The efficient solution of electromagnetic scattering for inhomogeneous media

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Abstract

We consider an electromagnetic scattering problem for inhomogeneous media. In particular, we focus on the numerical computation of the electromagnetic scattered wave generated by the interaction of an electromagnetic plane wave and an inhomogeneity in the corresponding propagation medium. This problem is studied in the VV polarization case, where some special symmetry requirements for the incident wave and for the inhomogeneity are assumed. This problem is reformulated as a Fredholm integral equation of second kind, which is discretized by a linear system having a special form. This allows to compute efficiently an approximate solution of the scattering problem by using iterative techniques for linear systems. Some numerical examples are reported.

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1. Introduction

We introduce the notation. Let $\mathbb{R}, \mathbb{C}$ be the sets of real numbers, and complex numbers, respectively. Let $\mathbb{R}^N, \mathbb{C}^N$ be the $N$-dimensional real Euclidean space, and the $N$-dimensional complex Euclidean space, respectively. Let $\underline{x}, \underline{y} \in \mathbb{R}^N$, we denote with $\underline{x}^t \underline{y}$ the Euclidean scalar product of $\underline{x}$ and $\underline{y}$, the superscript $t$ means transposed, $\| \underline{x} \|$ denotes the Euclidean norm of $\underline{x}$, and $\| \underline{x} \|_\infty$ denotes the usual infinity norm of $\underline{x}$. Let $\mathbb{S}^N = \{ \underline{x} \in \mathbb{R}^{N+1} : \| \underline{x} \| = 1 \}$. Let $z \in \mathbb{C}$, we denote with $\text{Re}(z), \text{Im}(z)$ the real part and the imaginary part of $z$, respectively. Finally, we denote with $\text{i}$ the imaginary unit.

We consider a time-harmonic electromagnetic wave propagating in the three-dimensional space. The interaction of this wave with an inhomogeneity in the propagation medium generates a scattered wave, that we suppose harmonic in time and with the same time-frequency of the incident wave. The computation of the scattered wave can be defined as the solution of a boundary value problem for the time-harmonic Maxwell equations [6, p. 238].

This problem can be reduced to a two-dimensional boundary value problem for a scalar unknown function when a special class of inhomogeneities and of incident waves is considered. More precisely, let $\underline{X} = (x_1, x_2, x_3)^t \in \mathbb{R}^3$ be a
generic point, we suppose that the refractive index of the medium is independent of \( x_3 \); note that the transverse section, with respect to \( x_3 \) direction, of the refractive index is denoted with \( n(x) \), \( x = (x_1, x_2)^t \in \mathbb{R}^2 \). Moreover, as incident waves, we consider electromagnetic plane waves having polarization vector parallel to the \( x_3 \) coordinate axis. Such symmetry assumptions are the so called VV polarization case, and imply that the electric field of the scattered wave has only one non-vanishing Cartesian component, i.e., the third component, and this component is independent of \( x_3 \), see [14] for details.

Finally, we assume that \( n(x) = 1 \) for \( x \in \mathbb{R}^2 \setminus D \), where \( D \) is a generic compact set; note that from standard arguments on electromagnetic theory we have \( n = 1 \) in the vacuum. Let \( u^s(x) \in \mathbb{C}, x \in \mathbb{R}^2 \) be the third component of the electric scattered field, function \( u^s \) is the solution of the following boundary value problem:

\[
\begin{align}
\Delta u^s(x) + k^2 n(x) u^s(x) &= k^2 m(x) u^i(x), \quad x \in \mathbb{R}^2, \\
\lim_{\|x\| \to +\infty} \sqrt{\|x\|}(\hat{x}^2 \nabla u^s(x) - i k u^s(x)) &= 0, \quad \hat{x} \in \mathbb{S}^1,
\end{align}
\]

where

\[
u^i(x) = e^{ik\hat{x}^2}, \quad x \in \mathbb{R}^2,
\]

and \( \Delta \) is the Laplacian operator, \( \nabla \) is the gradient operator, \( k > 0 \) is the wave number, \( \hat{x} = (\hat{x}_1, \hat{x}_2)^t \in \mathbb{S}^1 \) is the propagation direction of the incident wave \( u^i \), \( m(x) = 1 - n(x), x \in \mathbb{R}^2 \), and \( \hat{x} = x/\|x\| \in \mathbb{S}^1 \) for \( x \neq 0 \). We note that the support of \( m \) is contained in \( D \), and \( \text{Re}(m(x)) \leq 0, \text{Im}(m(x)) \leq 0, \hat{x} \in \mathbb{R}^2 \).

We consider the following problem:

**Problem 1.** Given \( k, \hat{x}, n \), compute the solution \( u^s \) of problem (1) and (2).

We note that the VV polarization case coincides with the well-known transverse magnetic waves. Similar reductions can be obtained for transverse electric waves. Such special configurations are frequently used in the electromagnetic theory, see [18, Chapter 6; 12, Chapter 8] for two very classical applications. These configurations are usually considered also in the electromagnetic scattering from the theoretical point of view as well as the experimental point of view, see [15,4] for two different examples. Thus two-dimensional scattering problems are object of intense studies, see [20,13,8,1,5] for some recent contributions; and, these studies have two main applications: the development of efficient solvers to use in the corresponding inverse problems, the validation of measurement techniques. Moreover, the relevance of two-dimensional scattering problems is due to the fact that they constitute a simplified environment to develop approximation methods, that usually can be easily generalized to the three-dimensional case, see [7] for an example.

We consider the numerical approximation of Problem 1. Many different methods for the solution of problem (1) and (2), or similar scattering problems, have been proposed in the scientific literature, such as for example finite difference approaches [3], finite element approaches [16]. We note that these general purpose methods cannot be applied directly to problem (1) and (2), being this problem defined on an unbounded domain. This difficulty can be overcome by substituting the Sommerfeld radiation condition at infinity with an absorbing boundary condition on an artificial boundary enclosing the support \( D \) of the inhomogeneity, see [10] for details. However, some specialized numerical methods allow to deal straightforwardly with the unbounded domain of problem (1) and (2), these methods are based on an integral formulation of problem (1) and (2), see [6, p. 234; 14] for details.

We propose a method to compute a numerical approximation of Problem 1. This method is based on a simple discretization of the integral formulation of problem (1) and (2). This discretization produces a linear system, and we show that the corresponding coefficient matrix can be approximated by a special decomposition, that allows to solve efficiently Problem 1.

Finally, we report some numerical examples, where the accuracy and the computational time of the proposed method are compared with the ones of a usual discretization method.

In Section 2 we give the integral formulation of problem (1) and (2), and we describe the method for the numerical solution of Problem 1. In Section 3 we report some numerical results. In Section 4 we give some conclusions and future possible developments of the present paper.
2. The approximation method

The integral formulation of problem (1) and (2) is the main ingredient in the proposed method. Let \( u^T = u^l + u^r \) be the total electric field, \( H_0^{(1)} \) be the Hankel function of first kind and order 0, see \[2, p. 358\] for details. We have the following Fredholm integral equation of second kind for \( u^T \):

\[
 u^T(x) = u^l(x) - \frac{k^2}{4} \int_D H_0^{(1)}(k\|x - y\|)u^T(y)m(y) \, dy, \quad x \in \mathbb{R}^2.
\]  

(4)

We note that integral equation (4) can be easily deduced from problem (1) and (2), since \( \nu/4H_0^{(1)} \) is the Green function of the Helmholtz operator with the Sommerfeld radiation condition, see \[6, p. 65\] for details.

Let \( \{Q_{i_1,i_2} \subset \mathbb{R}^2, i_1 = 1, 2, \ldots, v_1, i_2 = 1, 2, \ldots, v_2\} \) be a partition of \( D \), the integral equation (4) is discretized as follows:

\[
 u^T_{i_1,i_2} = u^l_{i_1,i_2} - \frac{k^2}{4} \sum_{j_1=1}^{v_1} \sum_{j_2=1}^{v_2} u^T_{j_1,j_2} m_{j_1,j_2} \int_{Q_{j_1,j_2}} H_0^{(1)}(k\|\tilde{x}_{j_1,j_2} - y\|) \, dy,
\]

(5)

where, for \( i_1 = 1, 2, \ldots, v_1, i_2 = 1, 2, \ldots, v_2, \tilde{x}_{i_1,i_2} \in \mathbb{R}^2 \) is the center of \( Q_{i_1,i_2} \), \( u^l_{i_1,i_2} = u^l(\tilde{x}_{i_1,i_2}), m_{i_1,i_2} = m(\tilde{x}_{i_1,i_2}) \), and \( u^T_{i_1,i_2} \) is an approximation of \( u^T(\tilde{x}_{i_1,i_2}) \). Note that Eq. (5) is obtained by considering a piece-wise constant approximation of \( u^T \) and \( m \) in \( D \), this approximation is related to the above mentioned partition of \( D \). When a complete knowledge of \( m \) is available we can easily obtain better-quality discretizations of (4), but this integral equation often needs to be solved by using only a partial knowledge of \( m \), such as for example in the Iterative Born Approximation for the Inverse Medium Problem we have to solve (4) several times with different approximations of the unknown function \( m \), see \[19\] for details. However Eq. (5) provides an accurate solution of (4) when suitable partitions of \( D \) are used, in particular, the fineness of these partitions must be proportionate to the variation rate of \( m \) and to the value of wave number \( k \).

Linear system (5) can be rewritten as follows:

\[
 Au = b,
\]

(6)

where \( b \in \mathbb{C}^{v_1v_2} \) contains \( u^l_{i_1,i_2}, i_1 = 1, 2, \ldots, v_1, i_2 = 1, 2, \ldots, v_2, u \in \mathbb{R}^{v_1v_2} \) contains \( u^T_{j_1,j_2}, j_1 = 1, 2, \ldots, v_1, j_2 = 1, 2, \ldots, v_2 \), and the entry of matrix \( A \), at row corresponding to indices \( i_1, i_2 \), and at column corresponding to \( j_1, j_2 \), contains \( \delta_{i_1,j_1}\delta_{i_2,j_2} + ik^2/4m_{i_1,j_2} \int_{Q_{j_1,j_2}} H_0^{(1)}(k\|\tilde{x}_{i_1,i_2} - y\|) \, dy \), where \( \delta \) is the Kronecker function.

In the remaining part of this section we propose an approximation of the coefficient matrix \( A \) of linear system (6). This approximation is computed from the properties of Hankel functions. In particular, we have

\[
 H_0^{(1)}(k\rho) = J_0(k\rho) + iY_0(k\rho), \quad \rho > 0,
\]

(7)

where \( J_0 \) is the Bessel function of order 0, and \( Y_0 \) is the Neumann function of order 0, see \[2, p. 358\] for details. From the usual power series expansion of these functions we have

\[
 J_0(k\rho) \approx \sum_{l=0}^{L} d_l \rho^{2l}, \quad \rho \geq 0,
\]

(8)

\[
 Y_0(k\rho) \approx \frac{2}{\pi} \left( \ln \left( \frac{k}{2} \right) + \frac{1}{2} \ln(\rho^2) + \gamma \right) \sum_{l=0}^{L} d_l \rho^{2l} - \frac{1}{\pi} \sum_{l=0}^{L} \psi_l d_l \rho^{2l}, \quad \rho > 0,
\]

(9)

\[
 d_l = \frac{(-1)^l k^{2l}}{(l!)^2 4^l}, \quad l \geq 0,
\]

(10)

where \( \gamma \approx 0.5772157 \) is the Euler–Mascheroni constant, and \( \psi_0 = 0, \psi_l = 2\sum_{j=1}^{l} 1/j, l \geq 1 \). We note that \( L > 0 \) is a truncation parameter, and approximations (8) and (9) become equalities for \( L = \infty \), see \[2, p. 360\] for details. For the
logarithm appearing in (9) we consider the following truncated expansion:

\[
\ln(\rho^2) \approx \ln(B) - \sum_{l=1}^{L} \frac{(B - \rho^2)^l}{lB^l} = \sum_{l=0}^{L} c_l \rho^{2l},
\]

\(c_l = \begin{cases} \ln(B) - \sum_{j=1}^{L} \frac{1}{j}, & l = 0, \\ (-1)^{l+1} B^{-l} \sum_{j=1}^{L} \frac{1}{j} \binom{j}{l}, & l > 0, \end{cases}\)

where \(B > 0\) is a suitable constant that depends on the size of \(D\). We note that the second power series expansion in (11) is obtained from the first one by expanding the binomials \((B - \rho^2)^l, l = 1, 2, \ldots, L\) and rearranging the corresponding power series.

From formulas (7)–(9), (11) we obtain

\[
H_0^{(1)}(k, \rho) \approx \sum_{l=0}^{L} t_l \rho^{2l}, \quad \rho > 0,
\]

\[
t_l = d_l + i \left[ \frac{2}{\pi} \ln \left( \frac{k}{2} \right) d_l + \frac{2}{\pi} \gamma d_l + \frac{1}{\pi} \left( \sum_{s=0}^{l} c_s d_{l-s} \right) - \frac{1}{\pi} \psi_1 d_l \right], \quad l \geq 0,
\]

that, substituted in (5), gives the following linear system:

\[
\tilde{u}_{i_1,i_2}^T = u_{i_1,i_2}^T - \frac{\kappa k}{4} \sum_{j_1=1}^{v_1} \sum_{j_2=1}^{v_2} \tilde{u}_{j_1,j_2}^T m_{j_1,j_2} \int_{Q_{j_1,j_2}} \sum_{l=0}^{L} t_l ||x|| dx - y ||x||^{2l} dy,
\]

\[i_1 = 1, 2, \ldots, v_1, \quad i_2 = 1, 2, \ldots, v_2.\]

where, for \(i_1 = 1, 2, \ldots, v_1, i_2 = 1, 2, \ldots, v_2, \tilde{u}_{i_1,i_2}^T\) is the approximation of \(u_{i_1,i_2}^T\) obtained as a consequence of the approximations (13) and (14).

We restrict our attention to coordinate partitions of \(D\), where \(Q_{i_1,i_2}, i_1 = 1, 2, \ldots, v_1, i_2 = 1, 2, \ldots, v_2\) are given by rectangles having edges parallel to the coordinate axes. Moreover, we use the following notations: \(Q_{i_1,i_2} = [a_{i_1}, b_{i_1}] \times [a_{i_2}, b_{i_2}], \xi = (\xi_{1,i_1}, \xi_{2,i_2}) \in \mathbb{R}^2, i_1 = 1, 2, \ldots, v_1, i_2 = 1, 2, \ldots, v_2.\) From (15) we obtain

\[
\tilde{u}_{i_1,i_2}^T = u_{i_1,i_2}^T - \frac{\kappa k}{4} \sum_{j_1=1}^{v_1} \sum_{j_2=1}^{v_2} \tilde{u}_{j_1,j_2}^T m_{j_1,j_2} \sum_{l=0}^{L} t_l \sum_{s=0}^{l} \binom{l}{s} \int_{a_{j_1}-\xi_{1,i_1}}^{b_{j_1}-\xi_{1,i_1}} y_1^s dy_1 \int_{a_{j_2}-\xi_{2,i_2}}^{b_{j_2}-\xi_{2,i_2}} y_2^{2l-2s} dy_2,
\]

\[i_1 = 1, 2, \ldots, v_1, \quad i_2 = 1, 2, \ldots, v_2.\]

so, calculating the integrals and rearranging the resulting terms, we obtain

\[
\tilde{u}_{i_1,i_2}^T = u_{i_1,i_2}^T - \frac{\kappa k}{4} \sum_{j_1=1}^{v_1} \sum_{j_2=1}^{v_2} \tilde{u}_{j_1,j_2}^T m_{j_1,j_2} \sum_{l=0}^{L} \sum_{s=0}^{l} \binom{l}{s} (b_{j_1}^l - a_{j_1}^l)(b_{j_2}^s - a_{j_2}^s)
\]

\[
\times \left( \sum_{p=\lfloor l/2 \rfloor + \lfloor s/2 \rfloor}^{L} \sum_{q=\lfloor l/2 \rfloor}^{P} \frac{p}{q} \frac{1}{2q + 1} \frac{1}{2p - 2q + 1} \binom{2q + 1}{l} \binom{2p - 2q + 1}{s} \right)
\]

\[
\times (-\xi_{1,i_1})^{2q+1-l} (-\xi_{2,i_2})^{2p+1-s}, \quad i_1 = 1, 2, \ldots, v_1, \quad i_2 = 1, 2, \ldots, v_2,
\]

where \([\cdot]\) denotes the integer part of \(\cdot\).
Linear system (17) is a discretization of integral equation (4), and this discretization can be considered as an approximation of (6), so we expect that the solution provided by (6) is more accurate than the one provided by (17). However linear system (17) can be solved efficiently due to the special structure of its coefficient matrix, which is given by the identity matrix plus 2\((L + 1)^2\) rank-one matrices. Such a structure can be easily seen from (17), where the coefficients in square brackets are given by a sum of several addenda and each addendum is a product of a factor depending only on the column indices, i.e., \(j_1, j_2\), and a factor depending only on the row indices, i.e., \(i_1, i_2\); we conclude that (17) can be rewritten as follows:

\[
(I + UV^t)\mathbf{u} = \mathbf{b},
\]

where \(U, V\) are suitable complex matrices having \(v_1v_2\) rows and \(2(L + 1)^2\) columns.

3. The numerical experience

We present some results obtained in the numerical solution of Problem 1 by using the method described in the previous section. In particular, problem (1) and (2) is considered for seven different inhomogeneities and two different wave numbers; for each one of these examples the numerical solution computed by linear system (6) is compared with the one computed by linear system (18).

In problem (1) and (2) we consider the following inhomogeneities (see Fig. 1):

\[
n_1(x) = \begin{cases} 2, & \|x\| < 0.5, \\ 1 & \text{otherwise}, \end{cases}
\]

\[
n_2(x) = \begin{cases} 3 + 2t, & \|x\| < 0.25, \\ 3 + t, & 0.25 \leq \|x\| < 0.5, \\ 1 & \text{otherwise}, \end{cases}
\]

\[
n_3(x) = \begin{cases} 2 + 3t, & \|x\|_{\infty} < 0.8, \\ 1 & \text{otherwise}, \end{cases}
\]

\[
n_4(x) = \begin{cases} 1 + t + (1 - i) \sin(\frac{\pi}{2}(x_1 + 1)) \sin(\frac{\pi}{2}(x_2 + 1)), & \|x\|_{\infty} < 1, \\ 1 & \text{otherwise}, \end{cases}
\]

\[
n_5(x) = \begin{cases} 1 + t + (1 - i)S(x_1)S(x_2), & \|x\|_{\infty} < \frac{9}{20}, \\ 1 & \text{otherwise}, \end{cases}
\]

\[
n_6(x) = \begin{cases} 1 + e^{-20(1/5-x_1^2-x_2^2)^2}, & \|x\|_{\infty} < 1, \\ 1 & \text{otherwise}, \end{cases}
\]

\[
n_7(x) = \begin{cases} 1 + t + te^{-17(1/2-x_1^2-x_2^2)^2}, & \|x\|_{\infty} < 1, \\ 1 & \text{otherwise}, \end{cases}
\]

where \(S(x) = \frac{1}{2}(\text{sign}(\cos(3\pi x)) - \frac{1}{2}) + 1, x \in \mathbb{R}, \text{sign}(x) = 1 \text{ for } x > 0, \text{sign}(0) = 0, \text{and sign}(x) = -1 \text{ for } x < 0; \) moreover, we choose \(k = 1, 2\) and \(z = (\cos \pi /3, \sin \pi /3)^t\).

Two different numerical methods are used for the solution of Problem 1:

**Method 1.** The integral formulation (4) of problem (1) and (2) is discretized by linear system (6), whose numerical solution is computed by using the biconjugate gradient method, see [17, p. 83] for details.

**Method 2.** The integral formulation (4) of problem (1) and (2) is discretized by linear system (18), whose numerical solution is computed by using the biconjugate gradient method, see [17, p. 83] for details.

We note that the discretization of integral equations usually produces dense linear systems, so they are generally solved by using direct methods. In this class of methods, linear system (18) can be solved more efficiently than (6) by
using the Sherman–Morrison formula, see [11, p. 50; 9] for details. However, the structure of linear system (18) can be profitably used also in an iterative method since the product of its coefficient matrix with a generic vector needs about $4\nu_1 \nu_2 (L + 1)^2$ multiplicative operations and a similar number of additive operations. For the coefficient matrix of linear system (6) we have $(\nu_1 \nu_2)^2$ multiplicative operations and $(\nu_1 \nu_2)^2$ additive operations, so we expect that Method 2 is more efficient than Method 1 when we can choose $L$ such that $L^2 \ll \nu_1 \nu_2$. Note that $L$ is a truncation parameter and the accuracy of discretization (18) depends on $L$, so the use of too small parameters $L$ is not recommended. The numerical results are obtained with the following choice of the previously introduced parameters: $B = 1.7$, $L = 15, 30$, $\nu_1 = \nu_2 = 70$; moreover, integrals appearing in (5) are evaluated by a Gaussian quadrature formula. These results are reported in Tables 1 and 2, where $T_1$ is the elapsed time in the solution of Problem 1 using Method 1, $T_2$ is the elapsed time in the solution of Problem 1 using Method 2, $E_1$ is the relative error in the solution computed by Method 1, $E_2$ is the relative error in the solution computed by Method 2, $E$ is the relative error between the solution computed by Method 1 and the one computed by Method 2. We note that for inhomogeneity $n = n_1$ problem (1), (2) can be easily rewritten in polar coordinates and can be explicitly solved by using the well-known separation variables techniques, so errors $E_1, E_2$ are computed from the knowledge of this exact solution. In particular, from error $E_1$, shown in Table 1, we have that Method 1 gives quite accurate results. Thus, in Table 2, the solutions provided by this method are used to evaluate the accuracy of the corresponding solutions computed with Method 2.
Table 1
The numerical result for inhomogeneity $n = n_1$; note that $s$ denotes seconds

<table>
<thead>
<tr>
<th>$k$</th>
<th>$T_1$ (s)</th>
<th>$E_1$</th>
<th>$T_2$ (s)</th>
<th>$E_2$</th>
<th>$L = 15$</th>
<th>$L = 30$</th>
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<tr>
<td>1</td>
<td>141</td>
<td>$2.0 \times 10^{-9}$</td>
<td>12</td>
<td>$1.1 \times 10^{-4}$</td>
<td>69</td>
<td>$4.9 \times 10^{-5}$</td>
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<td>2</td>
<td>197</td>
<td>$2.3 \times 10^{-9}$</td>
<td>17</td>
<td>$5.1 \times 10^{-3}$</td>
<td>99</td>
<td>$4.3 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

Table 2
The numerical result for inhomogeneities $n = n_2, n = n_3, \ldots, n = n_7$; note that $s$ denotes seconds

<table>
<thead>
<tr>
<th>$n$</th>
<th>$k$</th>
<th>$T_1$ (s)</th>
<th>$T_2$ (s)</th>
<th>$E$</th>
<th>$L = 15$</th>
<th>$L = 30$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_2$</td>
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<td>185</td>
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<td>93</td>
<td>$1.8 \times 10^{-4}$</td>
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<tr>
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<td>2</td>
<td>267</td>
<td>21</td>
<td>$1.2 \times 10^{-2}$</td>
<td>121</td>
<td>$5.8 \times 10^{-3}$</td>
</tr>
<tr>
<td>$n_3$</td>
<td>1</td>
<td>163</td>
<td>12</td>
<td>$4.4 \times 10^{-4}$</td>
<td>77</td>
<td>$7.8 \times 10^{-5}$</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>232</td>
<td>18</td>
<td>$5.1 \times 10^{-3}$</td>
<td>131</td>
<td>$2.5 \times 10^{-3}$</td>
</tr>
<tr>
<td>$n_4$</td>
<td>1</td>
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<td>$2.5 \times 10^{-3}$</td>
<td>174</td>
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</tbody>
</table>

Tables 1 and 2 show that Method 2 gives quite accurate results with a reduced computational time with respect to Method 1. However, relative errors $E_1, E_2$ and the computational times $T_1, T_2$ increase as wave number increases. This is a quite common behavior for such performance indices, in fact the solution of integral equation (4) becomes more and more difficult as $k$ increases. However, the high sensitivity of errors $E_1, E_2$, with respect to the value of $k$, is due to the use of approximations (9)–(12). Further developments of this method have to improve such approximations.

4. Conclusions

An electromagnetic scattering problem for an inhomogeneous medium is considered. This problem is studied in the VV polarization case, and it is reformulated as a Fredholm integral equation of second kind. This integral equation is discretized by a linear system having a special structure, that allows to compute efficiently an approximate solution of the scattering problem by the usual iterative techniques. The numerical results obtained with the proposed method are very interesting, so we are planning further investigations of such a method. In particular, it seems to be quite attractive when direct solvers cannot be used as consequence of the dimension of the discretized problem. Note that this is a quite common situation in practical problems, such as for example the three-dimensional version of the scattering problem presented in the paper.

References