Symplectic manifolds and formality

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For Steve Halperin on the occasion of his fiftieth birthday

Abstract


We study some questions about symplectic manifolds, using techniques of rational homotopy theory. Our questions and results focus around formality properties of symplectic manifolds. We assume the presence of a symplectic structure on a manifold, and establish extra conditions sufficient to imply formality. The conditions are phrased in terms of the minimal model. In addition we study the question of whether or not a manifold can admit a Kähler structure. We use our results to give examples of non-aspherical symplectic manifolds that do not admit a Kähler structure.

1. Introduction and notation

A symplectic manifold that does not admit a Kähler structure was first described in the literature by Thurston [23]. Subsequently a number of authors have described such manifolds, often using the criterion of formality (cf., for example, [4] and [2]). Typically these authors have constructed a symplectic manifold that is not formal, then invoked the well-known result of [5], that a Kähler manifold is formal, to conclude that the manifold does not admit a Kähler structure. Their examples of non-formal, symplectic manifolds all have non-trivial fundamental groups; indeed, most are nilmanifolds. This suggests the following question: Is every simply connected, compact, symplectic manifold a formal space? We are unable to answer this question in general; nonetheless it serves as a focus for our work.

An immediate difficulty encountered with this question is the lack of examples. The familiar simply connected, compact, symplectic manifolds, such as complex projective space $\mathbb{CP}^n$, are also Kähler manifolds. Hence they are formal by the result of [5] referred to above. In fact, it was relatively recently that McDuff gave the first example of a simply connected, compact, symplectic manifold that does not admit a Kähler structure [16]. One of her examples is described as follows: Let $M$ be the four-dimensional symplectic manifold described by Thurston in [23]. Take a...
symplectic embedding $i: M \to \mathbb{C}P^5$ of $M$ into $\mathbb{C}P^5$. Now blow up along this submanifold, to obtain a simply connected, compact symplectic 10-manifold, denoted $\mathbb{C}P^5$, that does not satisfy the hard Lefschetz theorem and hence does not admit a Kähler structure. This gives a special case of our question: Is $\mathbb{C}P^5$ formal? In general it seems quite difficult to analyse the homotopy-theoretic properties of the blow-up construction, and at present we cannot answer even this special case of our question.

In this paper, we take a different approach and prove several results which bear on the above question. We establish an affirmative answer to the question in some special cases. As illustrated by our examples, the approach that we develop is necessarily inconclusive as regards the general question. Still, our results suggest that a symplectic structure has a “formalising tendency” in a sense to be made clearer below.

We will prove the following:

**Corollary 2.3.** Let $X$ be a simply connected, compact manifold that has a pure minimal model. If $X$ admits a symplectic structure, then $X$ is formal.

**Corollary 2.6.** Let $X$ be a simply connected, compact, coformal manifold. If $X$ admits a symplectic structure, then $X$ is formal.

Actually, our results are more general than stated here, and apply to a larger class of spaces. We remark that a simply connected, compact manifold may have a pure minimal model and yet not be formal. Likewise, a simply connected, compact manifold may be coformal yet not formal. Thus our theorems illustrate that a symplectic structure entails formality in the presence of carefully chosen, but natural, rational homotopy-theoretic side-conditions. This is the sense in which a symplectic structure displays a formalising tendency.

These results do not require the manifold to be symplectic, but only that it resembles a compact symplectic manifold from the point of view of the rational cohomology algebra. Call such a rational cohomology algebra a symplectic algebra (see Section 2 for the precise definition). Then our approach can be explained as follows: We restrict attention to manifolds with certain kinds of minimal models, which are especially sensitive to the requirement that their cohomology algebra be a symplectic algebra. In particular, this requirement is then sufficient to imply formality (Theorem 2.1). Our results are not best possible, and at the end of Section 2 we suggest how they might be generalized. However, we give several examples (Examples 2.9, 2.10 and 2.12) to show the requirement that the rational cohomology algebra be a symplectic algebra is not alone sufficient to imply formality.

The remainder of the paper can be described as follows. The proof of Theorem 2.1 gives a little more information, and allows us to conclude the following result:

**Corollary 2.5.** Any simply connected, symplectic homogeneous space is a maximal rank homogeneous space.

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1Steve Halperin informs us that this result is contained in [8, Vol. III].
In Sections 3 and 4, we broaden our investigation and consider other situations in which requiring a certain structure in the rational cohomology algebra, for example a Lefschetz structure, is sufficient to imply formality. These results are intended to support the point of view adopted in Section 2, that imposing a rich structure on the rational cohomology algebra has implications for the rational homotopy— at least if one restricts attention to appropriate kinds of minimal models. In Section 3 we give a rational homotopy theoretic proof of the following:

**Theorem 3.1.** Let \( X \) be a compact \( K(\pi, 1) \) with \( \pi \) nilpotent. If \( H^*(X; \mathbb{Q}) \) is a Lefschetz algebra, then \( X \) has the homotopy type of a torus, and hence is formal.

From this theorem, we deduce the following corollary:

**Theorem 3.5.** If \( X \) is a nilmanifold and \( H^*(X; \mathbb{Q}) \) is a Lefschetz algebra, then \( X \) is diffeomorphic to a torus.

This is apparently a result due to Koszul, rediscovered by Benson and Gordon [2, Theorem A] (see [17]). The interesting aspect of Theorems 3.1 and 3.5, in the present context, is the proof that we give: We show that for a nilpotent \( K(\pi, 1) \), formality follows from the requirement that the cohomology algebra be a Lefschetz algebra. Our minimal model approach also applies in a slightly more general context. In Section 4 we use this approach to give examples of symplectic manifolds which are not \( K(\pi, 1) \)'s, and neither admit a Kähler structure nor satisfy the hard Lefschetz theorem.

We end this section by fixing notation and reviewing some ideas from rational homotopy theory (see [9], [10] and [11] for example). All vector spaces and algebras are non-negatively graded and over the rationals \( \mathbb{Q} \). We use the prefix DG to stand for differential graded. A **DG algebra** is a pair \((A, d_A)\), where \( A \) is a graded commutative, associative algebra and \( d_A \) is a degree + 1 differential of \( A \). Any DG algebra \((A, d_A)\) that we consider in this paper satisfies \( H^0(A, d_A) = \mathbb{Q} \) and \( H^n(A, d_A) \) is a finite-dimensional vector space for each \( n \). We denote the ideal of positive degree elements in an algebra \( A \) by \( A^+ \). If \( V \) is a vector space, then \( \Lambda V \) denotes the free graded commutative algebra generated by \( V \). If \( \{v_1, v_2, \ldots\} \) is a basis for \( V \), then we write \( V = \langle v_1, v_2, \ldots \rangle \) and \( \Lambda V = \Lambda(v_1, v_2, \ldots) \). A DG algebra is **minimal** if (1) as an algebra, \( A \cong \Lambda V \) for some \( V \) and (2) there is a basis \( V = \langle v_1, v_2, \ldots \rangle \) such that, for each \( j \), \( dv_j \in (A(v_1, \ldots, v_{j-1}))^+ (A(v_1, \ldots, v_{j-1}))^+ \). In particular, differentials of generators are decomposable in a minimal DG algebra. We will write a minimal DG algebra as \( A(V; d) \), or \( A(v_1, v_2, \ldots; d) \) if \( V = \langle v_1, v_2, \ldots \rangle \). Any DG algebra \((A, d_A)\) has a **minimal model**, i.e., a minimal DG algebra \( A(V; d) \) with a DG homomorphism \( \rho: A(V; d) \rightarrow (A, d_A) \) such that the induced homomorphism on cohomology \( \rho^* \) is an isomorphism.

A space \( X \) is called **nilpotent** if its fundamental group \( \pi_1(X) \) is a nilpotent group and the natural action of \( \pi_1(X) \) on \( \pi_n(X) \) is a nilpotent action. For example, simply connected spaces or \( K(\pi, 1) \)'s with \( \pi \) nilpotent are nilpotent spaces. Given a space \( X \), there is a functorially associated DG algebra of rational polynomial forms on \( X \),
denoted $A(X)$. A minimal model for a space $X$ is a minimal model for $A(X)$. A basic theorem of rational homotopy theory asserts that each nilpotent space $X$ has a minimal model, which contains all the rational homotopy information about the space. For example, if $A(V; d)$ is the minimal model of $X$, then $H^*(A(V; d)) \cong H^*(X; \mathbb{Q})$ and for $i > 1$, $V^i \cong \text{Hom}(\pi_i(X), \mathbb{Q})$.

A minimal DG algebra $A(V; d)$ is elliptic if the graded vector space $V$ and the cohomology $H^*(A(V; d))$ are finite-dimensional. In this case the homotopy Euler characteristic, defined as $\chi = \dim V^{\text{even}} - \dim V^{\text{odd}}$, becomes a salient rational homotopy invariant (see [10]). A minimal DG algebra $A(V; d)$ is pure if $d(V^{\text{even}}) = 0$ and $d(V^{\text{odd}}) \subseteq A(V^{\text{even}})$. We write a pure minimal DG algebra as $A(X, Y; d)$ with $X = V^{\text{even}}$ and $Y = V^{\text{odd}}$. A space is called elliptic, respectively pure, if its minimal model is elliptic, respectively pure. Pure spaces and elliptic spaces abound in homotopy theory. For example, any homogeneous space $G/H$ is both pure and elliptic, and also satisfies $\dim_\mathbb{Q}(X) = \text{rk}(H)$ and $\dim_\mathbb{Q}(Y) = \text{rk}(G)$ (see [11]). In this case therefore, $\chi(G/H) = \text{rk}(H) - \text{rk}(G)$.

A minimal DG algebra $A(V; d)$ is called formal if there is a DG homomorphism $\psi: A(V, d) \to H^*(A(V; d))$ that induces an isomorphism on cohomology. A space is called formal if its minimal model is formal. We will often use the following criterion for formality:

**Theorem** [5, Theorem 4.1]. A minimal DG algebra $A(V; d)$ is formal if, and only if, $V$ decomposes as a direct sum $V = C \oplus N$ with $d(C) = 0$ and $d$ injective on $N$, such that every closed element in the ideal generated by $N$ is exact. □

We say a graded algebra $\mathcal{A}$ has a second grading if there is a (graded) vector space decomposition $\mathcal{A} = \bigoplus_{r \in \mathbb{Z}} \mathcal{A}_r$, such that $\mathcal{A}_r: \mathcal{A}_s \subseteq \mathcal{A}_{r+s}$. A bigraded DG algebra is a DG algebra $(\mathcal{A}, d_{\mathcal{A}})$ with a second grading of $\mathcal{A}$ for which $d_{\mathcal{A}}: \mathcal{A}_r \to \mathcal{A}_{r-1}$. If $(\mathcal{A}, d_{\mathcal{A}})$ is a bigraded DG algebra, then the second grading on $\mathcal{A}$ carries over to the cohomology algebra $H^*(\mathcal{A}, d_{\mathcal{A}})$ giving it the structure of a bigraded algebra.

We often consider minimal DG algebras $A(V; d)$ which are bigraded. In this case, a second grading is determined by assigning lower degrees to basis elements for the vector space $V$, then extending multiplicatively to the whole algebra $A(V)$. In order that this gives $A(V; d)$ the structure of a bigraded DG algebra, it is necessary and sufficient that $d: V_r \to (A V)_{r-1}$ for each $r$. In general, we do not assume that our second gradings are non-negative. For minimal DG algebras, however, they will always take the form $V = \bigoplus_{j \geq 1} V_j$, with $\{r_j\}$ increasing, i.e., the lower degrees of the generators will be bounded below.

2. The formalising tendency of a symplectic structure

Let $H$ be a graded rational Poincaré duality algebra of top degree $2n$. We say $H$ is a symplectic algebra if there is an element $\omega \in H^2$ with $\omega^n$ non-zero in $H^{2n}$. We refer to
such an element \( \omega \) as a symplectic class for \( H \). We say \( H \) is a Lefschetz algebra if there is an element \( \omega \in H^2 \), such that each map of vector spaces \( H^r \to H^{r+s} \) given by multiplication with \( \omega^r \) is an isomorphism for \( r = 1, \ldots, n \). We refer to such an element \( \omega \) as a Kähler class for \( H \). Clearly, if \( \omega \) is a Kähler class for an algebra, then it is also a symplectic class. Note that a space with rational cohomology algebra a symplectic algebra, respectively a Lefschetz algebra, resembles a compact symplectic manifold, respectively a compact Kähler manifold, from the rational cohomology algebra point of view.

As suggested earlier, the results of this section apply not only to simply connected, compact symplectic manifolds but also to any simply connected space whose rational cohomology algebra is a symplectic algebra. We will refer to such a space as a rationally symplectic space, and state our theorems for such spaces.

The following is not the most general result that we can prove, but is a compromise between breadth of application and ease of understanding.

**Theorem 2.1.** Let \( \Lambda(V; d) \) be a bigraded minimal DG algebra. Write the second grading \( V = V_1 \oplus V_2 \oplus \cdots \) with \( r_1 < r_2 < \cdots \) and suppose it satisfies \( V^2 \subset V_{r_1} \) together with one of the following conditions:

(i) For \( r_1 \leq 0 \), if \( v \in V_j \) for \( j \geq 2 \), then \( r_j > \frac{1}{2} |v|r_1 \).

(ii) For \( r_1 > 0 \), \( V^2 = V_{r_1} \) and if \( v \in V_j \) for \( j \geq 2 \), then \( r_j > \frac{1}{2} |v|r_1 \).

If \( H^*(\Lambda(V; d)) \) is a symplectic algebra, then \( \Lambda(V; d) \) is formal. If, in addition, the second grading satisfies \( r_1 \neq 0 \), then \( \Lambda(V; d) \) is formal and \( H^*(\Lambda(V; d)) \) is generated by elements of degree 2.

**Proof.** We will apply the result of [5]. Write \( C = V_{r_1} \) and \( N = \bigoplus_{r \geq 2} V_r \). Then \( d(C) = 0 \), since \( (AV)_r = 0 \) for \( r < r_1 \). We show that \( d \) is injective when restricted to \( N \), and that if a closed element \( \eta \) is in the ideal generated by \( N \), denoted \( (N) \), then it is exact.

First observe that the second grading on \( V \) gives \( H^*(\Lambda(V; d)) = \bigoplus_r H^*(\Lambda(V; d)) \). Now \( H^*(\Lambda(V; d)) \) is a symplectic algebra and \( V^2 \subset V_{r_1} \). Therefore the symplectic class \( \omega \) has lower degree \( r_1 \) and \( H^{2n}(\Lambda(V; d)) = H^{2n}_{r_1}(\Lambda(V; d)) \). We claim that any element of \( (AV)^{2n} \) in the ideal \( (N) \) has lower degree greater than \( nr_1 \): From conditions (i) and (ii) we have, for a generator \( v \in V_j \), that \( r_j \geq \frac{1}{2} |v|r_1 \) with inequality for \( j \geq 2 \), i.e., for \( v \in N \). So a monomial \( v_1 v_2 \cdots v_k \in AV \) has lower degree \( \geq \frac{1}{2} |v_1|r_1 + \frac{1}{2} |v_2|r_1 + \cdots + \frac{1}{2} |v_k|r_1 \), with inequality if any generator is in \( N \). But

\[
\frac{1}{2} |v_1|r_1 + \frac{1}{2} |v_2|r_1 + \cdots + \frac{1}{2} |v_k|r_1 = \frac{|v_1 v_2 \cdots v_k|}{2} r_1.
\]

So any monomial \( X \) in \( (N) \) has lower degree strictly greater than \( \frac{1}{2} |X|r_1 \) and hence \( (N)_{2nr_1} = 0 \). This proves the claim. It now follows that if \( \eta \) is a closed element of degree \( 2n \) in \( (N) \), then \( \eta \) is exact. More generally, suppose that \( \eta \) is a closed element of degree \( k \) in the ideal \( (N) \), with \( k \leq 2n \). By Poincaré duality, if \([\eta]\neq 0\), then there is
a class \([\eta^*] \in H^{2n-k}(A(V; d))\) such that \([\eta] [\eta^*] \neq 0\) in \(H^{2n}(A(V; d))\). But \(\eta \eta^* \in (N)\) and so \([\eta] [\eta^*] \in \bigoplus_{r > m} H^{2n+r}(A(V; d)) = 0\), as observed above. Thus \([\eta] = 0\) and \(\eta\) is exact. Since there are no indecomposable boundaries, this argument also shows that there are no closed elements in \(N\); so \(d\) is injective when restricted to \(N\). It follows from [5, Theorem 4.1] that \(A(V; d)\) is formal.

The last assertion of the theorem is true trivially unless \(r_1 < 0\). In this case condition (i) holds, and the result follows from the observation, that a generator \(v \in V_{r_1}\) satisfies \(r_j \geq \frac{1}{2} |v| r_1\) with inequality for \(j \geq 2\) or for \(j = 1\) and \(|v| > 2\). The proof of the claim in the above argument now holds with \(N\) enlarged to include any elements of \(V_{r_1}\) in degrees greater than 2. In fact, using Poincaré duality as above, to show that \(d\) is injective when restricted to this enlarged \(N\), we conclude that \(V_{r_1}^k = 0\) for \(k \neq 2\). □

**Remark 2.2.** The hypotheses of Theorem 2.1 may seem rather specialized, but in fact they are satisfied in many cases of interest and we illustrate this in the corollaries below. More generally, we observe that a second grading on a DG algebra is easily translated into a *weighting*, in the sense of [6]. One simply gives an element in \(\mathcal{A}\) a weight of \(n + r\). Now the class of spaces that have weighted minimal models, equivalently bigraded minimal models, is known to be very large indeed. This class contains many spaces familiar to homotopy theorists, such as Eilenberg–Mac Lane spaces, \(H\)-spaces, homogeneous spaces, and is closed under sums and products. In Theorem 2.1, then, the restriction is not so much in requiring the space to have a bigraded minimal model, but rather that the lower grading be of a certain type.

We now apply Theorem 2.1 to obtain some interesting special cases.

**Corollary 2.3.** Let \(M\) be a simply connected space with a pure minimal model. If \(M\) is a rationally symplectic space, then \(M\) is formal and has only even-dimensional cohomology.

**Proof.** Suppose \(M\) has pure minimal model \(A(V; d) = A(X, Y; d)\) as in the Introduction, with \(X = V^{\text{even}}\) and \(Y = V^{\text{odd}}\). Setting \(X = V_0\) and \(Y = V_1\) gives \(A(V; d)\) the structure of a bigraded DG algebra. This second grading satisfies the hypotheses of Theorem 2.1 and hence \(M\) is formal. We note that the proof of Theorem 2.1 shows any class in \(H^*(A(V; d))\) is represented by an element of \(A(V_{r_1})\). In case \(M\) is pure, \(V_{r_1} = X\) is evenly graded, and the second assertion follows. □

**Remark 2.4.** A space may be pure but not elliptic, although the two are often associated. In case a pure space \(M\) is also elliptic, we obtain more information in the above results. Since \(H^{\text{odd}}(M; \mathbb{Q}) = 0\), it follows from [10, Theorem 1] that \(M\) must have homotopy Euler characteristic equal to zero. In particular, we obtain the following:

**Corollary 2.5.** Any simply connected, symplectic homogeneous space \(G/H\) is a maximal rank homogeneous space, i.e., \(\text{rk}(H) = \text{rk}(G)\).
Proof. A homogeneous space has a pure minimal model, so a simply connected, symplectic homogeneous space is formal by Corollary 2.3. Furthermore, a homogeneous space is elliptic, so as in Remark 2.4 it follows that the homotopy Euler characteristic is zero. But $G/H$ has $\chi_n = \text{rk}(H) - \text{rk}(G)$ (cf. [11, (7.4)]) and the corollary follows. \qed

Given a basis $V = \langle v_1, v_2, \ldots \rangle$, a differential $d$ of $AV$ can be described on each generator as $d(v_j) = d_2(v_j) + d_3(v_j) + \cdots$, where each $d_k(v_j)$ is a polynomial of length $k$. If $V$ has a basis for which $d(v_j) = d(v_j)$ on each generator, then we say $A(V; d)$ has a homogeneous length differential, of length $l$. With this vocabulary, we give the following corollary:

**Corollary 2.6.** Let $M$ be a simply connected space whose minimal model has a homogeneous length differential. If $M$ is a rationally symplectic space, then $M$ is formal and furthermore $H^*(M; \mathbb{Q})$ is generated as an algebra by elements of degree 2.

**Proof.** Suppose that $A(V; d)$ has differential of homogeneous length $l$, for $l \geq 2$. Assign each generator $v \in V$ a lower degree of $(v (1 - 2) - 1$. Then $V = V_{2(l-2)} \oplus V_{3(l-2)} \oplus \cdots$ with $V_{k(l-2)} = V_k$. One checks that this second grading of the generators gives $A(V; d)$ the structure of a bigraded DG algebra. This second grading satisfies the hypotheses of Theorem 2.1, with $r_1 \neq 0$, and the result follows. \qed

A particularly interesting case of Corollary 2.6 is when $M$ has minimal model with a homogeneous length-2 differential. Such a space is known as a coformal space and the property of coformality has a topological interpretation in terms of the rational homotopy of $M$. In addition, if $M$ is formal and coformal, then the rational cohomology algebra of $M$ is of a particularly restricted form. We state this case as a separate corollary:

**Corollary 2.7.** If $M$ is a coformal, simply connected, compact symplectic manifold, then $M$ is formal and its rational cohomology algebra has a presentation

$$H^*(M; \mathbb{Q}) \cong \frac{\mathbb{Q}[x_1, \ldots, x_r]}{(R_1, \ldots, R_s)},$$

with $|x_i| = 2$ and $|R_j| = 4$. \qed

**Remark 2.8.** In view of the restriction the corollary places on the cohomology algebra of a coformal symplectic manifold, one may expect a generic symplectic manifold not to be coformal. Interpreting this geometrically in the way alluded to above, one would expect a generic symplectic manifold to have non-trivial higher order Whitehead
products in its rational homotopy. Indeed, \( \mathbb{C}P^n \) satisfies the hypotheses of Corollary 2.6, is not coformal (for \( n > 1 \)) and has a non-trivial higher order Whitehead product.

We conclude this section with examples to show that a simply connected compact manifold may be rationally symplectic and yet be non-formal. An easy way to generate such examples is to form connected sums as in the following:

Example 2.9. Consider the minimal model \( A(x, y, c, x_1, x_2, x_3; d) \), where \(|x| = |y| = 2, |x_1| = |x_2| = |x_3| = c = 3\), and the differential is defined by \( d(x) = d(y) = d(c) = 0, \) \( d(x_1) = x^2, \) \( d(x_2) = xy \) and \( d(x_3) = y^2 \). This minimal DG algebra is not formal, as is easily checked, and has cohomology a Poincaré duality algebra of top dimension 10 [10]. By results from rational surgery theory [1], there is a simply connected, compact 10-manifold \( X \), such that \( X \) has this given minimal model. Now form the connected sum \( \mathbb{C}P^5 \# X \). This is not formal, since \( X \) is not formal, but is simply connected and has rational cohomology a symplectic algebra.

In examples such as the above, there is no general reason why the resulting manifold should admit a symplectic structure, since once of the summands is not even rationally symplectic. Our next example improves on this situation a little, in that the non-formal, simply connected compact manifold we describe actually has rational cohomology algebra isomorphic to that of a bona fide Kähler manifold.

Example 2.10. We sketch the example, since the details are quite involved. Let \( K = \mathbb{C}P^2 \times \mathcal{V} \), where \( \mathcal{V} \) is a hypersurface in \( \mathbb{C}P^4 \) defined by a single equation of degree 3. The rational cohomology algebra of \( \mathcal{V} \) is known, and it follows that \( K \) has rational cohomology algebra

\[
H^*(K; \mathbb{Q}) \cong \mathbb{Q}[\omega] \otimes \mathbb{Q}[x, a_1, \ldots, a_5, a_1^*, \ldots, a_5^*] / \mathcal{R}.
\]

with \(|\omega| = |x| = 2, |a_j| = |a_j^*| = 3\) and \( \mathcal{R} \) the ideal generated by \( \{xa_j, xa_j^*\}_{j=1, \ldots, 5}, \{a_ja_k, a_j^*a_k^*\}_{1 \leq j < k \leq 5}, \{a_ja_k^*\}_{j \neq k} \) and \( x^3 - (\sum_{j=1}^5 a_ja_j^*) \). Furthermore, since both \( \mathbb{C}P^2 \) and \( \mathcal{V} \) are Kähler manifolds, so is their product \( K \). In [15, 7.9], it is shown that there is a non-formal space with this cohomology algebra, using DG Lie algebra minimal models. Alternatively, one could use DG algebras and arrive at the same conclusion, following the methods of [12]. The computation involved in this latter approach, however, soon gets out of hand. In any case, there is a non-formal minimal model with cohomology isomorphic to \( H^*(K; \mathbb{Q}) \). Invoking results of rational surgery as in Example 2.9, the minimal model is that of a simply connected, compact 10-manifold which is not formal.

Remark 2.11. The previous example gives a simply connected, compact manifold that does not admit a Kähler structure, yet has rational cohomology algebra a Lefschetz
algebra. Indeed, the manifold has rational cohomology algebra that satisfies any property common to rational cohomology algebras of Kähler manifolds, since it is such an algebra. We observe that the analogous question for symplectic structure — viz. does a manifold that has rational cohomology algebra a symplectic algebra admit a symplectic structure? — is an open question (cf. the conjecture of Thurston’s in [23]).

The spaces of Examples 2.9 and 2.10 are not elliptic. In conjunction with Corollary 2.5, they raise the question of whether or not, for a simply connected, elliptic, compact manifold a symplectic cohomology algebra implies formality. Our last example shows that this is not, in general, the case.

**Example 2.12.** We describe a minimal DG algebra $A(V; d)$ that is elliptic, non-formal and yet has a symplectic cohomology algebra: Let $A(V; d) = A(x, y, w, z_1, z_2, z_3, \beta, \gamma; d)$, where $|x| = |y| = |w| = 2, |z_1| = |z_2| = |z_3| = |\beta| = 3, |\gamma| = 9$ and the differential is defined by $d(x) = d(y) = d(w) = 0, d(z_1) = x^2, d(z_2) = xy, d(z_3) = y^2, d(\beta) = xw, d(\gamma) = \omega^5 + (z_1 y - xz_2)(z_3 \omega - y\beta)$. One easily checks that $d$ is a differential. It follows from [10, Proposition 1] that $A(V; d)$ is elliptic. Now consider the criterion for formality from [5] used previously. In any decomposition $V = C \oplus N$ with $d(C) = 0$ and $d$ injective when restricted to $N$, we must have $V^2 \subset C$ and $V^5 \subset N$. But then $z_1 y - xz_2$ is a non-exact cycle in the ideal $(N)$, hence $A(V; d)$ is not formal.

Finally, we check that $H^*(A(V; d))$ is a symplectic algebra. According to [10, Theorem 3], $H^*(A(V; d))$ is a Poincaré duality algebra of dimension 18. It is sufficient, therefore, to check that $\omega^9$ is not exact in $A(V; d)$. We assume there is an element $\eta$ of degree 17 that satisfies $d\eta = \omega^9$, and arrive at a contradiction. Write $\eta = \gamma A + B$, for $A, B \in A(x, y, w, z_1, z_2, z_3, \beta)$. It follows that $A$ is a cocycle of degree 8. Further, write $A = a + A'$, where $a \in A(x, y, z_1, z_2, z_3)$ and $A' \in (\omega, \beta)$. Since the ideal $(\omega, \beta)$ is $d$-stable, and since $A$ is a cocycle, we have $d(a) = 0$ and $a$ is a cocycle of degree 8 in $A(x, y, z_1, z_2, z_3)$. This latter sub-DG algebra is elliptic and of dimension 7 (cf. [10, Proposition 1 and Theorem 3]), so $a = d(\alpha)$ for some $\alpha \in A(x, y, z_1, z_2, z_3)$. Thus we have

$$\eta = \gamma d(x) + \gamma A' + B = -d(\gamma \alpha) + \gamma A' + B,$$

with $A' \in (\omega, \beta)$ and $A', B' \in A(x, y, z_1, z_2, z_3, \beta)$. Now consider the equation $d\eta = \omega^9$ up to congruence modulo the ideal $(x^2, y)$. Since the subalgebra $A(x, y, z_1, z_2, z_3, \beta)$ has image under $d$ contained in the ideal $(x^2, y)$, we have $d(\gamma) A' \equiv \omega^9$, or

$$(\omega^5 - xz_2z_3\omega)A' \equiv \omega^9.$$

Plainly $A'$ must contain the term $\omega^4$, so $-xz_2z_3\omega A'$ contributes the term $-xz_2z_3\omega^4$ to the left-hand side. Recall that $A' \in (\omega, \beta)$, and so in particular does not contain a term in $xz_2z_3$. This gives the desired contradiction.
From the above it follows that $o^9$ is not a boundary and hence that $H^*(A(V; d))$ is a symplectic algebra. As in the previous examples, by [1] this minimal DG algebra may be realized as the minimal model of a simply connected, compact manifold which is elliptic and rationally symplectic, yet not formal.

The hypotheses of Theorem 2.1 are too restrictive for many purposes and we would like to relax them somewhat. Observe that the non-formal DG algebras in the previous examples do not admit a second grading. This leaves us with the following question:

**Question.** Let $M$ be a simply connected compact manifold whose minimal model is a bigraded DG algebra. If $M$ has cohomology algebra a symplectic algebra, then is it formal?

### 3. The Benson–Gordon theorem

In this section, we give a minimal model proof of a result due to Benson and Gordon [2, Theorem A] (see also [17, Proposition 5] and [13, (2.1) and (2.2)]). We believe our proof puts this result in the proper homotopy-theoretic framework. Indeed, the basic result (Theorem 3.1) is completely homotopy-theoretical modulo knowledge of torsionfree nilpotent groups (see [14]). It is only in applying a theorem of Mostow that we enter the world of Lie theory. If $\pi$ is a nilpotent group, then we shall say $K(\pi, 1)$ is a *nilpotent* $K(\pi, 1)$. We shall prove the following:

**Theorem 3.1.** Let $X$ be a compact, nilpotent $K(\pi, 1)$. If $H^*(X; \mathbb{Q})$ is a Lefschetz algebra, then $X$ has the homotopy type of a torus.

Recall that a nilmanifold is a quotient of a nilpotent Lie group by a discrete cocompact subgroup, which then must be finitely generated torsionfree — see [14] for example. By the well-known work of Malcev, if $X$ is a compact, nilpotent $K(\pi, 1)$, then there is a nilmanifold of the same homotopy type as $X$. We use the correspondence freely. For our work below, however, we only require some knowledge of torsionfree nilpotent groups and the concomitant effects on the structure of the minimal model of a nilmanifold.

A nilmanifold has a minimal model of the form $A(X; d) = A(x_1, \ldots, x_r; d)$ with $|x_j| = 1$ for each $j$ [14, 19]. If a nilmanifold is rationally symplectic, then its minimal model must have an even number of generators. This follows from the observation that the product of all the generators is the highest-degree non-zero element in the minimal model, which therefore is a cocycle. Furthermore, since the differential $d$ is decomposable, one easily checks that there are no non-zero boundaries in this highest degree. The product of all generators, therefore, is a cocycle representing the fundamental class. Since this class resides in an even degree under the assumption of symplecteness, the number of degree-1 generators must be that even degree.
We begin with preliminary results that apply to rationally symplectic nilmanifolds. Let $\Lambda(X; d) = \Lambda(x_1, \ldots, x_{2n}; d)$ be the minimal model of a rationally symplectic nilmanifold, and define a degree $-1$ derivation $\theta: \Lambda X \to \Lambda X$ on generators by

$$\theta(x_j) = \begin{cases} 0 & \text{if } j \neq 2n, \\ 1 & \text{if } j = 2n. \end{cases}$$

One easily checks that $\theta d = -dd$, so $\theta$ induces a degree $-1$ derivation on cohomology, $\tilde{\theta}: H^*(\Lambda X, d) \to H^{*-1}(\Lambda X, d)$. Also, note that if $\omega$ is a representative of the symplectic class, then it is a sum of products of pairs of degree-1 generators

$$\omega = \sum x_i, x_i^2.$$

Moreover, since $\omega^n = x_1 \cdots x_{2n}$, the fundamental class, the expression for $\omega$ must contain all degree-1 generators.

**Lemma 3.2.** Suppose that $(\Lambda X, d)$ is the minimal model of a rationally symplectic nilmanifold, and that $\omega \in H^2(\Lambda X, d)$ is a symplectic class. If $\theta$ is the derivation described above, then $\tilde{\theta}(\omega) \neq 0$.

**Proof.** Suppose a representative cocycle for $\omega$ is $x + yx_{2n}$, where $x, y \in \Lambda(x_1, \ldots, x_{2n-1})$ and $y$ has degree 1. Because the inductive construction of the minimal model precludes $x_{2n}$ from appearing in $dx_i$ for all $i < 2n$, it follows that $y$ must be a cocycle, and that $\tilde{\theta}(\omega) = [y]$. Furthermore, as observed above, $y$ must be non-zero because $\omega^n \neq 0$. But there are no non-zero boundaries in degree 1, and hence $[y] \neq 0$ in $H^1(\Lambda X, d)$. \qed

Next we give a result that concerns derivations on Poincaré duality algebras. This is essentially a result due to Thomas [22, p. 82].

**Lemma 3.3.** Let $H$ be a Poincaré duality algebra of dimension $r$ and let $\tilde{\theta}: H \to H$ be a degree $-1$ derivation. If $\tilde{\theta}(H^1) = 0$, then $\tilde{\theta}(H^r) = 0$.

**Proof.** Let $x \in H^r$ and $y \in H^1$, so that $xy = 0$. Then $0 = \tilde{\theta}(xy) = \tilde{\theta}(x)y$, since $\tilde{\theta}(y) = 0$ by assumption. Now $\tilde{\theta}(x) \in H^{r-1}$, yet annihilates $H^1$. It follows from Poincaré duality that $\tilde{\theta}(x) = 0$. \qed

With the above lemmas, Theorem 3.1 follows easily. The heart of the result is the following:

**Proposition 3.4.** Let $\Lambda(X; d)$ be the minimal model of a rationally symplectic compact nilpotent $K(\pi, 1)$ and let $\omega \in H^2(\Lambda(X; d))$ be a symplectic class. If $d \neq 0$, then $\omega^{2n-1}: H^1(\Lambda(X; d)) \to H^{2n-1}(\Lambda(X; d))$ is not injective.
Proof. Suppose $A(X; d) = A(x_1, \ldots, x_{2n}; d)$ and without loss of generality assume $d x_{2n} \neq 0$. If $\tilde{\theta}$ is the degree $-1$ derivation of $H^*(A(X; d))$ defined above, then $\tilde{\theta}(H^1) = 0$, since $\theta$ is zero on all cocycles of degree 1. Now Lemma 3.3 implies $\tilde{\theta}(\omega^n) = 0$. On the other hand, $\tilde{\theta}$ is a derivation and so $\tilde{\theta}(\omega^n) - n \omega^{n-1} \tilde{\theta}(\omega)$. From Lemma 3.2, we have that $\tilde{\theta}(\omega)$ is non-zero, and thus $\omega^{n-1}: H^1(A(X; d)) \to H^{2n-1}(A(X; d))$ has non-zero kernel. \[ \square \]

Proof of Theorem 3.1. $X$ is Lefschetz so Proposition 3.4 implies $d = 0$ and hence $X$ has the rational homotopy type of a torus. Now $K(\pi, 1)_\mathbb{Q} = K(\pi_\mathbb{Q}, 1)$, and $\pi_\mathbb{Q}$ is abelian. Since $X$ is compact, $\pi$ is finitely generated and torsion-free, hence $\pi$ is a finitely generated free abelian group. The homotopy type of $X$ is determined by the fundamental group, and the result follows. \[ \square \]

The Benson–Gordon theorem is somewhat sharper than was stated above. We finish this section with the sharper version. Say that a manifold $M$ dimension $2n$ is of Lefschetz type if there is a symplectic class $\omega \in H^2(M; \mathbb{Q})$, such that $\omega^{n-1}: H^1(M; \mathbb{Q}) \to H^{2n-1}(M; \mathbb{Q})$ is an isomorphism.

Theorem 3.5 [2, Theorem A]. A nilmanifold of Lefschetz type is diffeomorphic to a torus.

Proof. A nilmanifold is a $K(\pi, 1)$, for some torsion-free nilpotent $\pi$. By Theorem 3.1, the nilmanifold has the homotopy type of a torus. Now, by a result of Mostow's [18], nilmanifolds (more generally solvmanifolds) are classified up to diffeomorphism by their fundamental group. \[ \square \]

4. Extensions to non-aspherical spaces

In this section we consider forming twisted products of rationally symplectic spaces, so as to preserve non-formality and rational symplecticness. In this way, we extend results about aspherical spaces, i.e., nilmanifolds, to more complicated spaces. We will use the notion of a rational fibration (see [11]).

Theorem 4.1. Let $B$ be a rational Poincaré duality space of dimension $2n$, and suppose $\mathbb{C}P^{n-1} \to E \to B$ is a rational fibration. Then $E$ is formal if and only if $B$ is formal.

Proof. First note that the fibration is pure, by [22, Theorem 3]. Denote the minimal model of $B$ by $A(V; d_B)$. Then the fibration has minimal model $A(V; d_B) \twoheadrightarrow A(V, x, y; d_E) \to A(x, y; d_E)$, where $|x| = 2, |y| = 2n - 1$ and the differential in the total space is given by $d_E x = 0$ and $d_E y = x^n + \eta$ for some $d_E$-cocycle $\eta \in (A^+ V \otimes A(x))^2n$. Since $B$ is a rational Poincaré duality space, we can suppose $\eta$ is decomposable, and hence that $A(V, x, y; d_E)$ is the minimal model of $E$. 


Suppose that $E$ is formal. We will use the criterion of \cite[Theorem 4.1]{S} to show that $B$ is formal. Since $E$ is formal, there is a decomposition $V \oplus \langle x, y \rangle \cong C \oplus N$, with $d_E(C) = 0$, $d_E$ injective when restricted to $N$ and such that every closed element in (N) is $d_E$-exact. Now define $C_V = C \cap V$ and $N_V = N \cap V$. Clearly $V \cong C_V \oplus N_V$ and $d_B(C_V) = 0$. Furthermore, $d_B$ is injective when restricted to $N_V$, since $d_B$ and $d_E$ are identical on elements of $AV$. Now suppose that $x \in (N_V) \subset AV$ is closed. We show there is an element $\beta_B \in AV$ with $d_B(\beta_B) = x$. This is clearly true if $|x| > 2n$, by the assumption that $B$ has dimension $2n$. So first assume $|x| \leq 2n - 1$. Since $d_E(x) = 0$, there is an element $\beta_E \in A(V, x, y)$ with $|\beta_E| \leq 2n - 2$ and $d_E(\beta_E) = x$. So write $\beta_E = \beta_B + \beta_x$, with $\beta_B \in AV$ and $\beta_x \in AV \cdot \Lambda^+(x)$. But $d_E(\Lambda^+ \cdot \Lambda^+ (x)) \subset AV \cdot \Lambda^+(x)$ and it follows that $d_E(\beta_x) = 0$. Thus $d_E(x) = d_E(\beta_E) = d_B(\beta_B) = x$. Now assume that $|x| = 2n$. Again, there is a $\beta_E \in A(V, x, y)$ with $|\beta_E| = 2n - 1$ and $d_E(\beta_E) = x$. Write $\beta_E = \beta_B + \beta_x + \lambda y$ with $\beta_B$ and $\beta_x$ as before, and $\lambda \in \mathbb{Q}$. Then $\beta_x \in AV \cdot \Lambda^+(x)$ for degree reasons. Hence $d_E(\beta_x) \in \Lambda^+ \cdot V \cdot \Lambda^+(x)$ also. Since $d_E(\beta_B) = x \in AV$, it follows that $\lambda = 0$, $d_E(\beta_x) = 0$ and once again $d_E(\beta_E) = d_E(\beta_B) = d_B(\beta_B) = x$. Hence any $d_B$-closed element in $(N_V)$ is $d_B$-exact and $B$ is formal by \cite[Theorem 4.1]{S}.

On the other hand, suppose that $B$ is formal. Then the DG algebra $A(V, x; d_E)$ is also formal, since $d_E(x) = 0$. This DG algebra has cohomology algebra $H^\ast(A(V; d_E)) \otimes \mathbb{Q}[x]$, in which the class represented by $x^a + \eta$ is clearly not a zero-divisor. It now follows from \cite[Lemma 2.2]{20} that $A(V, x, y; d_E)$ and hence $E$ is formal. \hfill \Box

**Remark 4.2.** Given a rational fibration as in the hypotheses of Theorem 4.1, it is also the case that $H^\ast(E; \mathbb{Q})$ is a Lefschetz algebra if, and only if $H^\ast(B; \mathbb{Q})$ is a Lefschetz algebra. We omit the proof of this fact, since formality is a sufficient criterion for our immediate purposes.

We will apply Theorem 4.1 by using nilmanifolds for the base space $B$. It is well-known that if a nilmanifold is formal, then it is diffeomorphic to a torus. This can be seen in the spirit of Section 3 as follows: One shows that if a nilpotent $K(\pi, 1)$, $X$, is formal, then it has the rational homotopy type of a torus. Indeed, suppose $H^\ast(X; \mathbb{Q})$ has (top) dimension $n$. Then $X$ has minimal model $A(x_1, \ldots, x_n; d)$ with $|x_i| = 1$. A cocycle representative of the fundamental class is the product of all the generators $x_1 \cdots x_n$ (cf. Section 3). Recall the notation of \cite[Theorem 4.1]{5} from Section 1. Formality implies that $x_1 \cdots x_n$, and hence each of the generators $x_i$, is in the subalgebra generated by $C$, so $d$ is zero. The sharper statement for nilmanifolds is now obtained as in the proof of Theorem 3.1 and in Theorem 3.5. Thus we obtain the following:

**Corollary 4.3.** The total space of any rational fibration having fibre $\mathbb{C}P^{n-1}$ and base a non-toral nilmanifold of dimension $2n$ is not formal, and hence does not admit a Kähler structure. \hfill \Box

**Examples 4.4.** We construct examples of symplectic manifolds that are not $K(\pi, 1)$s, which are not formal and hence cannot admit a Kähler structure. They are the total
spaces of fibrations as in Corollary 4.3. Take any non-toral symplectic nilmanifold $B$ of dimension $2n$ together with a map of degree 1 to a sphere $S^{2n}$. Now, $BU(n)$, the classifying space of the unitary group $U(n)$, has the rational homotopy type of a product of $K(Q, 2j)$s, where each factor is "generated" by a universal Chern class. Hence, there exists a map from $S^{2n}$ to $BU(n)$ pulling back the Chern class $c_n$ to the fundamental class of the sphere (and, of course, pulling all other classes back to zero). By composing with the degree-1 map from the nilmanifold, we see that $c_n$ pulls back to the nilmanifold's fundamental class. Take the projectivization of the vector bundle given by the classifying map $B \to BU(n)$. This gives a fibration

$$\mathbb{C}P^{n-1} \to E \to B.$$ 

By [3, Chapter IV], Chern classes may be read off from the twisting relation in cohomology – exactly that which corresponds to the twisting relation of the differential of the minimal model. Hence, we obtain bundles with the appropriate minimal models. By [24, Theorem 3.3] we know that the total space is a symplectic manifold, yet it is neither Kähler nor Lefschetz.

Remark 4.5. From Theorem 4.1 we also see that examples of simply connected symplectic manifolds which are not formal cannot be constructed from the obvious $\mathbb{C}P^{n-1}$-fibrations as above, since starting with a formal base space $B$ will result in a formal total space $E$.

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