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On the existence of Hermitian positive definite solutions of the matrix equation $X^s + A^*X^{-t}A = Q$ ☆

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Abstract

In this paper, the existence of Hermitian positive definite solutions of the general nonlinear matrix equation $X^s + A^*X^{-t}A = Q$ is studied systematically and deeply. A new estimate of Hermitian positive definite solutions is derived. Based on a fixed point theorem, some new sufficient conditions and new necessary conditions for the existence of Hermitian positive definite solutions are obtained. In the end, a necessary and sufficient condition for the existence of a Hermitian positive definite solution is proved.

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1. Introduction

We consider the existence of Hermitian positive definite (HPD) solutions of the general nonlinear matrix equation

$$X^s + A^*X^{-t}A = Q, \tag{1.1}$$

where A is an $n \times n$ nonsingular matrix, Q is an $n \times n$ HPD matrix, s and t are positive integers. Nonlinear matrix equations of the form (1.1) often arise in control theory, ladder networks,

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dynamic programming, stochastic filtering and statistics, etc. (see [1,11,15,19] and the reference therein).

The existence of HPD solutions of Eq. (1.1) has been investigated in some special cases. In [3] it is assumed that A is a normal matrix, whereas in [16] it is assumed that A is a real matrix. Some necessary and sufficient conditions for the existence of HPD solutions of Eq. (1.1) with $s = t = 1$ have been derived by using the shorted operator approach [1] and the analytic factorization approach [4]. By means of the matrix decomposition method, Hasanov and Ivanov [9] and Zhan and Xie [19] presented some necessary and sufficient conditions for the existence of an HPD solution to Eq. (1.1) with $s = 1, t = n$ and $s = 1, t = 1$, but these conditions were not easily checked. And the matrix sequence theory were also used to study the existence of HPD solutions of the nonlinear matrix equations of type (1.1) (see [6,7,11,12,15]). Recently, fixed point theory techniques play a key role in investigating the existence of HPD solutions of the nonlinear matrix equations of type (1.1) (see [10,13,16,17,21]).

Based on Brouwer's fixed point theorem [22, Theorem 4.2.6] and Banach's fixed point theorem [22, Theorem 1.3.1], we investigate the existence and uniqueness of HPD solutions of Eq. (1.1) in this paper. In Section 2, we derive a new estimate of HPD solutions and a sufficient condition under which Eq. (1.1) has a unique HPD solution. In Section 3, under the assumption that $\lambda_1(A^*A) \leqslant \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}} \lambda_n^{\frac{t}{s}+1}(Q)$, we deepen some conclusions of Liu and Gao [16] and get some new results on the existence of HPD solutions. In Section 4, a necessary and sufficient condition for the existence of an HPD solution is given. Some results in [5,21] have been extended.

Throughout this paper, we write $B > O (B \geqslant O)$ if the matrix B is positive definite (semidefinite). If $B - C$ is positive definite (semidefinite), then we write $B > C (B \geqslant C)$. We use $\lambda_1(B)$ and $\lambda_n(B)$ to denote the maximal and minimal eigenvalues of an $n \times n$ HPD matrix B. We use $\|B\|$ and $\|B\|_F$ to denote the spectral norm and Frobenius norm of a matrix B, and we also use $\|b\|$ to denote l_2-norms of a vector b. We use X_S and X_L to denote the minimal and maximal HPD solution of Eq. (1.1), that is, for any HPD solution X of Eq. (1.1), then $X_S \leqslant X \leqslant X_L$. The symbol I denotes the $n \times n$ identity matrix. The symbol $\rho(B)$ denotes the spectral radius of B. Let $P(n)$ denote a set of $n \times n$ HPD matrices and $[B, C] = \{X|B \leqslant X \leqslant C\}$, $(B, C) = \{X|B < X < C\}$. For $B = (b_1, b_2, \ldots, b_n) = (b_{ij})$ and a matrix C, $B \otimes C = (b_{ij}C)$ is a Kronecker product and $vec(B)$ is a vector defined by $vec(A) = (a_1^{\mathrm{T}}, a_2^{\mathrm{T}}, \ldots, a_n^{\mathrm{T}})^{\mathrm{T}}$. In order to develop the paper, we need that

$$\mathrm{vec}(AXB) = (B^{\mathrm{T}} \otimes A)\mathrm{vec}(X) \quad \text{and} \quad \|\mathrm{vec}(X)\| = \|X\|_F,$$

where A, X and B are $n \times n$ complex matrix.

2. The general case

In this section, we give a new estimate of HPD solutions of Eq. (1.1). Based on fixed point theorems, we derive a sufficient condition for the existence of a unique HPD solution of Eq. (1.1). We begin with some lemmas.

Lemma 2.1 [18, Theorem 2.1]. *If Eq.* (1.1) *has an HPD solution X, then*

$$X \in ((AQ^{-1}A^*)^{\frac{1}{t}}, Q^{\frac{1}{s}}).$$

Lemma 2.2 [2]. *If $A \geqslant B > O$ (or $A > B > O$), then $A^\alpha \geqslant B^\alpha > O$ (or $A^\alpha > B^\alpha > O$) for all $\alpha \in (0, 1]$, and $B^\alpha \geqslant A^\alpha > O$ (or $B^\alpha > A^\alpha > O$) for all $\alpha \in [-1, 0)$.*

Lemma 2.3 [14, p. 656]. *For square nonsingular matrices A, B and C applications of Schur's lemma to the two matrices $A+BC$ and $A-BC$ yields that*

(i) $(A+BC)^{-1}=A^{-1}-A^{-1}B(I+CA^{-1}B)^{-1}CA^{-1}$,
(ii) $(A-BC)^{-1}=A^{-1}-A^{-1}B(CA^{-1}B-I)^{-1}CA^{-1}$.

Lemma 2.4 [8, Theorem 2.1]. *Let A and B be positive operators on a Hilbert space H such that $M_1I \geqslant A \geqslant m_1I > O$, $M_2I \geqslant B \geqslant m_2I > O$ and $B \geqslant A > O$. Then*

$$A^t \leqslant \left(\frac{M_1}{m_1}\right)^{t-1} B^t, \quad A^t \leqslant \left(\frac{M_2}{m_2}\right)^{t-1} B^t$$

hold for any $t \geqslant 1$.

So far, there are several estimates of HPD solutions of Eq. (1.1) and its special cases, such as, Theorem 2.1 in Yang [18] (i.e. Lemma 2.1), Theorem 4 in Ivanov [13] and Theorem 4 in Hasanov and Ivanov [10]. Now we give a new estimate which are sharper than that all of them.

Theorem 2.1. *If Eq. (1.1) has an HPD solution X, then*

$$X \in (M, N),$$

where

$$M=\left\{AQ^{-1}A^*+AQ^{-1}\left[\left(\frac{\lambda_1(A^{-*}QA^{-1})}{\lambda_n(A^{-*}QA^{-1})}\right)^{\frac{s-1}{t}}(A^{-*}QA^{-1})^{\frac{s}{t}}-Q^{-1}\right]^{-1}Q^{-1}A^*\right\}^{\frac{1}{t}},$$

$$N=\left[Q-\left(\frac{\lambda_n(Q^{-1})}{\lambda_1(Q^{-1})}\right)^{\frac{t-1}{s}}A^*Q^{-\frac{t}{s}}A\right]^{\frac{1}{s}}.$$

Proof. Let X be the HPD solution of Eq. (1.1), then from Lemma 2.1 it follows that

$$(AQ^{-1}A^*)^{\frac{1}{t}} < X < Q^{\frac{1}{s}}. \tag{2.1}$$

Applying Lemma 2.2 to (2.1) yields

$$Q^{-\frac{1}{s}} < X^{-1} < (AQ^{-1}A^*)^{-\frac{1}{t}}. \tag{2.2}$$

Since

$$\lambda_n^{\frac{1}{t}}(A^{-*}QA^{-1})I \leqslant (AQ^{-1}A^*)^{-\frac{1}{t}} = (A^{-*}QA^{-1})^{\frac{1}{t}} \leqslant \lambda_1^{\frac{1}{t}}(A^{-*}QA^{-1})I$$

and

$$\lambda_n^{\frac{1}{s}}(Q^{-1}) \leqslant Q^{-\frac{1}{s}} = (Q^{-1})^{\frac{1}{s}} \leqslant \lambda_1^{\frac{1}{s}}(Q^{-1}),$$

then applying Lemma 2.4 to (2.2) yields

$$Q^{-1} < \left(\frac{\lambda_1(A^{-*}QA^{-1})}{\lambda_n(A^{-*}QA^{-1})}\right)^{\frac{s-1}{t}}(A^{-*}QA^{-1})^{\frac{s}{t}}, \tag{2.3}$$

$$X^{-t} > \left(\frac{\lambda_n(Q^{-1})}{\lambda_1(Q^{-1})}\right)^{\frac{t-1}{s}} Q^{-\frac{t}{s}} \tag{2.4}$$

and

$$X^{-s} < \left(\frac{\lambda_1(A^{-*}QA^{-1})}{\lambda_n(A^{-*}QA^{-1})}\right)^{\frac{s-1}{t}} (A^{-*}QA^{-1})^{\frac{s}{t}}. \tag{2.5}$$

Rewriting Eq. (1.1) and combining (2.4), we have

$$Q - X^s = A^*X^{-t}A > \left(\frac{\lambda_n(Q^{-1})}{\lambda_1(Q^{-1})}\right)^{\frac{t-1}{s}} A^*Q^{-\frac{t}{s}}A,$$

which implies that

$$X < \left[Q - \left(\frac{\lambda_n(Q^{-1})}{\lambda_1(Q^{-1})}\right)^{\frac{t-1}{s}} A^*Q^{-\frac{t}{s}}A\right]^{\frac{1}{s}} = N. \tag{2.6}$$

On the other hand, Eq. (1.1) can also be rewritten as

$$X^t = A(Q - X^s)^{-1}A^*. \tag{2.7}$$

Applying Lemma 2.3 to (2.7) and combining (2.3) and (2.5) yield

$$\begin{aligned} X^t &= A(Q - X^s)^{-1}A^* \\ &= A[Q^{-1} - Q^{-1}(X^sQ^{-1} - I)^{-1}X^sQ^{-1}]A^* \\ &= AQ^{-1}A^* + AQ^{-1}(X^{-s} - Q^{-1})^{-1}Q^{-1}A^* \\ &> AQ^{-1}A^* + AQ^{-1}\left[\left(\frac{\lambda_1(A^{-*}QA^{-1})}{\lambda_n(A^{-*}QA^{-1})}\right)^{\frac{s-1}{t}} (A^{-*}QA^{-1})^{\frac{s}{t}} - Q^{-1}\right]^{-1} Q^{-1}A^*, \end{aligned}$$

which implies that

$$\begin{aligned} X &> \left\{AQ^{-1}A^* + AQ^{-1}\left[\left(\frac{\lambda_1(A^{-*}QA^{-1})}{\lambda_n(A^{-*}QA^{-1})}\right)^{\frac{s-1}{t}} (A^{-*}QA^{-1})^{\frac{s}{t}} - Q^{-1}\right]^{-1} Q^{-1}A^*\right\}^{\frac{1}{t}} \\ &= M. \end{aligned} \tag{2.8}$$

Combining (2.6) and (2.8), we get

$$X \in (M, N). \qquad \square$$

Remark 2.1. Comparing Theorem 2.1 with Lemma 2.1, it is easy to obtain that

$$(M, N) \subset ((AQ^{-1}A^*)^{\frac{1}{t}}, Q^{\frac{1}{s}}),$$

that is to say, our estimate of HPD solution of Eq. (1.1) is sharper than that of Yang [18].

Now we use an example to confirm the correctness of Theorem 2.1 and the sharpness of the bounds of HPD solutions of Eq. (1.1).

Example 2.1. Consider the matrix equation

$$X + A^{\mathrm{T}}X^{-1}A = I \tag{2.9}$$

with

$$A = \begin{bmatrix} 0.2000 & 0.2000 & 0.1000 \\ 0.2000 & 0.1500 & 0.1500 \\ 0.1000 & 0.1500 & 0.2500 \end{bmatrix}.$$

Here, A is normal and nonsingular. Therefore, from theorem 4.1 of Zhan and Xie [19] we obtain that Eq. (2.9) has a maximal HPD solution

$$X_L = \frac{1}{2}[I + (I - 4A^{\mathrm{T}}A)^{\frac{1}{2}}] \approx \begin{bmatrix} 0.8265 & -0.1684 & -0.1582 \\ -0.1684 & 0.8316 & -0.1633 \\ -0.1582 & -0.1633 & 0.8214 \end{bmatrix}$$

and a minimal HPD solution

$$X_S = \frac{1}{2}[I - (I - 4A^{\mathrm{T}}A)^{\frac{1}{2}}] \approx \begin{bmatrix} 0.1735 & 0.1684 & 0.1582 \\ 0.1684 & 0.1684 & 0.1633 \\ 0.1582 & 0.1633 & 0.1786 \end{bmatrix}.$$

After direct computations, we have

$$M \approx \begin{bmatrix} 0.1178 & 0.1128 & 0.1026 \\ 0.1128 & 0.1128 & 0.1077 \\ 0.1026 & 0.1077 & 0.1229 \end{bmatrix} \quad \text{and} \quad N \approx \begin{bmatrix} 0.9100 & -0.0850 & -0.0750 \\ -0.0850 & 0.9150 & -0.0800 \\ -0.0750 & -0.0800 & 0.9050 \end{bmatrix}.$$

It is easy to verify that X_L and X_S are in $[M, N]$. Hence, all HPD solutions of Eq. (2.9) are in $[M, N]$.

The next theorem describes a sufficient condition under which Eq. (1.1) has a unique HPD solution.

Theorem 2.2. *If* $(AQ^{-1}A^*)^{\frac{s}{t}} \leqslant Q$, $A^*X^{-t}A \leqslant Q - (AQ^{-1}A^*)^{\frac{s}{t}}$ *for all* $X \in [(AQ^{-1}A^*)^{\frac{1}{t}}, Q^{\frac{1}{s}}]$, *and*

$$p = \frac{t\|A\|_F^2}{s\lambda_n^{\frac{s+t}{t}}(AQ^{-1}A^*)} < 1,$$

then Eq. (1.1) *has a unique HPD solution.*

Proof. We consider the map $F(X) = (Q - A^*X^{-t}A)^{\frac{1}{s}}$ and let

$$X \in \Omega = \left\{X | (AQ^{-1}A^*)^{\frac{1}{t}} \leqslant X \leqslant Q^{\frac{1}{s}}\right\}.$$

Obviously, Ω is a convex, closed and bounded set and $F(X)$ is continuous on Ω. If $A^*X^{-t}A \leqslant Q - (AQ^{-1}A^*)^{\frac{s}{t}}$ for all $X \in \Omega$, then we have

$$Q^{\frac{1}{s}} \geqslant (Q - A^*X^{-t}A)^{\frac{1}{s}} \geqslant (Q - Q + (AQ^{-1}A^*)^{\frac{s}{t}})^{\frac{1}{s}} = (AQ^{-1}A^*)^{\frac{1}{t}},$$

i.e.

$$(AQ^{-1}A^*)^{\frac{1}{t}} \leqslant F(X) \leqslant Q^{\frac{1}{s}}.$$

Hence $F(\Omega) \subseteq \Omega$.

For arbitrary $X, Y \in \Omega$, we have

$$A^*X^{-t}A \leqslant Q - (AQ^{-1}A^*)^{\frac{s}{t}} \quad \text{and} \quad A^*Y^{-t}A \leqslant Q - (AQ^{-1}A^*)^{\frac{s}{t}},$$

i.e.

$$Q - A^*X^{-t}A \geqslant (AQ^{-1}A^*)^{\frac{s}{t}} \quad \text{and} \quad Q - A^*Y^{-t}A \geqslant (AQ^{-1}A^*)^{\frac{s}{t}}.$$

Hence

$$F(X) = (Q - A^*X^{-t}A)^{\frac{1}{s}} \geqslant (AQ^{-1}A^*)^{\frac{1}{t}} \geqslant \lambda_n^{\frac{1}{t}}(AQ^{-1}A^*)I, \tag{2.10}$$

$$F(Y) = (Q - A^*Y^{-t}A)^{\frac{1}{s}} \geqslant (AQ^{-1}A^*)^{\frac{1}{t}} \geqslant \lambda_n^{\frac{1}{t}}(AQ^{-1}A^*)I. \tag{2.11}$$

From (2.10) and (2.11) it follows that

$$\begin{aligned}
\|F(X)^s - F(Y)^s\|_F &= \left\|\sum_{i=0}^{s-1} F(X)^i(F(X) - F(Y))F(Y)^{s-1-i}\right\|_F \\
&= \left\|\text{vec}\left[\sum_{i=0}^{s-1} F(X)^i(F(X) - F(Y))F(Y)^{s-1-i}\right]\right\| \\
&= \left\|\sum_{i=0}^{s-1} \text{vec}[F(X)^i(F(X) - F(Y))F(Y)^{s-1-i}]\right\| \\
&= \left\|\sum_{i=0}^{s-1}(F(Y)^{s-1-i} \otimes F(X)^i)\text{vec}(F(X) - F(Y))\right\| \\
&\geqslant \sum_{i=0}^{s-1}(\lambda_n^{\frac{s-1-i}{t}}(AQ^{-1}A^*)\lambda_n^{\frac{i}{t}}(AQ^{-1}A^*))\|\text{vec}(F(X) - F(Y))\| \\
&= s\lambda_n^{\frac{s-1}{t}}(AQ^{-1}A^*)\|F(X) - F(Y)\|_F.
\end{aligned} \tag{2.12}$$

According to the definition of the map F, we have

$$F(X)^s - F(Y)^s = (Q - A^*X^{-t}A) - (Q - A^*Y^{-t}A) = A^*(Y^{-t} - X^{-t})A. \tag{2.13}$$

Combining (2.12) and (2.13), we have

$$\begin{aligned}
\|F(X) - F(Y)\|_F &\leqslant \frac{1}{s\lambda_n^{\frac{s-1}{t}}(AQ^{-1}A^*)}\|F(X)^s - F(Y)^s\|_F \\
&= \frac{1}{s\lambda_n^{\frac{s-1}{t}}(AQ^{-1}A^*)}\|A^*(Y^{-t} - X^{-t})A\|_F \\
&\leqslant \frac{\|A\|_F^2}{s\lambda_n^{\frac{s-1}{t}}(AQ^{-1}A^*)}\|Y^{-t} - X^{-t}\|_F \\
&= \frac{\|A\|_F^2}{s\lambda_n^{\frac{s-1}{t}}(AQ^{-1}A^*)}\left\|\sum_{i=1}^{t} Y^{-(t+1)+i}(X - Y)X^{-i}\right\|_F \\
&= \frac{\|A\|_F^2}{s\lambda_n^{\frac{s-1}{t}}(AQ^{-1}A^*)}\left\|\sum_{i=1}^{t}(X^{-i} \otimes Y^{-(t+1)+i})\text{vec}(X - Y)\right\| \\
&\leqslant \frac{\|A\|_F^2}{s\lambda_n^{\frac{s-1}{t}}(AQ^{-1}A^*)}\sum_{i=1}^{t}\|X^{-i} \otimes Y^{-(t+1)+i}\|\|X - Y\|_F.
\end{aligned}$$

Since $X, Y \in \Omega$, then we have

$$X^{-1} \leqslant (AQ^{-1}A^*)^{-\frac{1}{t}} \leqslant \lambda_n^{-\frac{1}{t}}(AQ^{-1}A^*)I = \frac{1}{\lambda_n^{\frac{1}{t}}(AQ^{-1}A^*)}I,$$
$$Y^{-1} \leqslant (AQ^{-1}A^*)^{-\frac{1}{t}} \leqslant \lambda_n^{-\frac{1}{t}}(AQ^{-1}A^*)I = \frac{1}{\lambda_n^{\frac{1}{t}}(AQ^{-1}A^*)}I.$$

Hence

$$\begin{aligned}\|F(X)-F(Y)\|_F &\leqslant \frac{\|A\|_F^2}{s\lambda_n^{\frac{s-1}{t}}(AQ^{-1}A^*)}\sum_{i=1}^{t}\|X^{-i}\otimes Y^{-(t+1)+i}\|\|X-Y\|_F \\ &\leqslant \frac{\|A\|_F^2}{s\lambda_n^{\frac{s-1}{t}}(AQ^{-1}A^*)}\frac{t}{\lambda_n^{\frac{t+1}{t}}(AQ^{-1}A^*)}\|X-Y\|_F \\ &= \frac{t\|A\|_F^2}{s\lambda_n^{\frac{s+t}{t}}(AQ^{-1}A^*)}\|X-Y\|_F \\ &= p\|X-Y\|_F.\end{aligned}$$

Since $p < 1$, we know that the map $F(X)$ is a contraction map in Ω. By Banach's fixed point theorem, the map $F(X)$ has a unique fixed point in Ω and this shows that Eq. (1.1) has a unique HPD solution in $[(AQ^{-1}A^*)^{\frac{1}{t}}, Q^{\frac{1}{s}}]$. Noting that Lemma 2.1, we know that Eq. (1.1) has a unique HPD solution. The theorem is proved. □

3. The case which satisfies $\lambda_1(A^*A)\leqslant\frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}}\lambda_n^{\frac{t}{s}+1}(Q)$

In this section, we first use the similar method mentioned in Liu and Gao [16] to derive a theorem for the existence of an HPD solution of Eq. (1.1) under the condition $\lambda_1(A^*A) < \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}}\lambda_n^{\frac{t}{s}+1}(Q)$. We continue to investigate HPD solutions of Eq. (1.1) when the condition doesn't satisfied. We get some new sufficient conditions and necessary conditions for the existence of HPD solutions of Eq. (1.1). The new results are illustrated by a numerical example.

We assume that A and Q satisfies

$$\lambda_1(A^*A) < \frac{s}{s+t}x_*^t\lambda_n(Q), \tag{3.1}$$

where $x_* = \left(\frac{t}{s+t}\lambda_n(Q)\right)^{\frac{1}{s}}$.

From (3.1) we get

$$\lambda_n(A^*A) < \frac{s}{s+t}x_{**}^t\lambda_1(Q), \tag{3.2}$$

where $x_{**} = \left(\frac{t}{s+t}\lambda_1(Q)\right)^{\frac{1}{s}}$.

Lemma 3.1. *Let*

$$f(x) = x^t(\theta - x^s), \quad \theta > 0,\ x \geqslant 0.$$

Then

(i) f *is increasing on* $\left[0, \left(\frac{t}{s+t}\theta\right)^{\frac{1}{s}}\right]$ *and decreasing on* $\left[\left(\frac{t}{s+t}\theta\right)^{\frac{1}{s}}, +\infty\right)$;

(ii) $f_{\max} = f\left(\left(\frac{t}{s+t}\theta\right)^{\frac{1}{s}}\right) = \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}}\theta^{\frac{t}{s}+1}$.

Consider the following equations

$$x^{s+t} - \lambda_n(Q)x^t + \lambda_1(A^*A) = 0, \tag{3.3}$$

$$x^{s+t} - \lambda_1(Q)x^t + \lambda_n(A^*A) = 0. \tag{3.4}$$

By (3.1) and Lemma 3.1, we know that Eq. (3.3) has two positive real roots $\alpha_2 < \beta_1$. We also get that Eq. (3.4) has two positive real roots $\alpha_1 < \beta_2$ from (3.2) and Lemma 3.1. It is easy to prove that

$$0 < \alpha_1 \leqslant \alpha_2 < x_* < \beta_1 \leqslant \beta_2.$$

We define matrix sets as follows:

$$\varphi_1 = \{X = X^* | \alpha_1 I \leqslant X \leqslant \alpha_2 I\},$$
$$\varphi_2 = \{X = X^* | \beta_1 I \leqslant X \leqslant \beta_2 I\},$$
$$\varphi_3 = \{X = X^* | \alpha_2 I \leqslant X \leqslant \beta_1 I\}.$$

Theorem 3.1. *Suppose that A and Q satisfy* (3.1), *i.e.*

$$\lambda_1(A^*A) < \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}}\lambda_n^{\frac{t}{s}+1}(Q).$$

Then

(i) *Eq.* (1.1) *has an HPD solution in* φ_1;
(ii) *Eq.* (1.1) *has a unique HPD solution in* φ_2;
(iii) *Eq.* (1.1) *has no HPD solution in* φ_3.

Proof. The proof is similar to that of Theorems 2.1 and 2.2 of Liu and Gao [16] and is omitted here.

Now we begin to discuss the case which does not satisfy (3.1).

Theorem 3.2. *If* $\lambda_1(A^*A) = \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}}\lambda_n^{\frac{t}{s}+1}(Q)$, *then Eq.* (1.1) *has an HPD solution.*

Proof. If $\lambda_n(A^*A) < \lambda_1(A^*A) = \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}}\lambda_n^{\frac{t}{s}+1}(Q)$, then Eq. (3.3) has a unique positive real root $\alpha = x_* = \beta_1$, and Eq. (3.4) has two positive real roots $\alpha_1 < \beta_2$. It is easy to prove that

$$0 < \alpha_1 \leqslant \alpha_2 = x_* = \beta_1 \leqslant \beta_2.$$

Let

$$\varphi_4 = \{X = X^* | \alpha_1 I \leqslant X \leqslant \alpha_2 I\},$$
$$\varphi_5 = \{X = X^* | \beta_1 I \leqslant X \leqslant \beta_2 I\}.$$

Note that φ_4 and φ_5 are bounded closed convex sets. Consider the map $G(X) = (A(Q - X^s)^{-1}A^*)^{\frac{1}{t}}$ which is continuous on φ_4.

For any $X \in \varphi_4$, we have

$$\begin{aligned}\lambda_n(G(X)) &= \lambda_n((A(Q - X^s)^{-1}A^*)^{\frac{1}{t}}) \\ &\geqslant \lambda_n^{\frac{1}{t}}(A(Q - \alpha_1^s I)^{-1}A^*) \\ &\geqslant \left[\frac{\lambda_n(AA^*)}{\lambda_1(Q) - \alpha_1^s}\right]^{\frac{1}{t}} \\ &= \alpha_1\end{aligned}$$

and

$$\begin{aligned}\lambda_1(G(X)) &= \lambda_1((A(Q - X^s)^{-1}A^*)^{\frac{1}{t}}) \\ &\leqslant \lambda_1^{\frac{1}{t}}(A(Q - \alpha_2^s I)^{-1}A^*) \\ &\leqslant \left[\frac{\lambda_1(AA^*)}{\lambda_n(Q) - \alpha_2^s}\right]^{\frac{1}{t}} \\ &= \alpha_2.\end{aligned}$$

Hence $G(X)$ maps φ_4 into φ_4, by Brouwer's fixed point theorem, we know that $G(X)$ has a fixed point on φ_4 which is an HPD solution of Eq. (1.1) on φ_4.

If $\lambda_n(A^*A) = \lambda_1(A^*A) = \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}} \lambda_n^{\frac{t}{s}+1}(Q)$, we will consider the following cases.

If $\lambda_n(Q) < \lambda_1(Q)$, we can obtain that Eq. (1.1) has an HPD solution by the method mentioned above.

On the other hand, if $\lambda_n(Q) = \lambda_1(Q)$, we have

$$0 < \alpha_1 = \alpha_2 = x_* = \beta_1 = \beta_2.$$

Thus, there exist positive real numbers a, q such that

$$A^*A = AA^* = a^2 I, \quad Q = qI.$$

Then the matrix $X = x_* I = (q\frac{t}{s+t})^{\frac{1}{s}} I$ is an HPD solution of Eq. (1.1). In fact, since $x_* = (q\frac{t}{s+t})^{\frac{1}{s}}$ is a solution of Eq. (3.3), i.e.

$$\left(q\frac{t}{s+t}\right)^{\frac{s+t}{s}} - q\left(q\frac{t}{s+t}\right)^{\frac{t}{s}} + a^2 = 0, \tag{3.5}$$

which implies that

$$a^2 = q^{1+\frac{t}{s}}\left(\frac{t}{s+t}\right)^{\frac{t}{s}}\left(\frac{s}{s+t}\right). \tag{3.6}$$

And

$$\begin{aligned}X^s + A^*X^{-t}A &= (x_*I)^s + A^*(x_*I)^{-t}A \\ &= q\frac{t}{s+t}I + A^*\left(q\frac{t}{s+t}\right)^{-\frac{t}{s}} A \\ &= q\frac{t}{s+t}I + \left(q\frac{t}{s+t}\right)^{-\frac{t}{s}} a^2 I.\end{aligned} \tag{3.7}$$

Combining (3.6) and (3.7), we have

$$\begin{aligned} X^s + A^*X^{-t}A &= q\frac{t}{s+t}I + \left(q\frac{t}{s+t}\right)^{-\frac{t}{s}} q^{1+\frac{t}{s}} \left(\frac{t}{s+t}\right)^{\frac{t}{s}} \left(\frac{s}{s+t}\right) I \\ &= qI \\ &= Q, \end{aligned}$$

i.e. $X = x_* I$ is an HPD solution of Eq. (1.1).

In a word, if $\lambda_1(A^*A) = \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}} \lambda_n^{\frac{t}{s}+1}(Q)$, Eq. (1.1) has an HPD solution. □

According to Theorems 3.1 and 3.2, we have the following result.

Corollary 3.1. *If* $\lambda_1(A^*A) \leqslant \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}} \lambda_n^{\frac{t}{s}+1}(Q)$, *then Eq.* (1.1) *has an HPD solution.*

Theorem 3.3. *If Eq.* (1.1) *has an HPD solution, then* $\lambda_n(A^*A) \leqslant \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}} \lambda_1^{\frac{t}{s}+1}(Q)$.

Proof. Let X be the HPD solution of Eq. (1.1), i.e.

$$X^s + A^*X^{-t}A = Q.$$

From

$$\begin{aligned} \lambda_1^s(X) = \lambda_1(X^s) &= \lambda_1(Q - A^*X^{-t}A) \\ &\leqslant \lambda_1(Q) - \lambda_n(A^*X^{-t}A) \\ &\leqslant \lambda_1(Q) - \frac{\lambda_n(A^*A)}{\lambda_1^t(X)} \end{aligned}$$

and Lemma 3.1, we have

$$\begin{aligned} \lambda_n(A^*A) &\leqslant \lambda_1(Q)\lambda_1^t(X) - \lambda_1^{s+t}(X) \\ &= \lambda_1^t(X)(\lambda_1(Q) - \lambda_1^s(X)) \\ &\leqslant \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}} \lambda_1^{\frac{t}{s}+1}(Q). \end{aligned}$$

The theorem is proved. □

Theorem 3.4. *Suppose that* $\lambda_1(A^*A) \leqslant \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}} \lambda_n^{\frac{t}{s}+1}(Q)$ *and* X *is an HPD solution of Eq.* (1.1), *then*

$$\begin{aligned} &\alpha_1 \leqslant \lambda_n(X) \leqslant \alpha_2 \text{ or } \beta_1 \leqslant \lambda_n(X) \leqslant \beta_2, \\ &\alpha_1 \leqslant \lambda_1(X) \leqslant \alpha_2 \text{ or } \beta_1 \leqslant \lambda_1(X) \leqslant \beta_2. \end{aligned}$$

Proof. The proof is similar to that of Theorem 2 of Zhang [21] and is omitted here. □

Remark 3.1. From Theorem 3.4 it follows that when the matrices A and Q satisfy

$$\lambda_1(A^*A) \leqslant \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}} \lambda_n^{\frac{t}{s}+1}(Q),$$

a necessary condition for existence an HPD solution X of Eq. (1.1) is that X satisfies

$$X \in [\alpha_1 I, \alpha_2 I] \cup [\beta_1 I, \beta_2 I] \cup \{X|\, \alpha_1 \leqslant \lambda_n(X) \leqslant \alpha_2, \beta_1 \leqslant \lambda_1(X) \leqslant \beta_2\}.$$

The following example confirms the correctness of Theorems 3.2–3.4.

Example 3.1. Consider the matrix equation

$$X + A^*X^{-2}A = Q, \tag{3.12}$$

with

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}.$$

It is easy to obtain that the matrices A and Q satisfy the condition of Theorem 3.2, i.e.

$$\lambda_1(A^*A) = 4 = \frac{1}{3}\left(\frac{2}{3}\right)^2 \lambda_n^3(Q).$$

By solving the Eqs. (3.3) and (3.4), we have

$$\alpha_1 = 0.4698, \quad \alpha_2 = 2, \quad \beta_1 = 2 \quad \text{and} \quad \beta_2 = 4.9593.$$

In fact, Eq. (3.12) exactly has the following four different HPD solutions

$$X_1 = \begin{bmatrix} 4.8284 & 0 \\ 0 & 2.8794 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 4.8284 & 0 \\ 0 & 0.6527 \end{bmatrix}$$

and

$$X_3 = \begin{bmatrix} 1.0000 & 0 \\ 0 & 2.8794 \end{bmatrix}, \quad X_4 = \begin{bmatrix} 1.0000 & 0 \\ 0 & 0.6527 \end{bmatrix}.$$

Evidently, these four solutions satisfy

$$X_1 \in [\beta_1 I, \beta_2 I], \quad X_4 \in [\alpha_1 I, \alpha_2 I]$$

and

$$X_2, X_3 \in \{X|0.4696 \leqslant \lambda_n(X) \leqslant 2, 2 \leqslant \lambda_1(X) \leqslant 4.9593\}.$$

And we also obtain that

$$\lambda_n(A^*A) = 1 < \frac{500}{27} = \frac{1}{3}\left(\frac{2}{3}\right)^2 \lambda_1^3(Q),$$

i.e. the matrices A and Q satisfy the conclusion of Theorem 3.3.

4. The case with $AA^* = A^*A$ and $AQ = QA$

In this section, we will investigate the existence of HPD solutions of Eq. (1.1) when $AA^* = A^*A$ and $AQ = QA$. We first give a property of commuting matrices. Then we prove a necessary condition for the existence of HPD solutions of Eq. (1.1). In the end, a necessary and sufficient condition for the existence of a Hermitian positive definite solution is given.

Lemma 4.1 [20, Theorem 3.2]. *Let B and C be square matrices of the same size. If $BC = CB$, then there exists a unitary matrix W such that W^*BW and W^*CW are both upper-triangular.*

Lemma 4.2. *Let A be an $n \times n$ nonsingular matrix and Q be an $n \times n$ Hermitian matrix. If $AQ = QA$ and $AA^* = A^*A$, then there exists a unitary matrix U and diagonal matrices T, $\widehat{T}$ such that*

$$U^*AU = T \quad \text{and} \quad U^*QU = \widehat{T}.$$

Proof. According to Lemma 4.1, if $AQ = QA$, then there exists a unitary matrix U such that

$$U^*AU = T \quad \text{and} \quad U^*QU = \widehat{T},$$

where T and $\widehat{T}$ are upper-triangular matrices.

Since Q is an Hermitian matrix, i.e. $Q = Q^*$, we have

$$\widehat{T} = U^*QU = U^*Q^*U = \widehat{T}^*.$$

Hence, $\widehat{T}$ is a diagonal matrix.

Since A is a nonsingular matrix and $A^*A = AA^*$, then we have

$$UT^*U^*UTU^* = A^*A = AA^* = UTU^*UT^*U^*,$$

i.e.

$$T^*T = TT^*.$$

Now let

$$T = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_{nn} \end{bmatrix}, \quad \text{then } T^* = \begin{bmatrix} \bar{t}_{11} & 0 & \cdots & 0 \\ \bar{t}_{12} & \bar{t}_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \bar{t}_{1n} & \bar{t}_{2n} & \cdots & \bar{t}_{nn} \end{bmatrix}.$$

Since $T^*T = TT^*$, then using that T^*T and TT^* have the same entries of diagonal line, we get

$$t_{ij} = 0 \ \ (i \neq j).$$

Hence, T is also a diagonal matrix. □

Theorem 4.1. *Let $AQ = QA$ and $AA^* = A^*A$, if Eq. (1.1) has an HPD solution X, then*

$$\lambda_1(A^*A) \leqslant \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}} \lambda_1^{\frac{t}{s}+1}(Q).$$

Proof. From Lemma 4.2, we obtain that if $AQ = QA$ and $A^*A = AA^*$, there exists a unitary matrix $U = (u_1, u_2, \ldots, u_n)$ such that

$$U^*AU = T \quad \text{and} \quad U^*QU = \widehat{T}, \tag{4.1}$$

where $T = \text{diag}(t_1, t_2, \ldots, t_n)$ and $\widehat{T} = \text{diag}(\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_n)$.

From (4.1) it follows that

$$Au_i = t_i u_i \quad \text{and} \quad Qu_i = \hat{t}_i u_i, \quad i = 1, 2, \ldots, n. \tag{4.2}$$

That is to say, t_i and $\hat{t}_i$ are the eigenvalue of A and Q respectively, and their corresponding eigenvectors are u_i.

Multiplying right side of Eq. (1.1) by u_i and left side by u_i^*, we have

$$u_i^* X^s u_i + |t_i|^2 u_i^* X^{-t} u_i = \hat{t}_i, \quad i = 1, 2, \ldots, n. \tag{4.3}$$

Since X is an HPD solution of Eq. (1.1), then there exists a unitary matrix V such that

$$X = V^* \Lambda V, \tag{4.4}$$

where $\Lambda = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$.

Let $Z = Vu_i = (z_1, z_2, \ldots, z_n)^{\mathrm{T}}$, then we have $Z^* Z = 1$.

Combining (4.3) and (4.4), we have

$$Z^* \Lambda^s Z + |t_i|^2 Z^* \Lambda^{-t} Z = \hat{t}_i Z^* Z, \quad i = 1, 2, \ldots, n$$

i.e.

$$|t_i|^2 = \frac{Z^*(\hat{t}_i I - \Lambda^s) Z}{Z^* \Lambda^{-t} Z} = \frac{\sum_{j=1}^n |z_j|^2 (\hat{t}_i - \sigma_j^s)}{\sum_{j=1}^n |z_j|^2 \sigma_j^{-t}}, \quad i = 1, 2, \ldots, n.$$

Without loss of generality, let $t_1 = \rho(A)$. Since $A^*A = AA^*$, then

$$\lambda_1(A^*A) = |t_1|^2 = \frac{\sum_{j=1}^n |z_j|^2 (\hat{t}_1 - \sigma_j^s)}{\sum_{j=1}^n |z_j|^2 \sigma_j^{-t}}. \tag{4.5}$$

From Lemmas 2.1 and 3.1, we have

$$\begin{aligned}(\hat{t}_1 - \sigma_j^s)\sigma_j^t &\leqslant \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}} (\hat{t}_1)^{\frac{t}{s}+1} \\ &\leqslant \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}} \lambda_1^{\frac{t}{s}+1}(Q).\end{aligned}$$

Then we have

$$\sum_{j=1}^n |z_j|^2 \left[\frac{(\hat{t}_1 - \sigma_j^s)\sigma_j^t - \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}} \lambda_1^{\frac{t}{s}+1}(Q)}{\sigma_j^t}\right] \leqslant 0,$$

i.e.

$$\sum_{j=1}^n |z_j|^2 \left[(\hat{t}_1 - \sigma_j^s) - \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}} \lambda_1^{\frac{t}{s}+1}(Q)\sigma_j^{-t}\right] \leqslant 0,$$

which implies

$$\sum_{j=1}^n |z_j|^2 (\hat{t}_1 - \sigma_j^s) \leqslant \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}} \lambda_1^{\frac{t}{s}+1}(Q) \sum_{j=1}^n |z_j|^2 \sigma_j^{-t},$$

i.e.

$$\frac{\sum_{j=1}^{n}|z_j|^2(\hat{t}_1-\sigma_j^s)}{\sum_{j=1}^{n}|z_j|^2\sigma_j^{-t}} \leqslant \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}}\lambda_1^{\frac{t}{s}+1}(Q). \tag{4.6}$$

Combining (4.5) and (4.6), we have

$$\lambda_1(A^*A) \leqslant \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}}\lambda_1^{\frac{t}{s}+1}(Q). \qquad \square$$

Example 4.1. Consider the matrix equation (3.12). It is easy to obtain that A and Q satisfy the conditions of Theorem 4.1. Hence

$$\lambda_1(A^*A) = 4 < \frac{500}{27} = \frac{1}{3}\left(\frac{2}{3}\right)^2\lambda_1^3(Q).$$

According to Corollary 3.1 and Theorem 4.1, we have the following result.

Corollary 4.1. *Let $A^*A = AA^*$ and $Q = bI$ $(b > 0)$. Then Eq.* (1.1) *has an HPD solution if and only if*

$$\lambda_1(A^*A) \leqslant \frac{s}{s+t}\left(\frac{t}{s+t}\right)^{\frac{t}{s}} b^{\frac{t}{s}+1}.$$

Remark 4.1. Theorem 11 of Engwerda [5] and Theorem 1 of Zhang [21] are special case of Corollary 4.1.

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