -\frac{\partial u_0}{\partial y_j} \rvert_{\zeta = z}, \\
    \frac{\partial v_0}{\partial y_j} \rvert_{\zeta = z} &= \frac{\partial u_0}{\partial x_j} \rvert_{\zeta = z}
\end{align*}
\]

and therefore

\[
\frac{\partial (u_0, v_0, \ldots, u_m, v_m)}{\partial (x, y, \ldots, x_n, y_n)} \rvert_{\zeta = z} = 4^n \left| \frac{\partial (\rho, h_1, \ldots, h_m)}{\partial (\zeta, \zeta + 1, \ldots, \zeta n)} (z) \right|^2 \neq 0 \quad (18)
\]

(this follows from Lemma 1.3.5 of Rudin [8]).

Now (18) implies (17) and completes the proof of the lemma.

Remarks. The coordinates \( t_1, \ldots, t_{2n} \) of Lemma 3 satisfy the following relation

\[
|\zeta - z|^2 \approx t_1^2 + \cdots + t_{2n}^2 \approx \sum_{k=1}^{2n} t_k^2 + \sum_{j=1}^{m} |h_j(z) - h_j(\zeta)|^2.
\]

Let us set

\[\varepsilon = \varepsilon(z) = |z - w(z)|.\]

Then, by Lemma 2, we have

\[|\rho(z) - \rho(w(z))| \approx \varepsilon^2\]
and
\[\sum_{j=1}^{m} |h_j(z) - h_j(w(z))| = \sum_{j=1}^{m} |h_j(z)| \approx \varepsilon\]
and therefore
\[|\zeta - z|^2 \approx \sum_{k=1}^{2g} t_k^2 + \varepsilon^2 \quad \text{if} \quad \zeta \in M\]  \hspace{1cm} (19)
(of course "\(\approx\)" in (19) is uniform in \(z\)). Also
\[|F(\zeta, z)| \approx |\text{Re} F(\zeta, z)| + |\text{Im} F(\zeta, z)| \geq |t_1| + |t_2| + |\zeta - z|^2\]
(this is well known and follows from the strict convexity) or
\[|F(\zeta, z)| \geq |t_1| + |t_2| + t_3^2 + \cdots + t_{2g}^2 + \varepsilon^2\]  \hspace{1cm} (20)
for \(\zeta \in M\) (close to \(z\)).
Also note that \(\zeta \in \partial M\) means \(t_i = 0\) provided that \(z \in \partial D - \partial M\).
It is important to keep in mind that the estimates that follow are uniform in \(z\).

**Lemma 4.** Fix an \(f \in A(M)\) and a \(z \in \partial D - \partial M\). If \(w = w(z)\) denotes the point of Lemma 1 then we have
\[\left|\frac{d}{d\lambda} (Ef)(w + \lambda(z - w))\right|_{\lambda = 1} \leq \|f\|_\infty \left[\int_{\zeta \in \partial M} \frac{|z - w|}{|F(\zeta, z)|^s} d\sigma(\zeta) + \int_{\zeta \in \partial M} \frac{\sum_{j=1}^{n} D_j \rho(\zeta)(z_j - w_j)}{|F(\zeta, z)|^{s+1}} d\sigma(\zeta)\right]\]  \hspace{1cm} (21)
\((d\sigma\) is the Euclidean volume element of \(\partial M\) and \(\|f\|_\infty = \sup_{\zeta \in M} |f(\zeta)|\)).

**Proof.** Recall that
\[Ef(\zeta) = c(n, m) \int_{\zeta \in \partial M} f(\zeta) \frac{\alpha(\zeta, \zeta) \wedge \beta(\zeta)}{|F(\zeta, \zeta)|^{s}} \quad \text{for} \quad \zeta \in \partial D \setminus \partial M.\]
Now the coefficient \(Q(\zeta, \zeta)\) of \(d\zeta_1 \wedge \cdots \wedge d\zeta_{s-1} \wedge \beta(\zeta)\) in \(\alpha(\zeta, \zeta) \wedge \beta(\zeta)/|F(\zeta, \zeta)|^s\) is
\[Q(\zeta, \zeta) = C \det \left[\gamma_j, \phi_{ij}, \cdots, \phi_{jm}, \frac{\partial \gamma_j}{\partial \zeta_1}, \cdots, \frac{\partial \gamma_j}{\partial \zeta_{s-1}}\right] \cdot (F(\zeta, \zeta))^{-s}.\]  \hspace{1cm} (22)
(C will denote a constant which depends only M and which may be different in different places.) It follows from (22) that

\[ Q(\zeta, w + \lambda(z - w)) = \frac{N(\zeta, w + \lambda(z - w))}{[F(\zeta, w + \lambda(z - w))]^s}, \]

where \( N \) is some smooth function. Now

\[ \left. \frac{\partial N}{\partial \lambda} \right|_{\lambda = 1} = O(|z - w|) \]

and

\[ \left. \frac{\partial}{\partial \lambda} \right|_{\lambda = 1} (F(\zeta, w + \lambda(z - w))) = \sum_{j=1}^{n} D_j \rho(\zeta)(z_j - w_j) \]

and therefore by (23) we obtain

\[ \left. \frac{\partial}{\partial \lambda} \right|_{\lambda = 1} (Q(\zeta, w + \lambda(z - w))) = O\left(\frac{|z - w|}{|F(\zeta, z)|^s}\right) + O\left(\frac{\sum_{j=1}^{n} D_j \rho(\zeta)(z_j - w_j)}{|F(\zeta, z)|^{s+1}}\right) \]

which implies (21) and completes the proof of the lemma.

Next note that

\[ \left| \sum_{j=1}^{n} D_j \rho(\zeta)(z_j - w_j) \right| = \left| \sum_{j=1}^{n} (D_j \rho(\zeta) - D_j \rho(w))(z_j - w_j) \right| \leq \varepsilon \cdot |\zeta - w| \leq \varepsilon (|\zeta - z| + |z - w|) \leq \varepsilon (\varepsilon + |\zeta - z|) \]

(recall \( \varepsilon = |z - w(z)| \)).

Now using the coordinates \( t_1, \ldots, t_{2n} \), their properties (19) and (20) together with (24) and Lemma 4, we can prove, exactly as in Henkin [7], the following

\[ \left. \frac{d}{d\lambda} \right|_{\lambda = 1} (Ef)(w + \lambda(z - w)) \leq C\|f\|_{\infty} \quad \text{for} \quad z \in (\partial D) \setminus (\partial M). \]

**Lemma 5.** We have

\[ \sup_{0 \leq \lambda \leq 1} \left. \frac{d}{d\lambda} \right|_{\lambda = 1} (Ef)(w + \lambda(z - w)) \leq C\|f\|_{\infty} \quad \text{for} \quad z \in (\partial D) \setminus (\partial M). \]

**Proof.** As in Henkin [7], let us consider

\[ \Delta = \{ \lambda \in \mathbb{C} : z(\lambda) = w + \lambda(z - w) \in D \}. \]
Since $D$ is convex so is $\Delta$ and $[0, 1] \subset \Delta$ and $0 \in \Delta$. Since $F(w(z), z) = 0$ we have

$$F(w(z), z(\lambda)) = 0. \quad (26)$$

Hence $w(z)$ is the point associated to $z(\lambda)$ by Lemma 1. Also if $\lambda \in \partial \Delta$ then $z(\lambda) \in \partial D$, i.e., $\rho(z(\lambda)) = 0$ and therefore, by Lemma 2 and $z \in \partial D$,

$$|z(\lambda) - w| \approx |\rho(z(\lambda)) - \rho(w)|^{1/2} = |\rho(w)|^{1/2} \approx \varepsilon. \quad (27)$$

But, by (27),

$$|\lambda| \varepsilon \geq |\lambda| |z - w| = |z(\lambda) - w| \approx \varepsilon \quad \text{if} \quad \lambda \in \partial \Delta.$$

Hence

$$|\lambda| \gtrsim 1 \quad \text{for} \quad \lambda \in \partial \Delta. \quad (28)$$

Now, by (25),

$$\left| \frac{d}{dt} (Ef)(w + \tau(z(\lambda) - w)) \right|_{\tau = 1} \lesssim \|f\|_{\infty}. \quad (29)$$

But (29) implies

$$\left| \frac{d}{d\lambda} (Ef)(w + \lambda(z - w)) \right| \lesssim \frac{1}{|\lambda|} \|f\|_{\infty}$$

and by (28)

$$\left| \frac{d}{d\lambda} (Ef)(w + \lambda(z - w)) \right| \lesssim \|f\|_{\infty} \quad (\lambda \in \partial \Delta)$$

which by the maximum principle (with respect to $\lambda$) gives

$$\left| \frac{d}{d\lambda} (Ef)(w + \lambda(z - w)) \right| \lesssim \|f\|_{\infty} \quad \text{for} \quad \lambda \in \Delta.$$

This proves Lemma 5.

*Proof of Theorem 1.* For $z \in \partial D - \partial M$ we have

$$|Ef(z) - Ef(w)| = \left| \int_0^1 \frac{d}{d\lambda} (Ef)(w + \lambda(z - w)) \ d\lambda \right|
\lesssim \|f\|_{\infty} \quad \text{(by lemma 5)}$$

and therefore

$$|Ef(z)| \lesssim \|f\|_{\infty}.$$
Hence by the maximum principle (its version on analytic varieties) we have

$$|Ef(z)| \leq C\|f\|_{\infty} \quad \text{for} \quad z \in \bar{D}\setminus M. \quad (30)$$

This proves (1). Finally it is clear that the constant $C$ in (30) is stable under small perturbations of the boundary of $D$. This completes the proof of Theorem 1.

Remarks. (i) One can go further and show (as in Henkin [7]) that $E$ maps $A(M)$ to $A(D)$ and (as in Elgueta [3] and Adachi [2]) that $E$ maps $C^\omega (\bar{M})$ to $C^\omega (\bar{D})$.

(ii) However, it seems nontrivial to study the operator $E$ in case $D$ is weakly-convex (the operator $E$ can be defined the same way as in the strictly-convex case). It would also be interesting to study the operator $E$, more generally, in the $H^p$-context.

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