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On the Relation between a Nonlinear Elliptic Equation and Its Uniform Approximation

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The qualitative behavior of the solution set of nonlinear elliptic boundary value problems has in some instances been studied by reducing the partial differential equation to a related algebraic equation. Although this procedure often gives a good picture of the bifurcation diagram, it can be quite wrong. In this paper some relationships between the solutions of the two problems are investigated.

1. INTRODUCTION

The equation

$$\begin{aligned} \Delta u + \lambda f(u) &= 0 \qquad \text{in } \Omega \\ \frac{\partial u}{\partial n} + \beta u &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{1}$$

in which Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and $\beta \in (0, \infty]$ ($\beta = \infty$ corresponds to the Dirichlet case $u = 0$ on $\partial\Omega$) arises in the study of heat-generation in chemically reacting bodies. Questions of existence of solutions and their behavior as the positive parameter λ is varied have been discussed by many authors, notably Lions [9] and Keller and Cohen [7]. Of particular interest in these investigations from a practical viewpoint are the discontinuities of the minimal branch of solutions and the spectrum, that is the set of values λ for which (1) has a positive solution. Indeed, the absence of positive solutions has often been equated with thermal runaway or infinite temperatures in finite time for the corresponding time-dependent problem; see Bebernes and Eberly [1] or Lacey [8].

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Several authors have attempted to obtain qualitative information of the solution set of (1) by considering the algebraic equation

$$\alpha = \bar{\lambda} f(\alpha); \tag{2}$$

see, for example, Boddington *et al.* [2], Graham-Eagle and Wake [4], and Gray and Wake [5]. Although this approach often gives a good idea of the main features of the bifurcation diagram of (1) it can be completely false, as is the case when $f(u) = e^u$ and Ω is the ball in three dimensions; see Joseph and Lundgren [6]. Note that (2) is domain independent.

A relation between (1) and (2) was given by Boddington *et al.* [2], who showed that (2) arises as a limiting case of (1) as $\beta \to 0$ and $\lambda \to 0$ in such a way that their ratio remains constant. Under suitable assumptions the solution of (1) tends to the uniform value α and the parameter $\bar{\lambda}$ is then

$$\bar{\lambda} = \lim(\lambda/\beta) \operatorname{volume}(\Omega)/\operatorname{area}(\partial\Omega).$$

This reduces (1) to a balance of heat production by exothermic reaction and heat loss due to Newton's law of cooling in the case of constant temperature distribution.

In this paper we give different interpretations of the relation between (1) and (2) and establish some connections between the bifurcation diagrams of both problems. In the case of increasing f this leads to a relationship between the minimal solution branches of both equations and the corresponding critical values of λ and $\bar{\lambda}$, and if $f(u)/u$ is decreasing the bifurcation diagram of (2) is shown to characterize (1) in complete detail.

2. Preliminaries

We assume throughout the paper that $\lambda > 0$, f is C^1 and that $f(0) > 0$. Further assumptions will be made as required. For the problems under consideration here we use the word spectrum to denote the set of $\lambda > 0$ for which a positive classical solution (u or α depending on context) exists.

By φ we always mean the unique solution of the problem

$$\Delta\varphi + 1 = 0 \qquad \text{in } \Omega$$

$$\frac{\partial\varphi}{\partial n} + \beta\varphi = 0 \qquad \text{on } \partial\Omega$$

and we define μ and ψ to be the first eigenvalue and corresponding eigenfunction of the linear equation

$$\Delta\psi + \mu\psi = 0 \qquad \text{in } \Omega$$
$$\frac{\partial\psi}{\partial n} + \beta\psi = 0 \qquad \text{on } \partial\Omega.$$

It is well known that φ and ψ are positive in Ω ($\bar{\Omega}$ if $\beta \neq \infty$) and that $\mu > 0$, and for definiteness we normalize so that $\|\psi\| = 1$, where $\|\ \|$ throughout denotes the supremum norm over Ω.

3. The Case of Increasing f

In this section we suppose that $f' \geqslant 0$. Consider the nonlocal problem

$$\begin{aligned} \Delta w + \lambda f(\|w\|) = 0 \qquad &\text{in } \Omega \\ \frac{\partial w}{\partial n} + \beta w = 0 \qquad &\text{on } \partial\Omega. \end{aligned} \tag{3}$$

It is clear that the solution w is given by

$$w = \alpha\varphi / \|\varphi\|$$

provided α solves the algebraic equation

$$\alpha = \lambda\ \|\varphi\|\ f(\alpha). \tag{4}$$

Thus we obtain an equation of type (2) with $\bar{\lambda} = \lambda\ \|\varphi\|$.

Theorem 1. *The spectrum of* (1) *includes the spectrum of* (4). *Moreover, the minimal solution of* (1) *for a given value of* λ *is dominated by the minimal solution of* (3).

Proof. Since zero is clearly a lower solution for (1), it suffices to show that w is an upper solution. But, the boundary condition is satisfied identically and by the monotonicity of f

$$\Delta w + \lambda f(w) = \lambda[f(w) - f(\|w\|)] \leqslant 0 \qquad \text{in } \Omega. \quad ∎$$

Theorem 2. *If u is a solution of* (1) *and* $\|u\| = \alpha$, *then* $\lambda \geqslant \alpha / \|\varphi\|\ f(\alpha)$.

Proof. The result is equivalent to the statement that if w is the solution of (3) with parameter λ^* and if $\|w\| = \|u\|$, then $\lambda^* \leqslant \lambda$. To prove this, suppose the contrary. Then the maximum principle applied to the inequalities

$$-\Delta(w-u)=\lambda^* f(\|w\|)-\lambda f(u)=\lambda^* f(\|u\|)-\lambda f(u)>0 \qquad \text{in } \Omega$$

$$\frac{\partial}{\partial n}(w-u)+\beta(w-u)=0 \qquad \text{on } \partial\Omega$$

shows that $w>u$ in Ω. Since $u\geqslant 0$ this contradicts $\|w\|=\|u\|$ and completes the proof. ∎

Remark. This shows that the region $\lambda<\alpha/\|\varphi\|\, f(\alpha)$ is off-limits to the bifurcation diagram $\alpha=\|u\|$ versus λ for (1).

The next result shows that the first discontinuity in the minimal branch of (3) occurs for a smaller value of λ than for (1) and so gives a lower bound for the critical explosion parameter λ_{cr}.

THEOREM 3. *The minimal branch of* (1) *is continuous for* λ *less than the first discontinuity in the minimal branch of* (3).

Proof. It suffices to show that the principal eigenvalue of the linearized equation corresponding to (1) is positive for such minimal solutions u. Accordingly let v, ν be the principal eigenfunction and eigenvalue of

$$\Delta v+\lambda f'(u)v+\nu v=0 \qquad \text{in } \Omega$$
$$\frac{\partial v}{\partial n}+\beta v=0 \qquad \text{on } \partial\Omega. \tag{5}$$

It is well known that v is positive in Ω. We show that $\nu>0$.

Since $d\lambda/d\alpha\geqslant 0$ on the minimal branch of (4) with equality possible only at isolated points, differentiating (4) with respect to α gives

$$f'(\alpha)\leqslant f(\alpha)/\alpha. \tag{6}$$

If α is the minimal solution of (4) for the given value of λ, then $\alpha\geqslant u$ in Ω by the remark following Theorem 1 and so the monotonicity of f implies from (5) that

$$\Delta v+\lambda\frac{f(u)}{u}v+\nu v>0 \qquad \text{in } \Omega.$$

Multiplying by u and integrating over Ω shows, applying Green's theorem to the first term,

$$\int_\Omega (u\,\Delta v + \lambda f(u) v + \nu u v)\,dx = \int_\Omega v(\Delta u + \lambda f(u) + \nu u)\,dx = \nu \int_\Omega uv\,dx > 0$$

and the proof is complete. ▮

Remark. If (4) has a only one solution for all λ in its spectrum, then the above proof shows that all solutions of (1) are stable (the linearized equation has positive principal eigenvalue). This implies that the minimal solution is a continuous function of λ in the spectrum. This observation is not new—uniqueness of solutions for (4) implies $f(u)/u$ is monotone decreasing (see Section 4).

Remark. It is not true that multiplicity of solutions of (4) implies the same of (1). A counterexample is provided by the Gelfand equation

$$\Delta u + \lambda e^u = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

which is known to have only one solution for each λ in the spectrum in the case that Ω is the ball in $\mathbb{R}^n$ with $n \geq 10$ (see Joseph and Lundgren [6]). Conversely, it is possible that (4) have a unique solution for some λ for which (1) has multiplicity; simply consider $n \leq 2$ in the above example and take for λ the supremum of the spectrum of (4). However, if (4) has at most one solution for every value of λ in the spectrum, then this is also true of (1); again this is true because the hypothesis implies $f(u)/u$ is decreasing.

We close this section by showing that the spectrum of (1) cannot be too much bigger than the spectrum of (4). In particular this leads to an upper bound for the critical value λ_{cr}.

THEOREM 4. *If λ is in the spectrum of* (1) *then $\lambda/\mu\,\|\varphi\|$ is no larger than the supremum of the spectrum of* (4).

Proof. Suppose u is a solution of (1) with parameter λ. Then by Green's theorem

$$\int_\Omega \lambda \psi f(u)\,dx = -\int_\Omega \psi\,\Delta u\,dx = -\int_\Omega u\,\Delta\psi\,dx = \int_\Omega \mu u \psi\,dx$$

or equivalently

$$\int_\Omega \psi f(u)\left(\lambda - \mu \frac{u}{f(u)}\right) dx = 0.$$

This implies the result. ▮

Remark. In particular both spectra are either bounded or not.

Remark. Combined with Theorem 2 this gives an indirect proof that $\mu \|\varphi\| \geqslant 1$. This is simple to see anyway from

$$0 = \int_\Omega (\psi \, \Delta\varphi - \varphi \, \Delta\psi) \, dx = \int_\Omega \psi(\mu\varphi - 1) \, dx.$$

EXAMPLE. For the unit sphere in $\mathbb{R}^n$ simple integration shows that

$$\varphi = \frac{1}{n}\left(\frac{1}{\beta} + \frac{1}{2}(1 - r^2)\right) \qquad \text{and so} \quad \|\varphi\| = \frac{1}{n}\left(\frac{1}{\beta} + \frac{1}{2}\right),$$

where r denotes distance from the center. It follows that the bifurcation diagram for Gelfand's equation

$$\Delta u + \lambda e^u = 0 \qquad \text{in } \Omega$$

$$\frac{\partial u}{\partial n} + \beta u = 0 \qquad \text{on } \partial\Omega$$

lies to the right of the curve

$$\lambda = \frac{2\beta}{2 + \beta} n\alpha e^{-\alpha}. \tag{7}$$

In particular the critical value of λ is bounded below by the maximum of (7), i.e.,

$$\lambda_{\text{cr}} \geqslant \frac{2\beta/e}{2 + \beta} n.$$

Using Theorem 4 an upper bound can be obtained in terms of the principal eigenvalue μ

$$\lambda_{\text{cr}} \leqslant \mu/e.$$

4. THE CASE OF DECREASING $f(u)/u$

If $f(u)/u$ is strictly monotone decreasing it is well known that (1) has a unique solution for each value of λ; see Cohen and Laetsch [3]. Let z be the solution of the nonlocal problem

$$\begin{aligned} \Delta z + \lambda \frac{f(\|z\|)}{\|z\|} z &= 0 \qquad \text{in } \Omega \\ \frac{\partial z}{\partial n} + \beta z &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{8}$$

Clearly the solution of (8) can be written in the form $z = \alpha\psi$ provided that

$$\alpha = \frac{\lambda}{\mu} f(\alpha). \tag{9}$$

Again we obtain equation (2), this time with $\bar{\lambda} = \lambda/\mu$.

We proceed to establish several results concerning the relation between the bifurcation diagrams of (1) and (9).

LEMMA 1. *The spectrum of* (9) *includes the spectrum of* (1). *Furthermore, for each* λ *in this spectrum,* $u \geq z$.

Proof. Fix λ and suppose (1) has a positive solution u. Since $f(\alpha)/\alpha \to \infty$ as $\alpha \downarrow 0$, to find a solution of (9) it clearly suffices to show that $\lambda f(\alpha)/\alpha \leq \mu$ for some positive α. We assert that $\alpha = \|u\|$ will do. To see this, note that the monotonicity of $f(u)/u$ implies that

$$\Delta u + \lambda \frac{f(\|u\|)}{\|u\|} u \leq \Delta u + \lambda \frac{f(u)}{u} u = 0 \qquad \text{in } \Omega.$$

Multiplying by ψ and integrating over Ω gives, applying Green's theorem,

$$\lambda \frac{f(\|u\|)}{\|u\|} \int_\Omega u\psi \, dx \leq -\int_\Omega \psi \, \Delta u \, dx = -\int_\Omega u \, \Delta\psi \, dx = \mu \int_\Omega u\psi \, dx.$$

Since u and ψ are positive in Ω this proves the first assertion of the theorem.

To prove $z \leq u$, assume the contrary. Let Ω_0 be the subset of Ω on which $z > u$. Then $z = u$ on $\partial\Omega_0$ so on the one hand

$$\int_{\Omega_0} (u \, \Delta z - z \, \Delta u) \, dx = \int_{\partial\Omega_0} \left(u \frac{\partial z}{\partial n} - z \frac{\partial u}{\partial n} \right) dS = \int_{\partial\Omega_0} u \left(\frac{\partial z}{\partial n} - \frac{\partial u}{\partial n} \right) dS \leq 0,$$

while on the other hand this also equals

$$\lambda \int_{\Omega_0} \left(zf(u) - u \frac{f(\|z\|)}{\|z\|} z \right) dx = \lambda \int_\Omega uz \left(\frac{f(\|z\|)}{\|z\|} - \frac{f(u)}{u} \right) dx > 0,$$

since $f(u)/u$ is strictly decreasing. This contradiction completes the proof. ∎

LEMMA 2. *The spectra of* (1) *and* (9) *coincide.*

Proof. By Lemma 1 there remains only to show that (1) has a solution whenever (9) does. We distinguish two cases depending on the behavior of f.

Case I. $f(U)=0$ for some positive U. Here the constant function U provides an upper solution for (1) and since zero is a lower solution it follows that (1) has a positive solution for every positive λ. Thus both problems have $\mathbb{R}^+$ for their spectrum.

Case II. $f(u)/u \to \kappa$ as $u \to \infty$. The spectrum of (9) is clearly the interval $(0, \mu/\kappa)$ (we define $\mu/\kappa = \infty$ if $\kappa = 0$). To show that this is also true for (1) fix $\lambda < \mu/\kappa$ and note that $f(u)/u \leqslant \mu/\lambda$ for sufficiently large u. Thus

$$\Delta(c\psi) + \lambda f(c\psi) = \lambda f(c\psi) - \mu c\psi = c\psi \left(\lambda \frac{f(c\psi)}{c\psi} - \mu \right) \leqslant 0 \qquad \text{in } \Omega$$

if c is large enough and $\beta \neq \infty$, and this shows that ψ is an upper solution for (1). If $\beta = \infty$ this upper solution fails because $c\psi$ does not become unbounded everywhere in Ω. In this case therefore we choose as upper solution $c\psi$ corresponding to some finite β and note that this works since

$$\psi = -\frac{1}{\beta} \frac{\partial \psi}{\partial n} \geqslant 0 \qquad \text{on } \partial\Omega.$$

In either case λ is in the spectrum of (1) and the theorem is proved. ∎

We now have the main result of the section.

THEOREM 5. *Equation* (1) *has a solution precisely when* (9) *does, and then* $u \geqslant \alpha\psi$. *Moreover, in all cases the bifurcation diagrams of both problems exhibit identical behavior at infinity.*

Proof. Only the final assertion is unproved. Again we distinguish the two cases of Lemma 2.

Case I. By the maximum principle $\|u\|$ cannot exceed U. Since clearly $\alpha \uparrow U$ as $\lambda \to \infty$ the same must be true of $\|u\|$ since $\|u\| \geqslant \alpha$.

Case II. Both solutions approach μ/κ from the left as $\lambda \to \infty$. ∎

Remark. If also f is increasing, then the bifurcation diagram of (1) lies between the solution curves of (9) and (4).

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