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The equivalence structure of descriptor representations of systems with possibly inconsistent initial conditions

U. Başer [a], J.M. Schumacher [b,*]

[a] *Department of Mathematics, Istanbul Technical University, 80626 Maslak, Istanbul, Turkey*
[b] *Department of Econometrics, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, Netherlands*

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Abstract

Necessary and sufficient conditions for minimality of descriptor representations of impulsive-smooth behaviors are derived. We obtain a complete set of transformations by which minimal descriptor representations that give rise to the same behavior can be transformed into each other. In particular this leads to a jump-behavioral interpretation of the notion of strong equivalence of descriptor representations.

Keywords: Descriptor representations; External equivalence; Impulsive-smooth behaviors; Inconsistent initial conditions

1. Introduction

Equivalence relations are a classical subject in the study of systems of linear differential and algebraic equations. They are used for instance to transform a given system to a canonical form so that various properties may be read off easily. Of course, one has to specify which equivalence transformations are allowed. One way to settle this is provided by the "behavioral approach" to systems theory, which has been developed in particular by Willems [18]. Briefly, the behavioral method of defining equivalence relations is the following.

Let a set of differential and algebraic equations be given. Suppose that some of the variables in these equations are marked as the ones that provide connections to

* Corresponding author.
E-mail addresses: febaser@cc.itu.edu.tr (U. Başer), jms@kub.nl (J.M. Schumacher).

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the outside world. Call these variables “external”, and let the remaining variables be labeled “internal”. Select a function space where solutions of the given set of differential and algebraic equations will be sought. In a suitable vector version of this space we can form the collection of all trajectories of the external variables for which there exists a corresponding trajectory of the external variables such that the combined trajectory is a solution of the given system. This collection is called the *external behavior*, or simply the *behavior*, of the system. Two systems are said to be *externally equivalent*, or just *equivalent*, if they give rise to the same behavior.

The behavioral definition of system equivalence is strongly focused on a designated set of external variables. This point of view can be motivated for instance in the context of control systems where the inputs and outputs can be thought of as external variables; representations involving only these variables, such as the transfer function, are indeed extensively used in control theory. Another motivation can be found in network modeling, by which we understand here the technique of modeling a complex system as the interconnection of smaller subsystems. In this context, the external variables of subsystem models are interpreted as the variables through which connection to other parts of the system may take place. From a modular point of view, each subsystem may be identified with its external behavior.

The behavioral definition of system equivalence can be applied in many different contexts, which may vary according to system specification as well as according to the function space that is being considered. Possible system specifications include first-order equations that are solved for the derivative, implicit first-order equations, higher-order equations, and so on. Concerning function spaces, one may for instance think of C^∞ functions, locally integrable functions, or spaces of generalized functions. In each of these cases, the notion of equivalence can be defined as above. However, the “conceptual” definition as given above is not very convenient for purposes such as reduction to a canonical form. For such purposes one needs a description of the notion of external equivalence in a concrete, operational form.

For linear systems, the problem of finding a concrete description of the operations of external equivalence has been solved in a number of cases; see for instance [3,7,12,18]. Typically, the transformations can be described in a nice way if it is assumed that they take place between representations that are minimal in an appropriate sense. For nonminimal systems, there are other transformations which produce an equivalent minimal representation. In this paper we shall be concerned with finding the concrete form of transformations under external equivalence for *descriptor representations* when solutions are considered in the space of *impulsive-smooth distributions* [6]. This particular setting can be motivated as follows.

Descriptor representations are systems of differential and algebraic equations that can be written in the form (see for instance [1,7])

$$E\dot{x} = Ax + Bu, \tag{1.1}$$

$$y = Cx + Du, \tag{1.2}$$

where E, A, B, C, and D are constant matrices. The matrix E may be singular or nonsquare. Equations of this form arise naturally when a system description is obtained by the coupling of equations of several subsystems. We will consider the variables y and u as external variables; the variable x, which is sometimes called the pseudo-state, will be taken as an internal variable.

In contrast to standard input/state/output equations, which are of the above form with E equal to the identity matrix, descriptor equations may allow *inconsistent* initial conditions, i.e. initial conditions that do not correspond to smooth solutions. Inconsistent initial conditions typically arise as a consequence of an externally or internally generated event, such as a switch being turned or a constraint becoming active; see for instance [14] for a discussion of mode changes in dynamical systems. An inconsistent initial condition must give rise to a state jump, which in the linear context may be suitably described in the language of distribution theory. Therefore, when looking at descriptor equations, it is natural to consider solutions that may include a distributional component having support at the initial time; a suitable setting is provided by the space of impulsive-smooth distributions [6].

Distributional solutions to implicit systems of linear equations and corresponding equivalence transformations have also been considered in [10], even for the case of time-varying coefficients, but with no distinction between external and internal variables. Note that an interpretation in which all variables are looked at as external is produced by (1.1) if one takes the special choice $C = I$ and $D = 0$.

Our development here is close to the papers [2,3] where the problem of external equivalence in the sense of impulsive-smooth distributions is considered for so-called *pencil* representations, i.e. representations of the forms

$$G\dot{z} = Fz, \tag{1.3}$$

$$w = Hz, \tag{1.4}$$

where w is a vector of external variables. In this paper we will make heavy use of the results in [2,3]. By focusing on descriptor representations, it becomes possible to make a comparison with equivalence notions for descriptor systems that have been proposed in the literature (for instance [11,15,16]). We shall indeed find that a certain transformation group that has been considered before can be interpreted as a group of transformations under external equivalence. As a consequence, this group of transformations obtains an interpretation in terms of jump behaviors. See the end of Section 5 for more discussion.

The paper is structured as follows. Section 2 is devoted to preliminaries. In particular, the definition of the space of impulsive-smooth distributions is recalled, two ways are given for associating an impulsive-smooth behavior to a descriptor representation, and a summary is given of the results in [2,3] that will be used in the present paper. To make use of the results in the cited papers, we need to establish a number of connections between pencil representations and descriptor representations; this is done in Section 3. In order to get a nice description of the relation of external equivalence, we need to restrict ourselves to descriptor representations that are minimal in an appropriate sense. The relevant minimality conditions are obtained in Section 4. The concrete description of the relation of external equivalence for minimal descriptor representations follows in Section 5. Section 6 briefly summarizes the conclusions.

2. Preliminaries

To describe jump behaviors mathematically, we use a simple fragment of the calculus of distributions, following the framework laid out in [6]. Throughout the paper, the first derivative of the Dirac distribution δ is denoted by p, and its kth derivative is denoted by p^k. $\mathbb{R}[s]$ denotes, as usual, the ring of polynomials in s with real coefficients. We denote by $\mathscr{C}(t_0, t_1)$ the set of restrictions of $\mathscr{C}^\infty(\mathbb{R})$-functions to (t_0, t_1) with $-\infty < t_0 < t_1 \leqslant \infty$. The product space $(\mathbb{R}[p] \times \mathscr{C}(t_0, t_1))^k$ is denoted by $\mathscr{C}^k_{\mathrm{imp}}(t_0, t_1)$; so elements v of this space consist of a polynomial part, which we shall refer to as the "purely impulsive part" $v_{\text{p-imp}}$ (representing a pulse at time t_0) and a function part, which is called the "smooth part" v_{sm}. We write $v = v_{\text{p-imp}} + v_{\mathrm{sm}}$; the summation is motivated by the fact that elements of $\mathscr{C}^k_{\mathrm{imp}}(t_0, t_1)$ may be identified with certain distributions (for further detail see [2,6]). On the basis of this identification, the convolution action of the operator p may be described by

$$pv = pv_{\text{p-imp}} + v_{\mathrm{sm}}(t_0^+) + \dot{v}_{\mathrm{sm}}.$$

For the purposes of the present paper, one might simply take this as the definition of the action of p. It is easily verified that for instance the scalar differential equation $\dot{v} = av$ with initial condition $v(0) = v_0$ can be expressed in the present framework by the formula $pv = av + v_0$. To alleviate the notation, explicit mention of the interval (t_0, t_1) will be suppressed in what follows.

We shall consider systems with external variables w, which will sometimes be distinguished into inputs u and outputs y. The variable w takes values in a finite-dimensional real vector space $\mathscr{W} = \mathscr{U} \times \mathscr{Y}$. The *behavior* $\mathscr{B}$ of a given system of differential and algebraic equations in internal and external variables is defined, as in [17,18], as the set of time trajectories of the external variables that are admitted by the system equations. In order to incorporate solutions that exhibit an initial jump, we shall consider solutions in the space $\mathscr{C}^k_{\mathrm{imp}}$ of impulsive-smooth distributions; in this we deviate from the setting of [17,18]. Behaviors defined in the space of impulsive-smooth distributions will be referred to as *impulsive-smooth behaviors* or *jump behaviors*.

A study of impulsive-smooth behaviors was based on the so-called *pencil* representations [2,3]. A pencil representation by itself is just a triple of real matrices satisfying certain size constraints. To such a triple one can associate a behavior in two different ways, which are referred to as the *conventional* style and the *unconventional* style in [2]. The two ways of generating a behavior are equally expressive (in the sense that a behavior generated by one can also be generated by the other, and vice versa); they differ though in the way in which the initial conditions are incorporated. Unconventional pencil representations are defined as follows.

Definition 2.1. For a matrix triple (F, G, H) $(F, G \in \mathbb{R}^{n\times(n+k)},\ H \in \mathbb{R}^{q\times(n+k)})$, the *unconventionally associated impulsive-smooth behavior* $\mathscr{B}_{\mathrm{u}}(F, G, H)$ is

$$\mathscr{B}_{\mathrm{u}}(F, G, H)=\{w \in \mathscr{C}^q_{\mathrm{imp}} \mid \exists z \in \mathscr{C}^{n+k}_{\mathrm{imp}},\ x_0 \in \mathbb{R}^n \\ \text{s.t. } pGz = Fz + x_0,\ w = Hz\}.$$

The triple (F, G, H) is said to constitute an *unconventional pencil representation* of the impulsive-smooth behavior $\mathscr{B} \subset \mathscr{C}^k_{\mathrm{imp}}$ if $\mathscr{B} = \mathscr{B}_{\mathrm{u}}(F, G, H)$.

The definition of conventional pencil representations is as follows.

Definition 2.2. For a matrix triple (F, G, H) $(F, G \in \mathbb{R}^{n\times(n+k)},\ H \in \mathbb{R}^{q\times(n+k)})$, the *conventionally associated impulsive-smooth behavior* $\mathscr{B}_{\mathrm{c}}(F, G, H)$ is

$$\mathscr{B}_{\mathrm{c}}(F, G, H)=\{w \in \mathscr{C}^q_{\mathrm{imp}} \mid \exists z \in \mathscr{C}^{n+k}_{\mathrm{imp}},\ z_0 \in \mathbb{R}^{n+k} \\ \text{s.t. } pGz = Fz + Gz_0,\ w = Hz\}.$$

The triple (F, G, H) is said to constitute a *conventional pencil representation* of the impulsive-smooth behavior $\mathscr{B} \subset \mathscr{C}^k_{\mathrm{imp}}$ if $\mathscr{B} = \mathscr{B}_{\mathrm{c}}(F, G, H)$.

For a brief illustration of the difference between conventional and unconventional representations, consider the situation in which $n = q = 1$, $k = 0$, $G = 0$, $F = 1$, and $H = 1$. The conventionally associated behavior is the one described by the equation $z = 0$, which is just the zero behavior. The unconventionally associated behavior is given by $z + x_0 = 0$, which generates the behavior that consists of all constant multiples of the delta distribution. This "pure delta" behavior can also be generated in the conventional way, but then one has to take n at least equal to 2; for instance, one may use the equations $pz_1 = z_2 + z_{10}$, $z_1 = 0$, $w = z_2$.

Analogously, we shall distinguish between conventional and unconventional *descriptor* representations given by quintuples (E, A, B, C, D) with $E, A \in \mathbb{R}^{n_1\times n_2}$, $B \in \mathbb{R}^{n_1\times m}$, $C \in \mathbb{R}^{p\times n_2}$, and $D \in \mathbb{R}^{p\times m}$). The unconventional form is given in the following way.

Definition 2.3. For a matrix quintuple (E, A, B, C, D), the *unconventionally associated impulsive-smooth behavior* $\mathscr{B}_{\mathrm{u}}(E, A, B, C, D)$ is

$$\mathscr{B}_{\mathrm{u}}(E, A, B, C, D) = \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \in \mathscr{C}^{p+m}_{\mathrm{imp}} \;\middle|\; \exists z \in \mathscr{C}^{n_2}_{\mathrm{imp}},\ x_0 \in \mathbb{R}^{n_1} \\ \text{s.t. } pEz = Az + Bu + x_0,\ y = Cz + Du \right\}.$$

The quintuple (E, A, B, C, D) is said to constitute an *unconventional descriptor representation* of the impulsive-smooth behavior $\mathscr{B} \subset \mathscr{C}^k_{\mathrm{imp}}$ if $\mathscr{B} = \mathscr{B}_{\mathrm{u}}(E, A, B, C, D)$.

The conventional form is the following.

Definition 2.4. For a matrix quintuple (E, A, B, C, D), the *conventionally associated impulsive-smooth behavior* $\mathscr{B}_{\mathrm{c}}(E, A, B, C, D)$ is

$$\mathscr{B}_{\mathrm{c}}(E, A, B, C, D) = \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \in \mathscr{C}^{p+m}_{\mathrm{imp}} \;\middle|\; \exists z \in \mathscr{C}^{n_2}_{\mathrm{imp}},\ z_0 \in \mathbb{R}^{n_2} \right.$$
$$\left. \text{s.t. } pEz = Az + Bu + Ez_0,\ y = Cz + Du \right\}.$$

The quintuple (E, A, B, C, D) is said to constitute a *conventional descriptor representation* of the impulsive-smooth behavior $\mathscr{B} \subset \mathscr{C}^k_{\mathrm{imp}}$ if $\mathscr{B} = \mathscr{B}_{\mathrm{c}}(E, A, B, C, D)$.

As soon as we associate behaviors to matrix tuples, we obtain a notion of *equivalence* [17,18]; we say that systems are *externally equivalent* if their associated behaviors are the same. Of course, the notion depends on the behavior that is being associated, and so one must distinguish between external equivalence in the sense of smooth behaviors, external equivalence in the sense of conventionally associated impulsive-smooth behaviors, and external equivalence in the sense of unconventionally associated impulsive-smooth behaviors. In this paper, we shall consider the latter two equivalences for descriptor representations, as a follow-up to [2,3] where a similar study was made for pencil representations.

A pencil representation (F, G, H) is said to be *minimal* if both the number of rows and the number of columns of the matrices F and G are minimal among the set of all triples equivalent to (F, G, H). Likewise, a descriptor representation (E, A, B, C, D) is *minimal* if both the number of rows and the number of columns of the matrices E and A are minimal among all equivalent descriptor representations. Note that the notion of minimality depends on the notion of equivalence that is being used, so that minimality in the sense of smooth behaviors is not the same as minimality in the sense of impulsive-smooth behaviors, and in the latter case one has to distinguish between conventional and unconventional representations. In the case of smooth behaviors, the difference between conventional and unconventional representations is not relevant because the equation $pGz = Fz + x_0$ does not give rise to smooth solutions unless $x_0 \in \operatorname{im} G$, and a similar remark can be made for descriptor representations.

As there are two indices to be minimized, it is not obvious that minimal pencil or descriptor representations exist at all. For the case of minimality in the sense of *smooth* behaviors, existence of minimal representations as well as necessary and sufficient conditions for minimality in terms of the system parameters were established in [8,9]. In these references, one can also find the transformations that relate equivalent minimal representations to each other. Similar results were obtained for pencil representations of impulsive-smooth behaviors in [3]. Here we will carry out the same program for descriptor representations of impulsive-smooth behaviors.

The development below will rely heavily on the minimality and equivalence results of [3]. For easy reference, we summarize these results here.

Theorem 2.5 [3, Theorem 4.2]. *A triple* (F, G, H) *is a minimal representation of its unconventionally associated impulsive-smooth behavior* $\mathscr{B}_\mathrm{u}(F, G, H)$ *if and only if the following conditions hold*:

(i) $sG - F$ *has full row rank as a rational matrix*,

(ii) $\left[\begin{smallmatrix} G \\ H \end{smallmatrix}\right]$ *has full column rank*,

(iii) $\left[\begin{smallmatrix} sG-F \\ H \end{smallmatrix}\right]$ *has full column rank for all* $s \in \mathbb{C}$.

Theorem 2.6 [3, Theorem 4.4]. *A triple* (F, G, H) *is a minimal representation of its conventionally associated impulsive-smooth behavior, if and only if it satisfies the conditions* (i)–(iii) *of Theorem* 2.5 *and the additional condition*

(iv) $F[\ker G] \subset \operatorname{im} G$.

The following concrete description of the relation of external equivalence in the sense of impulsive-smooth behaviors for minimal unconventional pencil representations was also given in [3].

Theorem 2.7 [3, Theorem 4.1]. *If the matrix triples* (F, G, H) *and* $(\tilde{F}, \tilde{G}, \tilde{H})$ *both satisfy conditions* (i)–(iii) *of Theorem* 2.5, *then* $\mathscr{B}_\mathrm{u}(F, G, H) = \mathscr{B}_\mathrm{u}(\tilde{F}, \tilde{G}, \tilde{H})$ *if and only if there exist constant nonsingular matrices S and T such that* $F = S\tilde{F}T^{-1}$, $G = S\tilde{G}T^{-1}$ *and* $H = \tilde{H}T^{-1}$.

An analogous result for conventional pencil representations was not given in [3]. This void is filled below (Theorem 5.6).

3. From pencil to descriptor form and vice versa

In this section we present algorithms for transforming descriptor representations of impulsive-smooth behaviors to pencil representations, and vice versa. These algorithms will be used in the next section where we derive results on the minimality of descriptor representations by using the known results for pencil representations that were mentioned above.

The connections between pencil representations and descriptor representations of smooth behaviors have been used in [8]. An algorithm is given for rewriting a pencil representation in descriptor form in such a way that minimality is preserved. In [9], an algorithm with a similar property for rewriting a descriptor representation in pencil form was presented and both algorithms were used for deriving minimality conditions for descriptor representations of smooth behaviors.

In pencil representations we have a set of external variables that are not distinguished in "inputs" and "outputs", whereas in descriptor representations we do have such an explicit distinction. We therefore shall also consider pencil representations in which the external variable space $\mathscr{W}$ has been split into an input space $\mathscr{U}$ and an output space $\mathscr{Y}$, with a corresponding decomposition of the matrix H, so that for instance the unconventional form becomes

$$\begin{aligned} pGz &= Fz + x_0 \\ y &= H_y z \\ u &= H_u z. \end{aligned} \tag{3.1}$$

A pencil representation of the above form will be denoted by the quadruple (F, G, H_y, H_u). No conditions are imposed a priori on the decomposition of external variables into inputs and outputs. In [13], the question is considered whether all decompositions are allowable given a certain type of representation (the *representability* problem). It turns out that the conventional and unconventional descriptor representations and the conventional descriptor representation allow any decomposition into inputs and outputs, but there is a condition that needs to be satisfied for representability in unconventional descriptor form; see the cited paper for details.

If we have a descriptor representation (E, A, B, C, D), it is always possible to obtain a corresponding pencil representation via the simple transformations

$$G = \begin{bmatrix} E & 0 \end{bmatrix}, \quad F = \begin{bmatrix} A & B \end{bmatrix}, \quad H = \begin{bmatrix} C & D \\ 0 & I \end{bmatrix}. \tag{3.2}$$

It will follow from the results below that this transformation preserves minimality in the case of unconventional representations but not in the case of conventional representations.

The following lemma is essential for the proof of external equivalence of representations of impulsive-smooth behaviors.

Lemma 3.1. *Let* $P(s) \in \mathbb{R}^{n\times m}[s]$ *and* $Q(s) \in \mathbb{R}^{q\times m}[s]$. *Consider*

$$\mathscr{B}(P, Q) := \{w \in \mathscr{C}^q_{\mathrm{imp}} \mid \exists z \in \mathscr{C}^m_{\mathrm{imp}},\ x_0 \in \mathbb{R}^n \text{ s.t. } x_0 = P(p)z,\ w = Q(p)z\}.$$

Moreover, assume that $P(s)$ *and* $Q(s)$ *have the following form (with respect to conformable partitionings):*

$$P(s) = \begin{bmatrix} P_1(s) & P_2(s) \\ 0 & P_3(s) \end{bmatrix}, \qquad Q(s) = \begin{bmatrix} 0 & Q_2(s) \end{bmatrix},$$

where $P_1(s)$ *has full row rank. Then,* $\mathscr{B}(P, Q) = \mathscr{B}(P_3, Q_2)$.

Proof. It is immediately seen from the definitions that $\mathscr{B}(P, Q) \subset \mathscr{B}(P_3, Q_2)$. To show the converse, let $w \in \mathscr{B}(P_3, Q_2)$ so that there exists an impulsive-smooth z_2 and a constant x_{20} such that $w = Q_2(p)z_2$ and $P_3(p)z_2 = x_{20}$. Since $P_1(s)$ has full row rank, we can partition $P_1(s)$ as

$$P_1(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \end{bmatrix},$$

where $P_{11}(s)$ is nonsingular. Using the fact that the operator $P_{11}(p)$ (as a mapping between spaces of vector-valued impulsive-smooth functions) is invertible (cf. [2,4,6]), we can define

$$z_{11} = -{P_{11}}^{-1}(p)P_2(p)z_2, \quad z_1 = \begin{bmatrix} z_{11} \\ 0 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 0 \\ x_{20} \end{bmatrix}.$$

With these definitions, we have $P_1(p)z_1 + P_2(p)z_2 = x_0$ and it follows that $w \in \mathscr{B}(P, Q)$. □

In the following we consider the transformation from pencil to descriptor representations, both in the conventional and in the unconventional case. First, let us consider the pencil representation given by Eq. (3.1). Decompose the internal variable space $\mathscr{Z}$ (the space on which F and G act) as $\mathscr{Z} = \mathscr{Z}_1 \oplus \mathscr{Z}_2 \oplus \mathscr{Z}_3$, where $\mathscr{Z}_2 = \ker G \cap \ker H_u$ and $\mathscr{Z}_2 \oplus \mathscr{Z}_3 = \ker G$. Accordingly, write

$$G = \begin{bmatrix} G_1 & 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} F_1 & F_2 & F_3 \end{bmatrix},$$
$$H_y = \begin{bmatrix} H_{y1} & H_{y2} & H_{y3} \end{bmatrix}, \quad H_u = \begin{bmatrix} H_{u1} & 0 & H_{u3} \end{bmatrix}. \tag{3.3}$$

The matrices G_1 and H_{u3} both have full column rank.

Algorithm 3.2. Consider the behavior $\mathscr{B}_\mathrm{u}(F, G, H_y, H_\mathrm{u})$. Assume that $H_u[\ker G] = \mathscr{U}$ and that the matrices F, G, H_y and H_u are of the form as in (3.3). The matrix H_{u3} is invertible (see the proof of Lemma 3.3 below). Define descriptor matrices by

$$\begin{aligned} &E = \begin{bmatrix} G_1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} \hat{F}_1 & F_2 \end{bmatrix}, \quad B = \hat{F}_3, \\ &C = \begin{bmatrix} \hat{H}_{y1} & H_{y2} \end{bmatrix}, \quad D = \hat{H}_{y3}, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} &\hat{F}_1 = F_1 - F_3 H_{u3}^{-1} H_{u1}, \quad \hat{F}_3 = F_3 H_{u3}^{-1}, \\ &\hat{H}_{y1} = H_{y1} - H_{y3} H_{u3}^{-1} H_{u1}, \quad \hat{H}_{y3} = H_{y3} H_{u3}^{-1}. \end{aligned} \tag{3.5}$$

Lemma 3.3. *Let (E, A, B, C, D) be a descriptor representation that results from applying Algorithm 3.2 to a pencil representation (F, G, H_y, H_u) satisfying $H_u[\ker G] = \mathscr{U}$. Then these two representations are externally equivalent as unconventional representations of impulsive-smooth behaviors.*

Proof. We note that the assumption $H_u[\ker G] = \mathscr{U}$ and the decomposition on the internal variable space imply $H_u[\ker G] = \operatorname{im}[0\ H_{u3}] = \mathscr{U}$. Therefore, H_{u3} is non-singular. Now, multiply F, G, H_y and H_u on the right by

$$T = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -H_{u3}^{-1}H_{u1} & 0 & H_{u3}^{-1} \end{bmatrix}. \tag{3.6}$$

Since the only operation that is involved in this algorithm is to choose another basis for the internal variable space, according to Theorem 2.7 we will obtain the following equivalent representation to the representation given in (3.1)

$$\begin{aligned} x_0 &= \begin{bmatrix} pG_1 - \hat{F}_1 & -F_2 & -\hat{F}_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \\ \begin{bmatrix} y \\ u \end{bmatrix} &= \begin{bmatrix} \hat{H}_{y1} & H_{y2} & \hat{H}_{y3} \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}. \end{aligned} \tag{3.7}$$

Thus, $\mathscr{B}_\mathrm{u}(E, A, B, C, D) = \mathscr{B}_\mathrm{u}(F, G, H_y, H_\mathrm{u})$. □

Algorithm 3.4. Consider the behavior $\mathscr{B}_\mathrm{c}(F, G, H_y, H_\mathrm{u})$. Assume that the matrices F, G, H_y and H_u are of the form as in (3.3). Then, by renumbering the u variables, we can write

$$H_{u1} = \begin{bmatrix} H_{11} \\ H_{21} \end{bmatrix}, \quad H_{u3} = \begin{bmatrix} H_{13} \\ H_{23} \end{bmatrix}, \tag{3.8}$$

where H_{23} is invertible (or empty, if $\ker G \subset \ker H_u$). Now, define descriptor matrices by

$$\begin{aligned} E &= \begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} \bar{F}_1 & F_2 \\ -\bar{H}_{11} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \bar{F}_3 & 0 \\ -\bar{H}_{13} & I \end{bmatrix}, \\ C &= \begin{bmatrix} \bar{H}_{y1} & H_{y2} \end{bmatrix}, \quad D = \begin{bmatrix} \bar{H}_{y3} & 0 \end{bmatrix}, \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} \bar{F}_1 &= F_1 - F_3H_{23}^{-1}H_{21}, \\ \bar{F}_3 &= F_3H_{23}^{-1}, \\ \bar{H}_{y1} &= H_{y1} - H_{y3}H_{23}^{-1}H_{21}, \\ \bar{H}_{11} &= H_{11} - H_{13}H_{23}^{-1}H_{21}, \\ \bar{H}_{y3} &= H_{y3}H_{23}^{-1}, \\ \bar{H}_{13} &= H_{13}H_{23}^{-1}. \end{aligned} \tag{3.10}$$

Lemma 3.5. *Let* (E, A, B, C, D) *be a descriptor representation that results from applying Algorithm* 3.4 *to a pencil representation* (F, G, H_y, H_u). *Then these two representations are externally equivalent as conventional representations of impulsive-smooth behaviors.*

Proof. Let us consider the representation given in (3.1) with behavior $\mathscr{B}_c(F, G, H_y, H_u)$ and assume that F, G, H_y and H_u are given of the form as in Eq. (3.3). Then, there exists an initial condition z_0 such that $x_0 = Gz_0$. By taking $H_{u1} = H_{21}$ and $H_{u3} = H_{23}$ in T and multiplying F, G, H_y and H_u on the right by T we will obtain a conventional representation which is equivalent to the representation in (3.1) with initial condition Gz_0, and by Lemma 3.1 it is also equivalent to the representation given below:

$$\begin{bmatrix} Gz_0 \\ 0 \end{bmatrix} = \begin{bmatrix} pG_1 - \bar{F}_1 & -F_2 & -\bar{F}_3 & 0 \\ \bar{H}_{11} & 0 & \bar{H}_{13} & -I \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ u_1 \end{bmatrix},$$

$$\begin{bmatrix} y \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \bar{H}_{y1} & H_{y2} & \bar{H}_{y3} & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ u_1 \end{bmatrix}.$$

Thus, $\mathscr{B}_c(E, A, B, C, D) = \mathscr{B}_c(F, G, H_y, H_u)$. $\square$

In the following, we will present two algorithms for obtaining a pencil representation from a descriptor representation for an impulsive-smooth behavior.

First, let us consider the descriptor representation. Decompose the descriptor space $\mathscr{X}_d$ (the space on which E and A act) as $\mathscr{X}_{d1} \oplus \mathscr{X}_{d2}$, where $\mathscr{X}_{d2} = \ker E$. Decompose the equation space $\mathscr{X}_e$ (the space that E and A map into) as $\mathscr{X}_{e1} \oplus \mathscr{X}_{e2} \oplus \mathscr{X}_{e3}$, where $\mathscr{X}_{e1} = \operatorname{im} E$ and $\mathscr{X}_{e1} \oplus \mathscr{X}_{e2} = \operatorname{im}[E\ \ B]$. Accordingly, write

$$E = \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix},$$
$$B = \begin{bmatrix} B_1 \\ B_2 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}. \tag{3.11}$$

Since the matrix B_2 is surjective, by renumbering the u variables we can write

$$\begin{bmatrix} B_1 \\ B_2 \\ 0 \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} D_1 & D_2 \end{bmatrix}, \tag{3.12}$$

where B_{22} is invertible.

Algorithm 3.6. Consider the behavior $\mathscr{B}_{\mathrm{u}}(E, A, B, C, D)$ and the matrices in (3.11) and (3.12). Define pencil matrices as follows:

$$
\begin{gathered}
F = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \bar{B}_{11} & 0 \\ -\bar{A}_{21} & -\bar{A}_{22} & -\bar{B}_{21} & -I \\ -A_{31} & -A_{32} & 0 & 0 \end{bmatrix}, \\
G = \begin{bmatrix} E_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad H = \begin{bmatrix} C_1 & C_2 & D_1 & D_2 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix},
\end{gathered} \tag{3.13}
$$

where

$$
\begin{aligned}
\bar{A}_{11} &= A_{11} - B_{12}B_{22}^{-1}A_{21}, \\
\bar{A}_{12} &= A_{12} - B_{12}B_{22}^{-1}A_{22}, \\
\bar{B}_{11} &= B_{11} - B_{12}B_{22}^{-1}B_{21}, \\
\bar{A}_{21} &= B_{22}^{-1}A_{21}, \\
\bar{A}_{22} &= B_{22}^{-1}A_{22}, \\
\bar{B}_{21} &= B_{22}^{-1}B_{21}.
\end{aligned} \tag{3.14}
$$

Lemma 3.7. *Let (F, G, H) be a pencil representation with behavior $\mathscr{B}_{\mathrm{u}}(F, G, H)$ that results from applying Algorithm* 3.6 *to a descriptor representation with behavior $\mathscr{B}_{\mathrm{u}}(E, A, B, C, D)$. Then, these two representations are externally equivalent.*

Proof. Let us consider a descriptor representation with behavior $\mathscr{B}_{\mathrm{u}}(E, A, B, C, D)$ determined by the equations:

$$pEz = Az + Bu + x_0, \tag{3.15}$$

$$y = Cz + Du. \tag{3.16}$$

Assume that descriptor matrices are given in the form as in Eqs. (3.11) and (3.12). Then, if we multiply Eq. (3.15) on the left by

$$S = \begin{bmatrix} I & -B_{12}B_{22}^{-1} & 0 \\ 0 & B_{22}^{-1} & 0 \\ 0 & 0 & I \end{bmatrix}, \tag{3.17}$$

we will obtain an equivalent representation. If we define F, G and H as in (3.2), then the descriptor representation can be regarded as a pencil representation (F, G, H) with behavior $\mathscr{B}_{\mathrm{u}}(F, G, H)$, where F, G, and H are of the form given in (3.13). Thus, the descriptor representation is externally equivalent to the pencil representation obtained by Algorithm 3.6. □

The following algorithm is similar to Algorithm 3.28 in [7].

Algorithm 3.8. Consider the behavior $\mathscr{B}_{\mathrm{c}}(E, A, B, C, D)$ and the matrices in Eqs. (3.11) and (3.12). Define pencil matrices as follows:

$$G = \begin{bmatrix} E_{11} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad F = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} & \bar{B}_{11} \\ A_{31} & A_{32} & 0 \end{bmatrix},$$
$$H_y = \begin{bmatrix} \bar{C}_1 & \bar{C}_2 & \bar{D}_1 \end{bmatrix}, \quad H_u = \begin{bmatrix} 0 & 0 & I \\ -\bar{A}_{21} & -\bar{A}_{22} & -\bar{B}_{21} \end{bmatrix}, \tag{3.18}$$

where

$$\bar{C}_1 = C_1 - D_2\bar{A}_{21}, \quad \bar{C}_2 = C_2 - D_2\bar{A}_{22}, \quad \bar{D}_1 = D_1 - D_2\bar{B}_{21} \tag{3.19}$$

and the other matrices are the same as in Eq. (3.15).

Lemma 3.9. *Let (F, G, H) be a pencil representation with behavior $\mathscr{B}_{\mathrm{c}}(F, G, H)$ that results from applying Algorithm* 3.8 *to a descriptor representation with behavior $\mathscr{B}_{\mathrm{c}}(E, A, B, C, D)$. Then, these two representations are externally equivalent.*

Proof. Consider a conventional descriptor representation $\mathscr{B}_{\mathrm{c}}(E, A, B, C, D)$. Then, for any $x_0 \in \operatorname{im} E$ there exists z_0 such that $x_0 = Ez_0$. Thus, if we follow the procedure given in the proof of the previous lemma, we will obtain the representation below, which is externally equivalent to a pencil representation with conventional behavior

$$\begin{bmatrix} E_{11}z_{10} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} pE_{11} - \bar{A}_{11} & -\bar{A}_{12} & -\bar{B}_{11} & 0 \\ -\bar{A}_{21} & -\bar{A}_{22} & -\bar{B}_{21} & -I \\ -A_{31} & -A_{32} & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ u_1 \\ u_2 \end{bmatrix}, \tag{3.20}$$

$$\begin{bmatrix} y \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} C_1 & C_2 & D_1 & D_2 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ u_1 \\ u_2 \end{bmatrix}. \tag{3.21}$$

Here, z_{10}, z_1, z_2 and u_1, u_2 are obtained by a suitable partitioning of z_0, z, and u respectively. By Lemma 3.1 this representation is equivalent to the following representation:

$$\begin{bmatrix} E_{11}z_{10} \\ 0 \end{bmatrix} = \begin{bmatrix} pE_{11} - \bar{A}_{11} & -\bar{A}_{12} & -\bar{B}_{11} \\ -A_{31} & -A_{32} & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ u_1 \end{bmatrix}, \tag{3.22}$$

$$\begin{bmatrix} y \\ u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \bar{C}_1 & \bar{C}_2 & \bar{D}_1 \\ 0 & 0 & I \\ -\bar{A}_{21} & -\bar{A}_{22} & -\bar{B}_{21} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ u_1 \end{bmatrix}. \tag{3.23}$$

Thus, $\mathscr{B}_{\mathrm{c}}(E, A, B, C, D) = \mathscr{B}_{\mathrm{c}}(F, G, H)$. □

Finally, we consider descriptor representations with a zero feedthrough term. The following algorithm is similar to Algorithm 3.36 in [7].

Algorithm 3.10. Let a descriptor representation be given by (E, A, B, C) (i.e. $D = 0$). Decompose the descriptor space X_d as $X_{d1} \oplus X_{d2} \oplus X_{d3}$, where $X_{d3} = A^{-1}[\mathrm{im}\, E] \cap \ker E$ and $X_{d2} \oplus X_{d3} = \ker E$. Decompose the equation space X_e as $X_{e1} \oplus X_{e2} \oplus X_{e3}$, where $X_{e1} = \mathrm{im}\, E$ and $X_{e2} = AX_{d2}$. Accordingly, write

$$\begin{bmatrix} E_{11}z_{01} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} pE_{11} - A_{11} & -A_{12} & -A_{13} \\ -A_{21} & -A_{22} & 0 \\ -A_{31} & 0 & 0 \end{bmatrix} z - \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} u \tag{3.24}$$

$$y = [C_1 \quad C_2 \quad C_3]z, \tag{3.25}$$

where E_{11} and A_{22} are nonsingular. Since

$$C_2 A_{22}^{-1}([-A_{21} \; -A_{22} \; 0]z - B_2 u) = 0,$$

we can write

$$y = \begin{bmatrix} C_1 - C_2 A_{22}^{-1} A_{21} & 0 & C_3 \end{bmatrix} z - C_2 A_{22}^{-1} B_2 u.$$

Now, define a descriptor representation $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ by

$$\begin{aligned} &\tilde{E} = \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} A_{11} & A_{13} \\ A_{31} & 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_1 \\ B_3 \end{bmatrix}, \\ &\tilde{C} = \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix}, \quad \tilde{D} = -C_2 A_{22}^{-1} B_2, \end{aligned} \tag{3.26}$$

where $\tilde{C}_1 = C_1 - C_2 A_{22}^{-1} A_{21}$, $\tilde{C}_2 = C_3$. Since A_{22} is nonsingular and the rows of z corresponding to the columns of A_{22} do not affect the behavior, then by Lemma 3.1 it is clear that

$$\mathscr{B}_{\mathrm{c}}(E, A, B, C) = \mathscr{B}_{\mathrm{c}}(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$$

and

$$\tilde{A}[\ker \tilde{E}] \subset \mathrm{im}\, \tilde{E}.$$

4. Minimality of descriptor representations

In this section, we discuss the minimality of both conventional and unconventional descriptor representations. We recall that a descriptor representation (E, A, B, C, D) is said to be minimal if both the number of rows and the number of columns of the matrices E and A are minimal among all equivalent descriptor representations.

In Lemma 3.3 above, we carried out the transition from unconventional pencil to unconventional descriptor representation under the condition $H_u[\ker G] = \mathscr{U}$. It will be important below that this property is preserved under a certain transformation as shown in the next lemma.

Lemma 4.1. *Let $G : \mathscr{Z} \to \mathscr{X}$ be of the form*

$$G = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}$$

with G_2 full column rank and consider $H_u = [H_{u1}\ H_{u2}]$: $\mathscr{Z} \to \mathscr{U}$. If $H_u[\ker G] = \mathscr{U}$, then also $H_{u1}[\ker G_1] = \mathscr{U}$.

Proof. Take $u \in \mathscr{U}$, then we may write

$$u = [H_{u1}\ H_{u2}] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad \text{with} \quad \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 0.$$

From $G_2 z_2 = 0$ it follows that $z_2 = 0$, so actually $u = H_{u1} z_1$ with $G_1 z_1 = 0$, i.e. $u \in H_{u1}[\ker G_1]$. $\square$

In the following lemma, we obtain already one part of the minimality conditions for unconventional descriptor representations.

Lemma 4.2. *A descriptor representation (E, A, B, C, D) is minimal in the sense of unconventional representations of impulsive-smooth behaviors only if the following condition holds*:

$[sE - A \quad -B]$ *has full row rank as a rational matrix.*

Proof. Define a pencil representation as in (3.2). Note that $H_u[\ker G] = \mathscr{U}$. Let E and A have size $n_1 \times n_2$, $\dim \mathscr{Y} = p$ and $\dim \mathscr{U} = m$. Then, G has size $n_1 \times (n_2 + m)$. If condition (i) does not hold, then $sG - F$ will not have full row rank as a rational matrix and hence a reduction is possible as in [3, Proof of Theorem 2.3] to a representation of size $\tilde{n}_1 \times (\tilde{n}_2 + m)$ with $\tilde{n}_1 < n_1$ and $\tilde{n}_2 \leqslant n_2$. By Lemma 4.1, we still have $\tilde{H}_\mathrm{u}[\ker \tilde{G}] = \mathscr{U}$ in the reduced representation. By Lemma 3.3, we can therefore find a descriptor representation of size $\tilde{n}_1 \times \tilde{n}_2$. Because $\tilde{n}_1 < n_1$ and $\tilde{n}_2 \leqslant n_2$, the representation that we started with is not minimal. $\square$

The full set of minimality conditions for unconventional descriptor representations can be stated as follows.

Theorem 4.3. *A descriptor representation (E, A, B, C, D) is a minimal representation of its unconventionally associated behavior $\mathscr{B}_\mathrm{u}(E, A, B, C, D)$ if and only if the following conditions hold*:

(i) $\begin{bmatrix} sE - A & -B \end{bmatrix}$ *has full row rank as a rational matrix,*

(ii) $\begin{bmatrix} E \\ C \end{bmatrix}$ *has full column rank,*

(iii) $\begin{bmatrix} sE - A \\ C \end{bmatrix}$ *has full column rank for all $s \in \mathbb{C}$.*

Proof. The necessity of condition (i) has already been shown in Lemma 4.2. The other conditions are shown to be necessary exactly as in the case of smooth behaviors (see [8, Proof of Lemma 4.7] for (ii) and [7, Proof of Theorem 4.12] for (iii)), by using the property given in Lemma 3.1. To prove the sufficiency, suppose that (E, A, B, C, D) satisfies (i)–(iii). Then, it is readily verified on the basis of Theorem 2.5 that the associated pencil representation defined by the equations in (3.2) is minimal. Hence, there can be no smaller descriptor representation of the same behavior. □

The analogous result for conventional representations is the following.

Theorem 4.4. *A descriptor representation* (E, A, B, C, D) *is a minimal representation of its conventionally associated behavior* $\mathscr{B}_c(E, A, B, C, D)$ *if and only if conditions* (i)–(iii) *of Lemma* 4.2 *and the additional condition*

(iv) $A[\ker E] \subset \operatorname{im} E$

are satisfied.

Proof. The proofs of conditions (i)–(iii) are similar to the proofs of the same conditions in Lemma 4.3, with the initial condition being taken as $x_0 = Ez_0$ since we now consider the conventional behavior $\mathscr{B}_c(E, A, B, C, D)$. To prove the necessity of (iv), apply Algorithm 3.10 to (E, A, B, C, D). Then, we have $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ as in (3.26) except that $\tilde{D} = D - C_2 A_{22}^{-1} B_2$. Because of the equality between $\mathscr{B}_c(E, A, B, C, D)$ and $\mathscr{B}_c(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ and the minimality of (E, A, B, C, D), the matrix A_{22}, which is given in Eq. (3.25) in Algorithm 3.10, should be empty. Thus (iv) holds. (Compare the argument in the proof of [7, Lemma 4.8].)

To prove sufficiency, suppose that (E, A, B, C, D) satisfies (i)–(iv). It can be verified that, when Algorithm 3.8 is applied to (E, A, B, C, D), the resulting pencil representation (F, G, H_y, H_u) defined by the matrices in (3.18) satisfies conditions (i)–(iv) of Theorem 2.6. Note in particular that condition (iv) implies $A_{32} = 0$ and condition (i) implies that A_{31} in F (in (3.18)) has full row rank. So, by Theorem 2.6 (F, G, H_y, H_u) is minimal with respect to the behavior $\mathscr{B}_c(F, G, H_y, H_u)$. Hence, there can be no smaller conventional descriptor representation having the same behavior. □

Comparing the result above to the minimality conditions for descriptor representations of smooth behaviors as given in [9], we see that the two sets of minimality conditions are identical, except that for minimality in the sense of smooth behaviors condition (i) above is replaced by the stronger requirement that the matrix $[E \;\; B]$ should have full row rank.

5. Equivalence of descriptor representations

In this section we obtain concrete descriptions of the transformations that relate minimal descriptor representations of impulsive-smooth behaviors, both in the conventional and in the unconventional case. We begin with some preparatory material.

Definition 5.1. The triples (F, G, H) and $(\tilde{F}, \tilde{G}, \tilde{H})$ are said to be *strongly similar* if there exist invertible matrices S and T such that

$$\begin{bmatrix} s\tilde{G} - \tilde{F} \\ \tilde{H} \end{bmatrix} = \begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} sG - F \\ H \end{bmatrix} T^{-1}. \tag{5.1}$$

Definition 5.2. The triples (F, G, H) and $(\tilde{F}, \tilde{G}, \tilde{H})$ are said to be *weakly similar* if there exist constant invertible matrices S and T and a constant matrix X such that

$$\begin{bmatrix} s\tilde{G} - \tilde{F} \\ \tilde{H} \end{bmatrix} = \begin{bmatrix} S & 0 \\ X & I \end{bmatrix} \begin{bmatrix} sG - F \\ H \end{bmatrix} T^{-1}. \tag{5.2}$$

Condition (5.2) is equivalent to the requirements $\tilde{F} = SFT^{-1}$, $\tilde{G} = SGT^{-1}$, $\tilde{H} = (H - XF)T^{-1}$ and $XG = 0$. It is straightforward to prove the following lemma.

Lemma 5.3. *Among matrix triples of equal dimensions, weak similarity is an equivalence relation.*

It is also easy to verify that the minimality conditions of Theorems 2.5 and 2.6 are similarity invariants, i.e. if a triple (F, G, H) satisfies the conditions of these theorems, then the same holds for any triple that is weakly similar to (F, G, H). The following lemma takes a little bit more effort.

Lemma 5.4. *Weakly similar triples generate the same conventional behavior.*

Proof. Assume that the triples (F, G, H) and $(\tilde{F}, \tilde{G}, \tilde{H})$ are weakly similar. Then there exist constant invertible matrices S and T and a constant matrix X such that (5.2) holds. Take $w \in \mathscr{B}_{\mathrm{c}}(F, G, H)$. Then by definition there exist $z \in \mathscr{C}_{\mathrm{imp}}^{n+k}$ and $z_0 \in \mathbb{R}^{n+k}$ such that

$$\begin{bmatrix} Gz_0 \\ w \end{bmatrix} = \begin{bmatrix} pG - F \\ H \end{bmatrix} z. \tag{5.3}$$

Define $\tilde{z} = Tz$ and $\tilde{z}_0 = Tz_0$. Since $XG = 0$, we can then write

$$\begin{aligned} \begin{bmatrix} p\tilde{G} - \tilde{F} \\ \tilde{H} \end{bmatrix} \tilde{z} &= \begin{bmatrix} S & 0 \\ X & I \end{bmatrix} \begin{bmatrix} pG - F \\ H \end{bmatrix} T^{-1}\tilde{z} = \begin{bmatrix} S & 0 \\ X & I \end{bmatrix} \begin{bmatrix} Gz_0 \\ w \end{bmatrix} \\ &= \begin{bmatrix} SGT^{-1}\tilde{z}_0 \\ w \end{bmatrix} = \begin{bmatrix} \tilde{G}\tilde{z}_0 \\ w \end{bmatrix}. \end{aligned} \tag{5.4}$$

It follows that $w \in \mathscr{B}_c(\tilde{F}, \tilde{G}, \tilde{H})$. So we have $\mathscr{B}_c(F, G, H) \subset \mathscr{B}_c(\tilde{F}, \tilde{G}, \tilde{H})$. Since weak similarity is an equivalence relation, it follows that, actually, equality must hold. $\square$

Below we will have occasion to use the following lemma, which relates conventional pencil representations to unconventional ones.

Lemma 5.5. *If a triple* (F, G, H) *satisfies the minimality conditions* (i)–(iv) *mentioned in Theorem* 2.6, *then* (F, G, H) *is weakly similar to a triple* $(\hat{F}, \hat{G}, \hat{H})$, *where*

$$\hat{F} = \begin{bmatrix} F_{11} & 0 \\ 0 & I \end{bmatrix}, \quad \hat{G} = \begin{bmatrix} G_{11} & G_{12} \\ 0 & 0 \end{bmatrix}, \quad \hat{H} = [H_1 \; 0] \tag{5.5}$$

in which $[G_{11} \; G_{12}]$ *has full row rank and*

$$\textit{no. of columns of } G_{12} = \operatorname{codim} \operatorname{im} G_{11}. \tag{5.6}$$

Moreover, the triple (F_{11}, G_{11}, H_1) *satisfies minimality conditions* (i)–(iii) *of Theorem* 2.5, *and we have*

$$\mathscr{B}_c(F, G, H) = \mathscr{B}_u(F_{11}, G_{11}, H_1). \tag{5.7}$$

Proof. Let U be a constant nonsingular matrix such that

$$UG = \begin{bmatrix} G_1 \\ 0 \end{bmatrix}, \tag{5.8}$$

where G_1 has full row rank, and define F_1 and F_2 by the comformable partitioning

$$UF = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}. \tag{5.9}$$

By minimality condition (i), the matrix F_2 must have full row rank. Then there exists a constant nonsingular matrix V such that

$$UFV = \begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix}, \tag{5.10}$$

where F_{22} is nonsingular. Let

$$G_1 V = [G_{11} \; G_{12}], \quad HV = [H_1 \; H_2] \tag{5.11}$$

with partitionings corresponding to those in (5.10). Since F_{22} is nonsingular, we can write down the following equation:

$$\begin{bmatrix} I & -F_{12}F_{22}^{-1} & 0 \\ 0 & F_{22}^{-1} & 0 \\ 0 & H_2 F_{22}^{-1} & I \end{bmatrix} \begin{bmatrix} sG_{11} - F_{11} & sG_{12} - F_{12} \\ 0 & -F_{22} \\ H_1 & H_2 \end{bmatrix} = \begin{bmatrix} sG_{11} - F_{11} & sG_{12} \\ 0 & -I \\ H_1 & 0 \end{bmatrix}. \tag{5.12}$$

By defining

$$\begin{bmatrix} 0 & -F_{12}F_{22}^{-1} \\ 0 & F_{22}^{-1} \end{bmatrix} U =: S, \quad [0\ H_2F_{22}^{-1}]U =: X, \quad V^{-1} =: T, \tag{5.13}$$

we will obtain

$$\begin{bmatrix} S & 0 \\ X & I \end{bmatrix} \begin{bmatrix} sG - F \\ H \end{bmatrix} T^{-1} = \begin{bmatrix} s\hat{G} - \hat{F} \\ \hat{H} \end{bmatrix}. \tag{5.14}$$

Thus, (F, G, H) and $(\hat{F}, \hat{G}, \hat{H})$ are weakly similar.

Because G_1 has full row rank, claim (5.6) is equivalent to saying that $\operatorname{im} G_1$ is the direct sum of $\operatorname{im} G_{11}$ and $\operatorname{im} G_{12}$, and that the columns of G_{12} are independent. This in turn is the same as saying that a vector z_2 satisfies $G_{12}z_2 \in \operatorname{im} G_{11}$ if and only if $z_2 = 0$. So, let us assume that z_2 is such that $G_{12}z_2 \in \operatorname{im} G_{11}$. Then there exists z_1 such that $G_{11}z_1 + G_{12}z_2 = 0$, i.e. $z := [z_1^{\mathrm{T}}\ z_2^{\mathrm{T}}]^{\mathrm{T}}$ belongs to $\ker \hat{G}$. By condition (iv) and weak similarity, we have $\hat{F}[\ker \hat{G}] \subset \operatorname{im} \hat{G}$. By the conformable partitionings of $\hat{F}$ and $\hat{G}$, the relation

$$\hat{F} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} F_{11}z_1 \\ z_2 \end{bmatrix} \in \operatorname{im} \hat{G}$$

implies $z_2 = 0$.

Due to the special structure of the matrices in (5.5), it is straightforward to verify that the triple (F_{11}, G_{11}, H_1) satisfies the minimality conditions for unconventional pencil representations. To prove the final claim, let $w \in \mathscr{B}_{\mathrm{c}}(F, G, H)$; then there exist $z_0 \in \mathbb{R}^{n+k}$ and $z \in \mathscr{C}_{\mathrm{imp}}^{n+k}$ such that $pGz = Fz + Gz_0$, $w = Hz$. From Eq. (5.5) we obtain

$$\begin{bmatrix} G_{11}z_{10} + G_{12}z_{20} \\ 0 \\ w \end{bmatrix} = \begin{bmatrix} pG_{11} - F_{11} & pG_{12} \\ 0 & -I \\ H_1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \tag{5.15}$$

where

$$V^{-1}z_0 =: \begin{bmatrix} z_{10} \\ z_{20} \end{bmatrix}, \quad V^{-1}z =: \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

It is clear from the equation above that $z_2 = 0$ and $w = H_1z_1$. Then, if we let $x_0 = G_{11}z_{10} + G_{12}z_{20}$ by Lemma 3.1 we have $w \in \mathscr{B}_{\mathrm{u}}(F_{11}, G_{11}, H_1)$. Conversely, since $[G_{11}\ G_{12}]$ has full row rank, for given x_0 it is always possible to find z_{10} and z_{20} such that $x_0 = G_{11}z_{10} + G_{12}z_{20}$ and (5.15) holds (setting $z_2 = 0$). Consequently, $w \in \mathscr{B}_{\mathrm{u}}(F_{11}, G_{11}, H_1)$ implies $w \in \mathscr{B}_{\mathrm{c}}(F, G, H)$. □

The following theorem completes the results in [3] on equivalence of minimal pencil representations of impulsive-smooth behaviors.

Theorem 5.6. *Suppose* (F, G, H) *and* $(\tilde{F}, \tilde{G}, \tilde{H})$ *both satisfy conditions* (i)–(iv) *of Theorem* 2.6. *Then,* $\mathscr{B}_{\mathrm{c}}(F, G, H) = \mathscr{B}_{\mathrm{c}}(\tilde{F}, \tilde{G}, \tilde{H})$ *iff* (F, G, H) *and* $(\tilde{F}, \tilde{G}, \tilde{H})$ *are weakly similar.*

Proof. The “if” part has already been proved in Lemma 5.4. So, let us prove the “only if” part. By Lemmas 5.4 and 5.5 and by Theorem 2.5, we may assume without loss of generality that

$$F = \begin{bmatrix} F_{11} & 0 \\ 0 & I \end{bmatrix} = \tilde{F}, \quad H = [H_1 \quad 0] = \tilde{H},$$
$$G = \begin{bmatrix} G_{11} & G_{12} \\ 0 & 0 \end{bmatrix}, \quad \tilde{G} = \begin{bmatrix} G_{11} & \tilde{G}_{12} \\ 0 & 0 \end{bmatrix}. \tag{5.16}$$

Because $[G_{11}\ \tilde{G}_{12}]$ has full row rank, we can write $G_{12} = G_{11}T_{12} + \tilde{G}_{12}T_{22}$ for certain matrices T_{12} and T_{22}, where T_{22} must be square (by property (5.6)). Suppose $T_{22}z_2 = 0$; then $G_{12}z_2 = G_{11}T_{12}z_2$ and it follows from (5.6) that $z_2 = 0$. So T_{22} must be invertible. Now note that

$$\tilde{F} = \begin{bmatrix} I & F_{11}T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} F_{11} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & T_{12} \\ 0 & T_{22} \end{bmatrix}^{-1}, \tag{5.17}$$

$$\tilde{G} = \begin{bmatrix} I & F_{11}T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} G_{11} & G_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & T_{12} \\ 0 & T_{22} \end{bmatrix}^{-1}, \tag{5.18}$$

$$\tilde{H} = \left([H_1 \quad 0] + [0 \quad H_1T_{12}] \begin{bmatrix} F_{11} & 0 \\ 0 & I \end{bmatrix} \right) \begin{bmatrix} I & T_{12} \\ 0 & T_{22} \end{bmatrix}^{-1}. \tag{5.19}$$

If we let

$$\begin{bmatrix} I & F_{11}T_{12} \\ 0 & T_{22} \end{bmatrix} =: S, \quad [0 \quad H_1T_{12}] =: X \quad \text{and} \quad \begin{bmatrix} I & T_{12} \\ 0 & T_{22} \end{bmatrix} =: T, \tag{5.20}$$

then

$$\begin{bmatrix} s\tilde{G} - \tilde{F} \\ \tilde{H} \end{bmatrix} = \begin{bmatrix} S & 0 \\ X & I \end{bmatrix} \begin{bmatrix} sG - F \\ H \end{bmatrix} T^{-1}. \qquad \square \tag{5.21}$$

We now can characterize the relations between minimal unconventional descriptor representations.

Theorem 5.7. *Let* (E, A, B, C, D) *and* $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ *be descriptor representations. Assume that both of them satisfy conditions* (i)–(iii) *of Theorem* 4.3. *Then*

$$\mathscr{B}_{\mathrm{u}}(E, A, B, C, D) = \mathscr{B}_{\mathrm{u}}(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) \tag{5.22}$$

if and only if there exist constant nonsingular matrices M and N and a constant matrix Y such that

$$\begin{bmatrix} sE - A & -B \\ C & D \end{bmatrix} = \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} s\tilde{E} - \tilde{A} & -\tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \begin{bmatrix} N & Y \\ 0 & I \end{bmatrix}. \tag{5.23}$$

Proof. To prove the "if" part let $w = \begin{bmatrix} y \\ u \end{bmatrix} \in \mathscr{B}_{\mathrm{u}}(E, A, B, C, D)$. Then there exist $z \in \mathscr{C}_{\mathrm{imp}}^{n_2}$ and $x_0 \in \mathbb{R}^{n_1}$ such that

$$\begin{bmatrix} x_0 \\ y \end{bmatrix} = \begin{bmatrix} pE - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix}. \tag{5.24}$$

It follows that

$$\begin{bmatrix} x_0 \\ y \end{bmatrix} = \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} p\tilde{E} - \tilde{A} & -\tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \begin{bmatrix} N & Y \\ 0 & I \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix} \tag{5.25}$$

and so $w \in \mathscr{B}_{\mathrm{u}}(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$. The reverse inclusion follows in the same way and so we have (5.22).

To prove the "only if" part let us assume that $\mathscr{B}_{\mathrm{u}}(E, A, B, C, D) = \mathscr{B}_{\mathrm{u}}(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$. By means of (3.2), let us define (F, G, H) from (E, A, B, C, D) and $(\tilde{F}, \tilde{G}, \tilde{H})$ from $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$. Since both (E, A, B, C, D) and $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ satisfy conditions (i)–(iii) of Theorem 4.3, both (F, G, H) and $\tilde{F}, \tilde{G}, \tilde{H})$ satisfy conditions (i)–(iii) of Theorem 2.5. Thus, both (F, G, H) and $(\tilde{F}, \tilde{G}, \tilde{H})$ are minimal representations of their unconventionally associated behaviors $\mathscr{B}_{\mathrm{u}}(F, G, H)$ and $\mathscr{B}_{\mathrm{u}}(\tilde{F}, \tilde{G}, \tilde{H})$, and also the following relations hold:

$$\begin{aligned} \mathscr{B}_{\mathrm{u}}(F, G, H) &= \mathscr{B}_{\mathrm{u}}(E, A, B, C, D) \\ &= \mathscr{B}_{\mathrm{u}}(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) = \mathscr{B}_{\mathrm{u}}(\tilde{F}, \tilde{G}, \tilde{H}). \end{aligned} \tag{5.26}$$

So, by Theorem 2.7 there exist constant nonsingular matrices S and T such that

$$F = S\tilde{F}T^{-1}, \quad G = S\tilde{G}T^{-1} \quad \text{and} \quad H = \tilde{H}T^{-1} \tag{5.27}$$

so that, by (3.2),

$$\begin{bmatrix} sE - A & -B \\ C & D \\ 0 & I \end{bmatrix} = \begin{bmatrix} S(s\tilde{E} - \tilde{A}) & -S\tilde{B} \\ \tilde{C} & \tilde{D} \\ 0 & I \end{bmatrix} T^{-1}. \tag{5.28}$$

Now, let

$$T^{-1} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}. \tag{5.29}$$

Then (5.28) and (5.29) imply

$$T_{21} = 0, \quad T_{22} = I. \tag{5.30}$$

Since T is nonsingular, T_{11} is nonsingular and we can define

$$T_{11} =: N, \quad T_{12} =: Y, \quad S =: M \tag{5.31}$$

to satisfy (5.23). □

The analogous result for conventional representations is the following.

Theorem 5.8. *Let* (E, A, B, C, D) *and* $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ *be descriptor representations. Assume that both of them satisfy conditions* (i)–(iv) *of Theorem* 4.4. *Then*

$$\mathscr{B}_{\mathrm{c}}(E, A, B, C, D) = \mathscr{B}_{\mathrm{c}}(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}) \tag{5.32}$$

if and only if there exist constant and nonsingular matrices M, N and constant matrices X and Y such that

$$\begin{bmatrix} M & 0 \\ X & I \end{bmatrix} \begin{bmatrix} sE - A & -B \\ C & D \end{bmatrix} = \begin{bmatrix} s\tilde{E} - \tilde{A} & -\tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \begin{bmatrix} N & Y \\ 0 & I \end{bmatrix}. \tag{5.33}$$

Proof. For the "if" part, take $w = \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathscr{B}_{\mathrm{c}}(E, A, B, C, D)$. By definition, there exist a constant z_0 and an impulsive-smooth z such that

$$\begin{bmatrix} Ez_0 \\ y \end{bmatrix} = \begin{bmatrix} pE - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} z \\ u \end{bmatrix}. \tag{5.34}$$

Note that (5.33) implies that $ME = \tilde{E}N$ and $XE = 0$. Therefore, it follows from (5.33) that

$$\begin{bmatrix} \tilde{E}Nz_0 \\ y \end{bmatrix} = \begin{bmatrix} p\tilde{E} - \tilde{A} & -\tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \begin{bmatrix} Nz + Yu \\ u \end{bmatrix} \tag{5.35}$$

so that $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathscr{B}_{\mathrm{c}}(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$. The argument is completed as in the proof of the previous theorem.

In order to prove the "only if" part, let us assume that (E, A, B, C, D) and $(\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ satisfy conditions (i)–(iv) of Theorem 4.4 and their conventionally associated behaviors are the same. Next, apply Algorithm 3.8 to both of them; this yields conventionally externally equivalent pencil representations (F, G, H) and $(\tilde{F}, \tilde{G}, \tilde{H})$ which are minimal. Then by Theorem 5.6 they are weakly similar and so there exist constant invertible matrices S and T and a constant matrix X such that

$$\begin{bmatrix} s\tilde{G} - \tilde{F} \\ \tilde{H} \end{bmatrix} = \begin{bmatrix} S & 0 \\ X & I \end{bmatrix} \begin{bmatrix} sG - F \\ H \end{bmatrix} T^{-1}. \tag{5.36}$$

We may assume that both descriptor representations are in the form (3.11)–(3.12) with $E_{11} = I$ and $B_{22} = -I$. Then (5.36) may be written in further detail as

$$\begin{bmatrix} S_1 & S_2 & 0 & 0 & 0 \\ S_3 & S_4 & 0 & 0 & 0 \\ X_1 & X_2 & I & 0 & 0 \\ X_3 & X_4 & 0 & I & 0 \\ X_5 & X_6 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} sI - A_{11} & -A_{12} & 0 \\ -A_{31} & 0 & 0 \\ C_1 + D_2A_{21} & C_2 & D_1 + D_2B_{21} \\ 0 & 0 & I \\ A_{21} & 0 & B_{21} \end{bmatrix}$$
$$= \begin{bmatrix} sI - \tilde{A}_{11} & -\tilde{A}_{12} & 0 \\ -\tilde{A}_{31} & 0 & 0 \\ \tilde{C}_1 + \tilde{D}_2\tilde{A}_{21} & \tilde{C}_2 & \tilde{D}_1 + \tilde{D}_2\tilde{B}_{21} \\ 0 & 0 & I \\ \tilde{A}_{21} & 0 & \tilde{B}_{21} \end{bmatrix} \begin{bmatrix} T_1 & T_2 & T_3 \\ T_4 & T_5 & T_6 \\ T_7 & T_8 & T_9 \end{bmatrix}.$$

It now follows immediately that the matrices S_3, X_1, X_3, X_5, T_2, T_3, T_8 must all be zero matrices, and $T_9 = I$. Then, since S and T are nonsingular, S_1, S_4, T_1, and T_5

are nonsingular. Also note that $T_7 = -X_4 A_{31}$. After tedious but in principle straightforward calculations, it can be verified that (5.33) is satisfied with

$$
\begin{aligned}
M &:= \begin{bmatrix} S_1 & S_1 B_{12} - \tilde{B}_{12} & S_2 - \tilde{B}_{12} X_6 + \tilde{B}_{11} X_4 \\ 0 & I & -X_6 + \tilde{B}_{21} X_4 \\ 0 & 0 & S_4 \end{bmatrix}, \\
N &:= \begin{bmatrix} T_1 & 0 \\ T_4 & T_5 \end{bmatrix}, \\
X &:= \begin{bmatrix} 0 & -D_2 + \tilde{D}_2 & X_2 - \tilde{D}_2 X_6 - \tilde{D}_1 X_4 \end{bmatrix}, \\
Y &:= \begin{bmatrix} 0 & 0 \\ T_6 & 0 \end{bmatrix}. \qquad \square
\end{aligned}
\tag{5.37}
$$

With the above result, we have completed the program of characterizing the transformations of external equivalence in the sense of impulsive-smooth behaviors for pencil and descriptor representations. The equivalence relation (5.33) is well-known in the literature; it was introduced by Verghese et al. [16] under the name of *strong equivalence operation*. The same transformation group was used earlier for descriptor representations with zero feedthrough term by Van der Weiden and Bosgra [15], who used the name *restricted system equivalence*. Compare also Rosenbrock's notion of *strict system equivalence* [11, p. 52] which uses polynomial matrices in a format similar to (5.33). The above theorem provides a motivation for the notion of strong equivalence in terms of impulsive-smooth behaviors. The transformation group (5.23) that we have found for unconventional descriptor representations has, to our knowledge, not been considered before.

Our results generate two possible ways of describing versions of the "space of linear input/output systems": the collection of quintuples (E, A, B, C, D) satisfying the minimality conditions of Theorem 4.4 modulo the transformation group of Theorem 5.8, and the collection of quintuples (E, A, B, C, D) satisfying the minimality conditions of Theorem 4.3 modulo the transformation group of Theorem 5.7. It follows from the results of [13] that the two objects so defined are not the same; the second space contains only systems that have "Dirac free inputs" in the sense of [13], whereas the systems in the first space are not subject to such a restriction. Both spaces can be seen as extensions of spaces considered traditionally, such as the space of rational matrices.

The first author who gave a motivation for strong equivalence from an intrinsically defined notion of equivalence was Grimm [5]. The minimality conditions obtained by Grimm were the same as the ones mentioned in Theorem 4.4 above, except that requirement (i) is replaced in his paper by the stronger condition that the matrix $[sE - tA \quad -B]$ should have full row rank for *all* pairs of complex numbers $(s, t) \neq (0, 0)$. This condition can be interpreted as a controllability condition. A weaker notion of minimality (so one that is satisfied in a wider class of systems) was used by Kuijper and Schumacher [9]; they used external equivalence in the sense of smooth behaviors, which leads to minimality conditions (i)–(iv) of Theorem 4.4 with condition (i) replaced by the requirement that $[E\ \ B]$ should have full row rank. This requirement can be interpreted as a condition of "controllability at infinity". The operations relating minimal representations (in the sense of smooth behaviors) to each other were again identified in [9] as the operations of strong equivalence. The result above gives an interpretation of strong equivalence that goes even further, since it applies to systems satisfying the conditions of Theorem 4.4 as such; note that the condition that $[sE - A \quad -B]$ should have full row rank as a rational matrix is equivalent to requiring that the matrix $[sE - tA \quad -B]$ should have full row rank for *some* pair of complex numbers (s, t). Condition (i) as given in Theorem 4.4 is no longer a controllability condition but rather a nonredundancy condition, as it requires that none of the equations given by the rows of $pEz = Az + Bu + Ez_0$ should be obtainable from the other equations by differentiating and taking linear combinations.

6. Conclusions

In this paper we have discussed minimality and equivalence of descriptor representations for impulsive-smooth behaviors. As can be expected, the minimality conditions are weaker than those for descriptor representations of smooth behaviors; in particular, no form of controllability is required for minimality in the sense of impulsive-smooth behaviors. In the case of conventional representations, minimal representations turn out to be related by operations of strong equivalence as defined in [16]. We have therefore given a motivation for strong equivalence that applies to a wider class of systems than the classes considered earlier in [5,9]. The operations that we found for minimal unconventional representations have to our knowledge not been considered before. We have also identified the transformation group that describes the relations between minimal conventional pencil representations of impulsive-smooth behaviors, thus completing the results in [3].

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