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On the projections of the Dirichlet probability distribution on symmetric cones

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ABSTRACT

In this paper, we introduce the Riesz–Dirichlet distribution on a symmetric cone as an extension of the Dirichlet distribution defined by the Wishart distribution. We also show that some projections of these distributions related to the Pierce decomposition are also Dirichlet.

1. Introduction and preliminaries

It is well known that the real Dirichlet distribution is derived from the gamma distribution defined by

$$\gamma_{p,\sigma}(dy) = \exp(-\sigma y) y^{p-1} \sigma^p \big(\Gamma(p)\big)^{-1} \mathbf{1}_{(0,+\infty)}(y)\, dy,$$

where $\sigma > 0$ is the scale parameter and $p > 0$ is the shape parameter. In fact, if $Y_1, \dots, Y_q$ are independent random variables with respective gamma distributions $\gamma_{p_1,\sigma}, \dots, \gamma_{p_q,\sigma}$, and if we define

$$S = Y_1 + \cdots + Y_q \quad \text{and} \quad X = \left(\frac{Y_1}{S}, \dots, \frac{Y_q}{S}\right),$$

then the distribution of X is called the Dirichlet distribution with parameters $(p_1, \dots, p_q)$ and is denoted $D_{(p_1,\dots,p_q)}$. For the definition of multivariate analogs of the real Dirichlet distribution, the gamma distribution is replaced by the Wishart distribution on symmetric matrices and the ordinary division in real numbers is replaced by a division algorithm (see [6] or [1]). An interesting question within the framework of the Wishart distribution is: “Are the variables obtained from a Wishart–Dirichlet random variable by some projections and some inversions of the matrix margins are still Dirichlet?”. The aim of the paper is to give an answer to this question. We will consider it in a more general setting. In fact the Wishart distribution represent a particular example of the more general Riesz distribution on the cone of positive symmetric matrices or on any symmetric cone. The definition of these distributions is based on the notion of generalized power in a Jordan algebra which reduces to the ordinary determinant in a particular situation (see [3]). We first use the class of Riesz distribution and an appropriate division algorithm to introduce an extension of the class of the Wishart–Dirichlet distributions which we call the class of Riesz–Dirichlet distributions. We then show that some variables related to the Pierce decomposition of the Riesz–Dirichlet and in particular of the Wishart–Dirichlet distribution are also Dirichlet. Our

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results are presented in the general framework of Jordan algebras and symmetric cones with the emphasis upon the algebra of symmetric matrices.

Recall that a Euclidean Jordan algebra is a finite dimensional Euclidean space E of dimension n, with a scalar product $\langle x, y\rangle$ and a bilinear map

$$E \times E \to E, \qquad (x, y) \mapsto xy$$

which verify some specific properties (for more details, see Faraut and Korányi [2]). A Euclidean Jordan algebra is said to be simple if it does not contain a nontrivial ideal. Actually to each Euclidean simple Jordan algebra, one attaches the set of Jordan squares

$$\overline{\Omega} = \{x^2;\ x \in E\}.$$

Its interior Ω is a symmetric cone, i.e. a cone which is self dual that is

$$\Omega = \{x \in E;\ \langle x, y\rangle > 0\ \forall y \in \overline{\Omega} \setminus \{0\}\}$$

and homogeneous, i.e. the group $G(\Omega)$ of linear automorphisms preserving Ω acts transitively on Ω. We denote by G the identity component of $G(\Omega)$ and $K = G \cap O(E)$, where $O(E)$ is the orthogonal group of E. An element k of K satisfies $k(xy) = k(x)k(y)$. In particular, $ke = e$ and this equality characterizes K, that is $K = \{g \in G;\ ge = e\}$. For each $x \in E$, we define $L(x) : E \to E$ by $L(x)y = xy$, $y \in E$ and the trace of x is $\mathrm{trace}(x) = \langle x, e\rangle$. The inner product on E is then given by $\langle x, y\rangle = \mathrm{trace}(xy)$. Consider the map $P(x) : E \to E$, defined by

$$P(x) = 2L(x)^2 - L(x^2).$$

Then the map $x \mapsto P(x)$ is called the quadratic representation of E. An element $c \in E$ is idempotent if $c^2 = c$. A scalar α is an eigenvalue of $c \in E$ if there exists a nonzero $x \in E$ such that $cx = \alpha x$. If c is idempotent then it can be shown that its eigenvalues must be equal to $1, 1/2$ or 0 (see [2, p. 62]). The corresponding eigenspaces are respectively denoted by $E(c, 0)$, $E(c, 1/2)$ and $E(c, 1)$ and the decomposition

$$E = E(c, 0) \oplus E(c, 1/2) \oplus E(c, 1)$$

is called the Peirce decomposition of E with respect to c. An idempotent c is primitive if it is nonzero and is not expressible as the sum of two nonzero idempotents. Two idempotents c_1 and c_2 are orthogonal if $c_1c_2 = 0$. A maximal system of orthogonal primitive idempotents is called a Jordan frame. It may be shown that any Jordan frame has the same number, r, of elements; and r is called the rank of Ω. If $\{c_1, \dots, c_r\}$ is a Jordan frame, then $c_1 + \cdots + c_r = e$, the identity element in E. Let us choose and fix a Jordan frame $\{c_1, \dots, c_r\}$ in E and define a collection of subspaces, for $1 \leqslant i, j \leqslant r$

$$E_{ij} = \begin{cases} E(c_i, 1) = \mathbb{R}c_i & \text{if } i = j, \\ E(c_i, \frac{1}{2}) \cap E(c_j, \frac{1}{2}) & \text{if } i \neq j. \end{cases}$$

Then (see [2, Theorem IV.2.1]) we have $E = \bigoplus_{i \leqslant j} E_{ij}$, each E_{jj}, for $j = 1, \dots, r$, is a one-dimensional subalgebra. Further, the subspaces E_{ij}, for $i, j = 1, \dots, r$ with $i \neq j$, all have a common dimension, called the Peirce constant, denoted by d. The constant d is independent of the choice of Jordan frame. It may be shown that n, d and r are related by the formula

$$n = r + r(r-1)\frac{d}{2}.$$

For $1 \leqslant k \leqslant r$, let P_k the orthogonal projection on the Jordan subalgebra

$$E^k = E(c_1 + \cdots + c_k, 1),$$

$\det^{(k)}$ the determinant in the subalgebra E^k and, for x in E, $\Delta_k(x) = \det^{(k)}(P_k(x))$. Then Δ_k is called the principal minor of order k with respect to the Jordan frame $(c_i)_{1 \leqslant i \leqslant r}$. For $s = (s_1, \dots, s_r) \in \mathbb{R}^r$, and x in Ω, we write

$$\Delta_s(x) = \Delta_1(x)^{s_1 - s_2}\Delta_2(x)^{s_2 - s_3} \cdots \Delta_r(x)^{s_r}.$$

This is the generalized power function. Note that if $x = \sum_{i=1}^r \lambda_i c_i$, then $\Delta_s(x) = \lambda_1^{s_1}\lambda_2^{s_2} \cdots \lambda_r^{s_r}$ and that $\Delta_s(x) = (\det x)^\lambda$ if $s = (\lambda, \dots, \lambda)$ with $\lambda \in \mathbb{R}$. It is also easy to see that $\Delta_{s+s'}(x) = \Delta_s(x)\Delta_{s'}(x)$. In particular, if $m \in \mathbb{R}$ and $s + m = (s_1 + m, \dots, s_r + m)$, we have $\Delta_{s+m}(x) = \Delta_s(x)(\det x)^m$.

In the case where E is the space of $r \times r$ real symmetric matrices, all these concepts are familiar. The Jordan product xy of two symmetric matrices x and y is defined by $\frac{1}{2}(x.y + y.x)$ where $x.y$ is the ordinary product of the matrices x and y, the cone Ω is the cone of $r \times r$ positive definite symmetric matrices, and its closure $\overline{\Omega}$ is the cone of positive $r \times r$ positive semi-definite symmetric matrices and $d = 1$. Also if $x = (x_{ij})_{1 \leqslant i, j \leqslant r}$ is an (r, r)-symmetric positive definite matrix, then $P_k(x) = (x_{ij})_{1 \leqslant i, j \leqslant k}$ and $\Delta_k(x) = \det(x_{ij})_{1 \leqslant i, j \leqslant k}$.

2. Riesz–Dirichlet distributions on symmetric cones

As we mentioned above, for the definition of the Dirichlet distribution on a symmetric cone, we need a notion of quotient. In general, there is not a single way to define a quotient. For example, in the matrix case, if Y is a positive definite matrix, one can write $Y = Y^{\frac{1}{2}}Y^{\frac{1}{2}}$ and define the ratio X by Y as $Y^{-\frac{1}{2}}XY^{-\frac{1}{2}}$. We can also use the Cholesky decomposition $Y = UU^*$, where U is a lower triangular matrix and define the ratio as $(U^{-1})X(U^{-1})^*$. More generally, a division algorithm on a symmetric cone is a measurable map g from Ω into G such that, for all x in Ω, $g(x)(x) = e$. The two examples given above for symmetric matrices are the most usual and most important. The definition of the first in any symmetric cone is based on the quadratic representation, it is given by $g(x) = P(x^{-\frac{1}{2}})$. The second algorithm takes its values in the triangular group T. For the definition of T, we need to introduce some other facts concerning a Jordan algebra. For x and y in E, let $x \,\Box\, y$ denote the endomorphism of E defined by

$$x \,\Box\, y = L(xy) + \big[L(x), L(y)\big] = L(xy) + L(x)L(y) - L(y)L(x). \tag{2.1}$$

If c is an idempotent and if z is an element of $E(c, \frac{1}{2})$,

$$\tau_c(z) = \exp(2z \,\Box\, c)$$

is called a Frobenius transformation, it is an element of the group G.

Given a Jordan frame $(c_i)_{1\leqslant i\leqslant r}$, the subgroup of G

$$T = \left\{ \tau_{c_1}\big(z^{(1)}\big) \cdots \tau_{c_{r-1}}\big(z^{(r-1)}\big) P\left(\sum_{i=1}^{r} a_i c_i\right),\ a_i > 0,\ z^{(j)} \in \bigoplus_{k=j+1}^{r} E_{jk} \right\}$$

is called the triangular group corresponding to the Jordan frame $(c_i)_{1\leqslant i\leqslant r}$. It is an important result [2, p. 113, Prop. VI.3.8] that the symmetric cone Ω of the algebra E is parameterized by the set

$$E_+ = \left\{ u = \sum_{i=1}^{r} u_i c_i + \sum_{i<j} u_{ij},\ u_i > 0 \right\}. \tag{2.2}$$

More precisely, if

$$t_u = \tau_{c_1}\big(z^{(1)}\big) \cdots \tau_{c_{r-1}}\big(z^{(r-1)}\big) P\left(\sum_{i=1}^{r} u_i c_i\right) \tag{2.3}$$

where $z_{ij} = \frac{u_{ij}}{u_i}$, $i < j$ and $z^{(j)} = \sum_{k=j+1}^{r} z_{jk}$, then the map $u \mapsto t_u(e)$ is a bijection from E_+ into Ω with a Jacobian equal to $2^r \prod_{i=1}^{r} u_i^{1+d(r-i)}$. Also, for all x in E, we have

$$\Delta_k\big(t_u(x)\big) = u_1^2 \cdots u_k^2 \Delta_k(x) = \Delta_k\big(t_u(e)\big)\Delta_k(x). \tag{2.4}$$

It is shown that, for each b in Ω, there exists a unique t in the triangular group T such that $b = t(e)$. Hence the map

$$g : \Omega \to T; \qquad b \mapsto t^{-1} \tag{2.5}$$

realizes a division algorithm. This algorithm is the most appropriate for the division of Riesz random variable and so we will use it in all what follows.

Now consider the absolutely continuous Wishart distribution on Ω, with shape parameter $p > \frac{d}{2}(r-1)$ and scale parameter σ,

$$W_{p,\sigma}(dx) = \frac{(\det \sigma)^p}{\Gamma_\Omega(p)} \exp\big(-\langle \sigma, x\rangle\big)(\det x)^{p-\frac{n}{r}} \mathbf{1}_\Omega(x)\, dx. \tag{2.6}$$

Let $p_1, \ldots, p_q$ be in $(\frac{d}{2}(r-1), +\infty)$, $q \geqslant 2$, and let σ be in Ω. If $Y_1, \ldots, Y_q$ are mutually independent random variables in E with Wishart distributions $W_{p_1,\sigma}, \ldots, W_{p_q,\sigma}$, respectively, and if $S = Y_1 + \cdots + Y_q$, then the distribution of the random variable $(X_1, \ldots, X_q) = (g(S)Y_1, \ldots, g(S)Y_q)$ is called the Wishart–Dirichlet distribution on E with parameters $(p_1, \ldots, p_q)$, it is denoted by $D_{(p_1,\ldots,p_q)}$. It is proved in [1] that the random variable $(X_1, \ldots, X_{q-1})$ is independent of S and its distribution does not depend on the parameter σ. In fact, the vector $(X_1, \ldots, X_{q-1})$ has the density

$$\big(B_\Omega(p_1, \ldots, p_q)\big)^{-1} (\det x_1)^{p_1-\frac{n}{r}} \cdots \det(e - x_1 - \cdots - x_{q-1})^{p_q-\frac{n}{r}}$$

where $B_\Omega(p_1, \ldots, p_q)$ is the beta function defined by

$$B_\Omega(p_1, \ldots, p_q) = \frac{\Gamma_\Omega(p_1) \cdots \Gamma_\Omega(p_q)}{\Gamma_\Omega(p)},$$

where $p = p_1 + \cdots + p_q$.

We come now to the definition of the Riesz–Dirichlet distribution. It relies on the following fundamental theorem proved by Hassairi et al. [4]. Recall that the absolutely continuous Riesz distribution on a symmetric cone is defined by these authors by

$$R(s,\sigma)(dx) = \frac{1}{\Gamma_\Omega(s)\Delta_s(\sigma^{-1})} e^{-\langle\sigma,x\rangle} \Delta_{s-\frac{n}{r}}(x)\mathbf{1}_\Omega(x)\,dx,$$

where σ is in Ω, $s=(s_1,\dots,s_r)$ is in $\mathbb{R}^r$ such that $s_i>(i-1)\frac{d}{2}$ for $1\leqslant i\leqslant r$, and

$$\Gamma_\Omega(s) = (2\pi)^{\frac{n-r}{2}} \prod_{j=1}^{r} \Gamma\left(s_j-(j-1)\frac{d}{2}\right).$$

The distribution $R(s,\sigma)$ reduces to the Wishart given in (2.6), when $s_1=\dots=s_r=p$.

Theorem 2.1. *Let $Y_1,\dots,Y_q$ be q independent Riesz random variables with the same σ, $Y_j\sim R(s^j,\sigma)$, where $s^j=(s_1^j,\dots,s_r^j)$ $\forall 1\leqslant j\leqslant q$. If we set $S=Y_1+\dots+Y_q$ and $X_j=g(S)(Y_j)$, then*

(i) *S is a Riesz random variable, $S\sim R(\sum_{j=1}^q s^j,\sigma)$ and is independent of $(X_1,\dots,X_{q-1})$.*
(ii) *The density of the joint distribution of $(X_1,\dots,X_{q-1})$ with respect to the Lebesgue measure is*

$$\frac{\Gamma_\Omega(\sum_{j=1}^q s^j)}{\prod_{j=1}^q \Gamma_\Omega(s^j)} \prod_{j=1}^{q-1} \Delta_{s^j-\frac{n}{r}}(x_j)\Delta_{s^q-\frac{n}{r}}\big(e-(x_1+\dots+x_{q-1})\big)$$

where $x_j\in\Omega$, $1\leqslant j\leqslant q-1$ and $e-\sum_{j=1}^{q-1}x_j\in\Omega$.

Definition 2.1. The distribution of $(X_1,\dots,X_q)$ is called the Riesz–Dirichlet distribution on E with parameters $(s^1,\dots,s^q)$, it is also denoted by $D_{(s^1,\dots,s^q)}$.

Note that, if $s_k^j=p_j$; $1\leqslant k\leqslant r$, then $D_{(s^1,\dots,s^q)}$ is nothing but the Wishart–Dirichlet distribution $D_{(p_1,\dots,p_q)}$. Also for $q=2$, that is if X and Y are two independent random variables; $X\sim R(s,\sigma)$ and $Y\sim R(s',\sigma)$,then we have that the random variable $Z=g(X+Y)(X)$ is independent of $X+Y$ and has the density

$$\big(B_\Omega(s,s')\big)^{-1}\Delta_{s-\frac{n}{r}}(z)\ \Delta_{s'-\frac{n}{r}}(e-z)\mathbf{1}_{\Omega\cap(e-\Omega)}(z)\,dz,$$

where $B_\Omega(s,s')$ is the beta function defined on the symmetric cone Ω (see [2, p. 130]) by $B_\Omega(s,s')=\frac{\Gamma_\Omega(s)\Gamma_\Omega(s')}{\Gamma_\Omega(s+s')}$. The distribution of the random variable $Z=g(X+Y)(X)$ is called the beta-Riesz distribution with parameters s and s'.

3. The Projection of a Riesz–Dirichlet distribution

In this section, we state and prove our main results concerning the projections of a Riesz–Dirichlet distribution on a symmetric cone. Recall that $P_k(x)$ denotes the orthogonal projection of an element x of E on the Jordan subalgebra $E^k=E(c_1+\dots+c_k,1)$. We will denote by Ω^k the symmetric cone of E^k. We also denote by e^k the unity in E^k, by T^k the corresponding triangular group with respect to the Jordan frame $(c_i)_{1\leqslant i\leqslant k}$, and we set g^k the division algorithm defined by the Cholesky decomposition in the cone Ω^k. Similarly, these objects are defined for the subalgebra $E_j=E(c_{r-j+1}+\dots+c_r,1)$, $1\leqslant j\leqslant r-1$. The projection on E_j is denoted P_j^* and the symmetric cone of E_j is denoted Ω_j, it is parameterized by the set $(E_+)_j$. Also e_j, T_j, K_j and g_j denote respectively the unity, the triangular group, the orthogonal group, and the division algorithm corresponding to E_j.

Our first theorem shows that the direct orthogonal projection of a Riesz–Dirichlet distribution on E^k is still Riesz–Dirichlet.

Theorem 3.1. *Let $X=(X_1,\dots,X_q)$ be a Riesz–Dirichlet random variable with distribution $D_{(s^1,\dots,s^q)}$. Then for all $1\leqslant k\leqslant r$, the random variable $(P_k(X_1),\dots,P_k(X_q))$ has a Dirichlet distribution on $E(c_1+\dots+c_k,1)$ with distribution $D_{(\underline{s}^1,\dots,\underline{s}^q)}$, where $\underline{s}^i=(s_1^i,\dots,s_k^i)$, $\forall 1\leqslant i\leqslant q$.*

Our second main result uses the inversion in a symmetric cone and the orthogonal projection P_j^* on the subalgebra $E_j=E(c_{r-j+1}+\dots+c_r,1)$, $1\leqslant j\leqslant r-1$.

Theorem 3.2. *Let $X=(X_1,\dots,X_q)$ be a Riesz–Dirichlet random variable with distribution $D_{(s^1,\dots,s^q)}$, and let $1\leqslant j\leqslant r-1$. Setting $S_j=\sum_{l=1}^q(P_j^*(X_l^{-1}))^{-1}$, we have that*

$$\big(g_j(S_j)\big(P_j^*\big(X_1^{-1}\big)\big)^{-1},\dots,g_j(S_j)\big(P_j^*\big(X_q^{-1}\big)\big)^{-1}\big)$$

has a Riesz–Dirichlet distribution on the algebra $E(c_{r-j+1}+\cdots+c_r,1)$ with distribution $D_{(\bar{s}^1-(r-j)\frac{d}{2},\dots,\bar{s}^q-(r-j)\frac{d}{2})}$, where $\bar{s}^i=(s^i_{r-j+1},\dots,s^i_r)$, $\forall 1\leqslant i\leqslant q$.

Remarks. (i) Particular statements of Theorems 3.1 and 3.2 may be given replacing the Riesz–Dirichlet distributions by the ordinary Wishart–Dirichlet distribution.

(ii) From Theorem 3.1, we have in particular that $(P_1(X_1),\dots,P_1(X_q))$ is a real Dirichlet random variable with parameters $(s_1^1,\dots,s_1^q)$.

(iii) Theorem 3.2 implies that

$$\left(\frac{(P_1^*(X_1^{-1}))^{-1}}{\sum_{l=1}^q(P_1^*(X_l^{-1}))^{-1}},\dots,\frac{(P_1^*(X_q^{-1}))^{-1}}{\sum_{l=1}^q(P_1^*(X_l^{-1}))^{-1}}\right)$$

is a real Dirichlet random variable with parameters $(s_r^1-(r-1)\frac{d}{2},\dots,s_r^q-(r-1)\frac{d}{2})$.

The rest of the paper is devoted to the proofs of Theorems 3.1 and 3.2. For this we need to establish two results concerning the projections P_k and P_j^* which are important in their own rights in the framework on Jordan algebras and their symmetric cones.

Proposition 3.1. *Let u and v be in Ω and let $x=g(u+v)(u)$. Then, for $1\leqslant k\leqslant r$, we have*

$$P_k(x)=g^k\big(P_k(u)+P_k(v)\big)\big(P_k(u)\big).$$

Proof. As $u+v\in\Omega$, then there exist α in E_+ such that

$$u+v=t_\alpha(e),$$

so that

$$x=t_\alpha^{-1}(u).$$

Using the fact that

$$P_k\big(t_u(x)\big)=t_{P_k(u)}\big(P_k(x)\big), \tag{3.1}$$

(see [2, p. 114]), we get

$$P_k(x)=t^{-1}_{P_k(\alpha)}\big(P_k(u)\big)$$

and

$$P_k(u+v)=t_{P_k(\alpha)}\big(e^k\big).$$

Then

$$g^k\big(P_k(u)+P_k(v)\big)=t^{-1}_{P_k(\alpha)}.$$

Hence

$$P_k(x)=g^k\big(P_k(u)+P_k(v)\big)\big(P_k(u)\big).\qquad\square$$

Proposition 3.2. *Let u and v be in Ω and let $x=g(u+v)(u)$. Then, for $1\leqslant j\leqslant r$, we have*

$$\big(P_j^*\big(x^{-1}\big)\big)^{-1}=g_j\big(\big(P_j^*\big((u+v)^{-1}\big)\big)^{-1}\big)\big(P_j^*\big(u^{-1}\big)\big)^{-1}.$$

Proof. As $(u+v)\in\Omega$, there exists a unique α in E_+, such that $u+v=t_\alpha(e)$, and we have (see [3]) that

$$\big(P_j^*(u+v)^{-1}\big)^{-1}=t_\alpha(e_j). \tag{3.2}$$

Since $(P_j^*(u+v)^{-1})^{-1}\in\Omega_j$ then there exists a unique β_j in $(E_+)_j$ such that

$$\big(P_j^*(u+v)^{-1}\big)^{-1}=t_{\beta_j}(e_j). \tag{3.3}$$

Comparing (3.2) and (3.3), we obtain that

$$t_{\beta_j}^{-1} \circ t_{\alpha|_{E_j}} \in T_j \cap K_j,$$

and since the only orthogonal transformation which is triangular with positive diagonal entries is the identity (see [2, p. 111]), then

$$t_{\alpha|_{E_j}} = t_{\beta_j}.$$

On the other hand it is proved in [4] that

$$t_\alpha\big(\big(P_j^*\big(x^{-1}\big)\big)^{-1}\big) = \big(P_j^*\big(u^{-1}\big)\big)^{-1}.$$

Since $(P_j^*(x^{-1}))^{-1} \in E_j$, one obtains

$$\big(P_j^*\big(x^{-1}\big)\big)^{-1} = t_{\beta_j}^{-1}\big(\big(P_j^*(u)^{-1}\big)^{-1}\big).$$

Using (3.3), we get

$$\big(P_j^*\big(x^{-1}\big)\big)^{-1} = g_j\big(\big(P_j^*(u+v)^{-1}\big)^{-1}\big)\big(P_j^*\big(u^{-1}\big)\big)^{-1}. \qquad \square$$

This concludes the proof of Proposition 3.2 and we are now ready to prove the theorems.

Proof of Theorem 3.1. Suppose that the distribution of X is $D_{(s^1,\dots,s^q)}$. Then there exist $Y_1,\dots,Y_q$ independent Riesz random variables with the same scale parameter σ and respective shape parameters $s^1,\dots,s^q$ such that, if $S = Y_1 + \cdots + Y_q$, we have $X = (X_1,\dots,X_q) = (g(S)Y_1,\dots,g(S)Y_q)$.

From Proposition 3.1, we obtain that

$$\big(P_k(X_1),\dots,P_k(X_q)\big) = \left(g^k\left(\sum_{i=1}^q P_k(Y_i)\right)P_k(Y_1),\dots,g^k\left(\sum_{i=1}^q P_k(Y_i)\right)P_k(Y_q)\right).$$

We now use a result due to Hassairi et al. [5] which says that for all $1 \leqslant i \leqslant q$, $P_k(Y_i)$ is a Riesz random variable with parameters $\underline{s}^i$ and $\sigma_1 - P(\sigma_{12})\sigma_0^{-1}$, where σ_1, σ_{12}, σ_0 are the peirce components with respect to $c_1 + \cdots + c_k$ of σ. Since $P_k(Y_1),\dots,P_k(Y_q)$ are independent, one obtains that $(P_k(X_1),\dots,P_k(X_q))$ has the Dirichlet distribution $D_{(\underline{s}^1,\dots,\underline{s}^q)}$. $\square$

Proof of Theorem 3.2. Suppose that the distribution of X is $D_{(s^1,\dots,s^q)}$. Then there exist $Y_1,\dots,Y_q$ independent Riesz random variables with the same scale parameter σ and respective shape parameters $s^1,\dots,s^q$ such that, if $S = Y_1 + \cdots + Y_q$, we have $X = (X_1,\dots,X_q) = (g(S)Y_1,\dots,g(S)Y_q)$.

From Proposition 3.2, we have that for all $1 \leqslant i \leqslant q$

$$\big(P_j^*\big(X_i^{-1}\big)\big)^{-1} = g_j\big(\big(P_j^*(Y_1+\cdots+Y_q)^{-1}\big)^{-1}\big)\big(P_j^*\big(Y_i^{-1}\big)\big)^{-1}.$$

Then

$$S_j = g_j\big(\big(P_j^*(Y_1+\cdots+Y_q)^{-1}\big)^{-1}\big)\left(\sum_{i=1}^q \big(P_j^*\big(Y_i^{-1}\big)\big)^{-1}\right). \tag{3.4}$$

As $(P_j^*(Y_1+\cdots+Y_q)^{-1})^{-1} \in \Omega_j$, then there exist γ_j in $(E_+)_j$ such that

$$\big(P_j^*(Y_1+\cdots+Y_q)^{-1}\big)^{-1} = t_{\gamma_j}(e_j). \tag{3.5}$$

On the other hand $S_j \in \Omega_j$, there exist ν_j in $(E_+)_j$, such that

$$S_j = t_{\nu_j}(e_j).$$

This with (3.4) and (3.5) imply that

$$\sum_{i=1}^q \big(P_j^*\big(Y_i^{-1}\big)\big)^{-1} = t_{\gamma_j}(S_j) = t_{\gamma_j} \circ t_{\nu_j}(e_j). \tag{3.6}$$

In fact

$$g_j(S_j)\big(P_j^*\big(X_i^{-1}\big)\big)^{-1} = \big(t_{\nu_j}^{-1} \circ t_{\gamma_j}^{-1}\big)\big(\big(P_j^*\big(Y_i^{-1}\big)\big)^{-1}\big).$$

From (3.6), we immediately get

$$g_j(S_j)\big(P_j^*\big(X_i^{-1}\big)\big)^{-1} = g_j\left(\sum_{i=1}^q \big(P_j^*\big(Y_i^{-1}\big)\big)^{-1}\right)\big(P_j^*\big(Y_i^{-1}\big)\big)^{-1}. \tag{3.7}$$

From (3.7), we obtain that

$$\begin{aligned}&\big(g_j(S_j)\big(P_j^*\big(X_1^{-1}\big)\big)^{-1},\dots,g_j(S_j)\big(P_j^*\big(X_q^{-1}\big)\big)^{-1}\big)\\&=\left(g_j\left(\sum_{l=1}^{q}\big(P_j^*\big(Y_l^{-1}\big)\big)^{-1}\right)\big(P_j^*\big(Y_1^{-1}\big)\big)^{-1},\dots,g_j\left(\sum_{l=1}^{q}\big(P_j^*\big(Y_l^{-1}\big)\big)^{-1}\right)\big(P_j^*\big(Y_q^{-1}\big)\big)^{-1}\right).\end{aligned}$$

We know use a result due to Hassairi et al. [5] which says that for all $1 \leqslant i \leqslant q$, $(P_j^*(Y_i^{-1}))^{-1}$ is a Riesz random variable with parameters $\bar{s}^i - (r-j)\frac{d}{2}$ and σ_0, where σ_1, σ_{12}, σ_0 are the peirce components with respect to $c_{r-j+1} + \cdots + c_r$ of σ. Since $(P_j^*(Y_1^{-1}))^{-1}, \dots, (P_j^*(Y_q^{-1}))^{-1}$ are independent, one obtains that

$$\big(g_j(S_j)\big(P_j^*\big(X_1^{-1}\big)\big)^{-1},\dots,g_j(S_j)\big(P_j^*\big(X_q^{-1}\big)\big)^{-1}\big)$$

has the Dirichlet distribution $D_{(\bar{s}^1-(r-j)\frac{d}{2},\dots,\bar{s}^q-(r-j)\frac{d}{2})}$. $\square$

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