On Meromorphically Starlike Functions and Functions Meromorphically Starlike with Respect to Symmetric Conjugate Points

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In a previous paper, M.-P. Chen, Z.-R. Wu, and Z.-Z. Zou (1996, J. Math. Anal. Appl. 201, 25–34) developed a method, using some operators, to deal with functions holomorphic and starlike with respect to symmetric conjugate points in the unit disc. Now the same method can be employed to functions meromorphic in the punctured disc $0 < |z| < 1$. Especially, a structural representation of such functions is obtained.

Key Words: meromorphically starlike; Hadamard products (or convolution); operators.

1. INTRODUCTION

Let $\mu$ denote the class of functions $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$ which are analytic and univalent in the punctured unit disk

$$\dot{\Delta} := \{ z : 0 < |z| < 1 \} = \Delta \setminus \{0\},$$

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and which have a simple pole at the origin \((z = 0)\) with residue 1 here. Let us denote by \(P\) the class of functions \(p(z)\) which are regular in \(\Delta\) and satisfy the conditions \(p(0) = 1\) and \(\text{Re}(z) > 0\) in \(\Delta\).

A function \(f(z) \in \mu\) is said to be in the subclass \(MS^*\) of meromorphically starlike in \(\Delta\) if it satisfies the condition

\[
\text{Re}\left(\frac{-zf'(z)}{f(z)}\right) > 0, \ z \in \Delta.
\]

Furthermore, a function \(f(z) \in \mu\) is said to be in the subclass \(MK\) of meromorphically convex in \(\Delta\) if and only if \(F(z) = -zf'(z)\) belongs to the class \(MS^*\).

We now define two operators \(D\) and \(T\) as follows [1]:

1. The operator \(T\). For \(f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \in \mu\), let

\[
Tf(z) = \frac{1}{2}\{f(z) - \overline{f(-z)}\} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{2}\left[a_n - (-1)^n \overline{a_n}\right]z^n.
\]

2. The operator \(D\). For \(f \in \mu\) and \(n\) a positive integer, let

\[
D^n f(z) = f(z), \quad Df(z) = zf'(z), \ldots,
\]

\[
D^{n+1} f(z) = D(D^n f(z)), \quad n = 1, 2, 3, \ldots.
\]

It is easily seen that \(D\) and \(T\) are well-defined on \(\mu\) and have the following properties:

1. Let \(\lambda \in C, f, g \in \mu\). Then

\[
D\{\lambda f + (1 - \lambda)g\} = \lambda Df + (1 - \lambda)Dg,
\]

and

\[
T\{\lambda f + (1 - \lambda)g\} = \lambda Tf + (1 - \lambda)Tg.
\]

2. \(DT = TD\).

3. \(TT = T\).

Furthermore, we define a new class \(MS^*_\alpha(x)\) (see [1, 2]).

**Definition.** A function \(f \in \mu\) with \(f(z)/zf'(z) \neq 0\) in \(\dot{\Delta}\) is said to be meromorphically \(\alpha\)-starlike with respect to symmetric conjugate points, if it
satisfies
\[
\text{Re}\left\{ -\frac{D(\alpha D + (1 + \alpha)D^p)f(z)}{(\alpha D + (1 + \alpha)D^p)\overline{f}(z)} \right\} > 0, \quad z \in \hat{\Delta},
\]
for some $\alpha \geq 0$. This class is denoted by $MS^*_\alpha(\alpha)$.

For classes $MS^*$ and $MK$, which are discussed in [3, 4], in this paper some new results of these classes are obtained. Moreover, the new class $MS^*_\alpha(\alpha)$ is discussed too and some properties of this class such as coefficient estimates and a structural formula are obtained.

Let us adopt the symbol $D_\alpha = \alpha D + (1 + \alpha)D^p$, and $f^*(z) = D_\alpha f(z) = (1 + \alpha)f(z) + azf'(z)$. We see that $f \in MS^*_\alpha(\alpha)$ is equivalent to $f^*(z) \in MS^*_{\alpha'}(0) = MS^*_c$.

2. SOME RESULTS OF THE CLASS $MS^*$

**Theorem 1.** If $f(z) = \frac{1}{z} + \sum_{n=1}^\infty a_n z^n \in \mu$ and the condition
\[
\sum_{n=1}^\infty n|a_n| \leq 1,
\]
is satisfied then $f \in MS^*$.

**Proof.** Set
\[
p(z) = \begin{cases} -\frac{Df(z)}{f(z)}, & z \in \hat{\Delta} \\ 1, & z = 0. \end{cases}
\]
It is sufficient to prove that
\[
|p(z) - 1| < |p(z) + 1|, \quad z \in \Delta.
\]
From (3), we obtain
\[
\left| \sum_{n=1}^\infty (n + 1) a_n z^{n+1} \right| < 1 + \sum_{n=1}^\infty |a_n|, \quad z \in \Delta,
\]
and
\[
\left|2 - \sum_{n=1}^{\infty} (n - 1)a_nz^{n+1}\right| > 2 - \sum_{n=1}^{\infty} (n - 1)|a_n| \\
\geq 1 + \sum_{n=1}^{\infty} |a_n|, \quad z \in \Delta. \quad (7)
\]

Since
\[
|p(z) - 1| = \left|\sum_{n=1}^{\infty} (n + 1)a_nz^{n+1}\right| / \left|1 + \sum_{n=1}^{\infty} a_nz^{n+1}\right|, \quad z \in \hat{\Delta},
\]
and
\[
|p(z) + 1| = \left|2 - \sum_{n=1}^{\infty} (n - 1)a_nz^{n+1}\right| / \left|1 + \sum_{n=1}^{\infty} a_nz^{n+1}\right|, \quad z \in \hat{\Delta},
\]
by (6) and (7), we see that inequality (5) holds in \(\hat{\Delta}\). The case \(z = 0\) is trivial. Hence \(p(z) \in P\). So the proof of Theorem 1 is complete.

**Corollary 1.** If \(f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_nz^n \in \mu\) then \(f \in MK\) when \(\sum_{n=1}^{\infty} n^2|a_n| \leq 1\).

**Theorem 2.** If \(f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_nz^n \in MS^*, \) then
\[
\sum_{n=1}^{\infty} n|a_n|^2 \leq 1. \quad (8)
\]

The estimate (8) is sharp and the equality is attained for the function \(f_0(z) = \frac{1}{z} + a_1z\), where \(|a_1| = 1\).

**Proof.** Since \(f \in MS^*\), hence the inequality (5) holds. This implies that
\[
\left|\sum_{n=1}^{\infty} (n + 1)a_nz^{n+1}\right| < \left|2 - \sum_{n=1}^{\infty} (n - 1)a_nz^{n+1}\right|, \quad z \in \Delta.
\]

Therefore for every \(r \in (0, 1)\) and letting \(z = re^{i\theta}, 0 \leq \theta \leq 2\pi\), we have
\[
\int_{0}^{2\pi} \left|\sum_{n=1}^{\infty} (n + 1)a_nz^{n+1}\right|^2 d\theta \leq \int_{0}^{2\pi} \left|2 - \sum_{n=1}^{\infty} (n - 1)a_nz^{n+1}\right|^2 d\theta, \quad (9)
\]
and hence
\[ \sum_{n=1}^{\infty} n|a_n|^2 r^{2(n+1)} \leq 1. \]  
(10)

Let \( r \to 1 - 0 \); (10) yields (8).

It is easily seen that \( f_0(z) \in MS^* \). Thus we complete the proof of Theorem 2.

**COROLLARY 2.** Let \( f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \in \mu \). If \( f \in MK \), then 
\[ \sum_{n=1}^{\infty} n^3 |a_n|^4 \leq 1. \]

By means of Theorems 1, 2, Corollaries 1, 2, and the Cauchy–Schwarz inequality we obtain immediately the following theorem.

**THEOREM 3.** Let \( f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \) and \( g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \).

(i) If \( f, g \in MS^* \), then their Hadamard product \((f \ast g)(z)\) defined by
\[ (f \ast g)(z) := \frac{1}{z} + \sum_{n=1}^{\infty} a_n b_n z^n \]
is in the class \( MS^* \);

(ii) If \( f \in MS^* \), \( g \in MK \), then \( f \ast g \in MK \).

3. THE CLASS \( MS^*_{sc}(\alpha) \) AND HADAMARD PRODUCTS

First, by using the same method as Lemma 2 in [1], we obtain immediately a result as follows:

**THEOREM 4.** If \( f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \in MS^*_{sc} \), then \( Tf \in MS^* \).

Since \( Tf(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{2} (a_n - (-1)^n \overline{a_n}) z^n \), hence from Theorems 2, 4, we obtain

**COROLLARY 3.** If \( f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \in MS^*_{sc} \), then
\[ \sum_{n=1}^{\infty} \frac{n}{4} |a_n - (-1)^n \overline{a_n}|^2 \leq 1, \]  
(11)
i.e.,
\[ \sum_{n=1}^{\infty} n(|a_n|^2 - (-1)^n \text{Re}a_n^2) \leq 2. \]  
(12)

**THEOREM 5.** Let \( \alpha \geq 0 \). If \( f \in MS^*_{sc}(\alpha) \), then \( D_\alpha Tf \in MS^* \) and \( Tf \in MS^*_{sc}(\alpha) \).
**Proof.** Since \( f \in MS^*_{\alpha}(\alpha) \) if and only if \( f^* = D_{\alpha}f = (\alpha D + (1 + \alpha)D^*)f \in MS^*_{\alpha} \), then we can see from Theorem 4 that \( Tf^* = TD_{\alpha}f \in MS^* \).

Further, by using \( TD = DT \) we get \( TD_{\alpha} = D_{\alpha}T \). Hence, we have

\[
D_{\alpha}Tf(z) = TD_{\alpha}f(z) \in MS^*.
\]

Moreover, \( TT = T \) yields

\[
\text{Re} \left\{ \frac{-D(\alpha D + (1 + \alpha)D^*)Tf(z)}{(\alpha D + (1 + \alpha)D^*)TTf(z)} \right\} = 0, \quad z \in \hat{\Delta}.
\]

This means that \( Tf \in MS^*_{\alpha}(\alpha) \). The proof of Theorem 5 is complete.

**THEOREM 6.** If \( f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \in \mu \) and the condition

\[
\sum_{n=1}^{\infty} \sqrt{n^2 + 1} |a_n| \leq 1
\]

is satisfied then \( f \in MS^*_{\alpha} \).

**Proof.** Let \( p(z) = -\frac{p'f}{f'f}, z \in \hat{\Delta}, \) and \( p(0) = 1 \). We need only to prove that

\[
|p(z) - 1| < |p(z) + 1|, \quad z \in \Delta.
\]

After some computations, we see that we need only

\[
\left| \sum_{n=1}^{\infty} \left( \left( n + \frac{1}{2} \right) a_n - (-1)^n \frac{1}{2} \bar{a}_n \right) z^{n+1} \right| < 2 - \sum_{n=1}^{\infty} \left( \left( n - \frac{1}{2} \right) a_n + (-1)^n \frac{1}{2} \bar{a}_n \right) z^{n+1}, \quad z \in \Delta.
\]

It is sufficient to prove that

\[
\sum_{n=1}^{\infty} \left( \left| \left( n + \frac{1}{2} \right) a_n - (-1)^n \frac{1}{2} \bar{a}_n \right| + \left| \left( n - \frac{1}{2} \right) a_n + (-1)^n \frac{1}{2} \bar{a}_n \right| \right) \leq 2.
\]
Therefore after some easy computations, for \( n \) odd or even, we have
\[
\left| \left( n + \frac{1}{2} \right) a_n - \left( -1 \right)^{n} \frac{1}{2} a_n \right| + \left| \left( n - \frac{1}{2} \right) a_n + \left( -1 \right)^{n} \frac{1}{2} a_n \right| 
\leq 2\sqrt{n^2 + 1 |a_n|}.
\] (16)

From (13) and (16), we see that inequality (15) holds. This completes the proof of Theorem 6.

**Remark 1.** If the \( a_n(n = 1, 2, 3, \ldots) \) are reals and satisfy the condition
\[
\sum_{n=1}^{\infty} n|a_n| \leq 1, \tag{17}
\]
then \( f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \in MS^*_c. \)

Now, we prove a structural formula in the class \( MS^*_c. \)

**THEOREM 7.** A function \( f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \in MS^*_c \) exists if and only if there exist a function \( p \in P \) and a function \( G \in MS^* \) with real coefficients such that \( G \) satisfies
\[
- \frac{DG}{G} = \frac{1}{2} \left( p(iz) + \overline{p(i\bar{z})} \right), \quad z \in \hat{\Delta}, \tag{18}
\]
and
\[
f'(z) = \frac{i p(z) G(-iz)}{z}, \quad z \in \hat{\Delta}. \tag{19}
\]

**Proof.** Suppose \( f \in MS^*_c. \) Then there exists a function \( p \in P \) such that
\[
-f'(z) = \frac{p(z) Tf(z)}{z}, \quad z \in \hat{\Delta}, \tag{20}
\]
where \( p(0) = 1, \ p(z) = - \frac{DG}{G}, (z \in \hat{\Delta}), \) and \( Tf(z) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{2} (a_n - (-1)^{n} \overline{a_n}) z^n. \) By Theorem 4, \( Tf(z) \in MS^* \), and hence
\[
Re \left( - \frac{DTf(z)}{Tf(z)} \right) = \begin{cases} 
1/z - \sum_{n=1}^{\infty} (n/2) (a_n - (-1)^n \overline{a_n}) z^n \\
1/z + \sum_{n=1}^{\infty} (1/2) (a_n - (-1)^n \overline{a_n}) z^n
\end{cases} > 0, \quad z \in \hat{\Delta}.
\]
Letting 

\[ s(z) = \frac{1 - \sum_{n=1}^{\infty} \left( \frac{n}{2} \right) (a_n - (-1)^n \bar{a}_n) z^n}{1 + \sum_{n=1}^{\infty} \left( \frac{n}{2} \right) (a_n - (-1)^n \bar{a}_n) z^n} , \]

then

\[ s(z) < \frac{1 + z}{1 - z}, \quad z \in \Delta. \]

Since \( \frac{1}{1-z} < \frac{1}{1-\bar{z}} (z \in \Delta) \), hence

\[
\begin{align*}
\text{s}(iz) &= s(z) \frac{1}{1 - iz} \\
&= \frac{1 - \sum_{n=1}^{\infty} \left( \frac{n}{2} \right) (a_n - (-1)^n \bar{a}_n) i^{n+1} z^{n+1}}{1 + \sum_{n=1}^{\infty} \left( \frac{n}{2} \right) (a_n - (-1)^n \bar{a}_n) i^{n+1} z^{n+1}} < \frac{1 + z}{1 - z}. \quad \text{(21)}
\end{align*}
\]

From (21), we see that

\[
G(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{2} (a_n - (-1)^n \bar{a}_n) i^{n+1} z^n = T\text{f}(z) \ast \left( \frac{1}{z} - \frac{z}{1+i z} \right) \]

is in the class \( MS^* \) with real coefficients. Therefore

\[
T\text{f}(z) = G(z) \ast \left( \frac{1}{z} - \frac{z}{1+i z} \right) = -iG(iz). \quad \text{(23)}
\]

From (20) and (23), we obtain (19).

We now show that the function \( G \) satisfies the condition (18).

Since \( -\frac{D\text{f}(\bar{z})}{D\text{f}(z)} = p(z) (z \in \hat{\Delta}) \), we obtain

\[
\overline{p(-\bar{z})} = -\frac{D(-f(-\bar{z}))}{T\text{f}(z)} ,
\]

and

\[
-\frac{\overline{D\text{f}(z)}}{T\text{f}(z)} = -\frac{1}{2} \left( \frac{D(f(z)) + D(-f(-\bar{z}))}{T\text{f}(z)} \right) = \frac{1}{2} \left( p(z) + \overline{p(-\bar{z})} \right).
\]
Moreover, we have
\[
- \frac{zG'(z)}{G(z)} = \left( - \frac{DTf(z)}{Tf(z)} \right) \ast \frac{1}{1 - iz}
\]
\[
= \frac{1}{2} \left( p(z) + \overline{p(-z)} \right) \ast \frac{1}{1 - iz}
\]
\[
= \frac{1}{2} \left( p(iz) + \overline{p(i\overline{z})} \right), \quad z \in \hat{\Delta}.
\]
Since \( p \in P \), hence the condition (18) holds.

Conversely, for \( f(z) \in \mu \), if there exist a function \( p \in P \) and a function \( G \in MS^\infty \) with real coefficients such that conditions (18) and (19) hold, then we can show that \( f \in MS^\infty \).

To do this, we need only show that \( Tf(z) = -iG(-iz) \). From (18) we get
\[
iG'(-iz) = \frac{1}{2} \left( p(z) + \overline{p(-z)} \right) \frac{G(-iz)}{z},
\]
hence
\[
iG(-iz) - iG(-iz_0) = \int_{z_0}^z G'(-it)dt
\]
\[
= -i \int_{z_0}^z \frac{1}{2} \left( p(t) + \overline{p(-t)} \right) \frac{G(-it)}{t} \, dt,
\]
where \( z_0 \in \hat{\Delta} \).

From (19)
\[
f(z) - f(z_0) = i \int_{z_0}^z \frac{p(t)G(-it)}{t} \, dt.
\]
It is easy to verify that
\[
Tf(z) - Tf(z_0) = i \int_{z_0}^z \frac{p(t) + \overline{p(-t)}}{2} \frac{G(-it)}{t} \, dt.
\]
Since
\[
Tf(z_0) = -iG(-iz_0),
\]
from (24) and (25), we obtain
\[
Tf(z) = -iG(-iz).
\]
The proof of Theorem 7 is complete.
THEOREM 8. A function $f \in MS^*_\alpha(\alpha)$, $\alpha > 0$, if and only if there exists a function $p \in P$ and $G \in MS^*$ with real coefficients satisfying condition (18) such that

$$f'(z) = \begin{cases} 
\frac{\gamma - 1}{z^{\gamma+1}} \int_0^z p(t)G(-it)t^{\gamma-1} \, dt + \frac{p'(0)}{1+\alpha}z^{-1}, \\
0 < \alpha < 1, \\
-1 + i\frac{\gamma - 1}{z^{\gamma+1}} \int_0^z \left( p(t)G(-it) + \frac{1}{it} \right)t^{\gamma-1} \, dt \\
+ \frac{p'(0)}{1+\alpha}z^{-1}, \\
\alpha \geq 1,
\end{cases} \quad z \in \hat{\Delta}, \quad (26)$$

where $\gamma = \frac{1}{\alpha} + 1 > 1$.

Proof. Since $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \in MS^*_\alpha(\alpha)$, $\alpha > 0$ if and only if

$$D_\alpha f(z) = (1 + \alpha)f(z) + \alpha zf'(z)$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty} \left( 1 + (n + 1)\alpha \right) a_n z^n \in MS^*_\alpha.$$

From (19) in Theorem 7 we have

$$(D_\alpha f(z))' = i\frac{p(z)G(-iz)}{z}.$$

Now, let

$$g_\alpha(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z^n}{1 + (n + 1)\alpha} = \frac{1}{z} + \frac{\gamma - 1}{z^\gamma} \int_0^z \frac{t^\gamma}{1-t} \, dt.$$

Then we have

$$-z(D_\alpha f(z))' * g_\alpha(z) = \frac{1}{z} - \sum_{n=1}^{\infty} na_n z^n = -zf'(z),$$

which implies

$$-zf'(z) = -ip(z)G(-iz) * \left( \frac{1}{z} + \frac{\gamma - 1}{z^\gamma} \int_0^z \frac{t^\gamma}{1-t} \, dt \right)$$

$$= \frac{1}{z} + \left( -ip(z)G(-iz) - \frac{1}{z} - p'(0) \right) * \frac{\gamma - 1}{z^\gamma} \int_0^z \frac{t^\gamma}{1-t} \, dt.$$
which yields (26). The proof of Theorem 8 is complete.

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