Shadowing properties of $\mathcal{L}$-hyperbolic homeomorphisms

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Abstract

We introduce the notion of $\mathcal{L}$-hyperbolic homeomorphisms on compact metric spaces as a strict generalization of Axiom A diffeomorphisms and prove that the notion is equivalent to expansive homeomorphisms having the shadowing property and to Ruelle’s Smale spaces. Furthermore, for $\mathcal{L}$-hyperbolic homeomorphisms, both the Lipschitz shadowing property and the average shadowing property are shown. © 2001 Elsevier Science B.V. All rights reserved.

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Introduction

Let $X$ be a compact metric space and let $f : X \to X$ be a homeomorphism onto itself. Fix any metric $d$ compatible with the topology of $X$. As usual, a sequence $\{x_i\}_{i \in \mathbb{Z}}$ of points in $X$ is called a $\delta$-pseudo-orbit ($\delta > 0$) of $f$ if $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. We say that $f$ has the shadowing property if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $\delta$-pseudo-orbit $\{x_i\}_{i \in \mathbb{Z}}$, there exists $y \in X$ satisfying $d(f^i(y), x_i) < \varepsilon$ ($\forall i \in \mathbb{Z}$). This property is independent of the compatible metric for $X$ and invariant under a topological conjugacy. We say that $f : X \to X$ is expansive if there is a constant $e > 0$ such that $d(f^i(x), f^i(y)) \leq e$ ($\forall i \in \mathbb{Z}$) implies $x = y$. Such a number $e$ is called an expansive constant. The expansivity (although not $e$) is also independent of the compatible metric chosen for $X$. These properties are very often appearing in several branch of the theory of dynamical systems, and it is well known that every Axiom A diffeomorphism restricted to the non-wandering set possesses them. Especially, they are usually playing an important
role in the investigation of the stability theory and the ergodic theory of Axiom A diffeomorphisms (see [4]).

For \( \varepsilon > 0 \), define the local stable set and local unstable set of \( x \) by

\[
W^s_\varepsilon(x, d) = \{ y \in X : d(f^n(x), f^n(y)) \leq \varepsilon, \forall n \geq 0 \},
\]

\[
W^u_\varepsilon(x, d) = \{ y \in X : d(f^{-n}(x), f^{-n}(y)) \leq \varepsilon, \forall n \geq 0 \}.
\]

The following fact is already known by combining theorems proved in [11,14,16].

**Fact.** For a homeomorphism \( f \) on a compact metric space \( X \), the following conditions are mutually equivalent:

(i) \( f \) is expansive and has the shadowing property,

(ii) we can find a compatible metric \( d \) for \( X \) such that

(H.1) there is \( \varepsilon_0 > 0 \) such that for every \( 0 < \varepsilon \leq \varepsilon_0 \), there exists \( \delta > 0 \) so that

\( W^u_\varepsilon(x, d) \cap W^s_\varepsilon(y, d) \) consists of a single point \( r(x, y) \) whenever \( d(x, y) < \delta \) (\( x, y \in X \)),

(H.2) there are \( 0 < \varepsilon_0, \mu < 1 \) such that for all \( x \in X \) and \( n \geq 0 \),

\[
\begin{align*}
    d(f^n(x), f^n(y)) &\leq \mu^n d(x, y) & \text{if } y \in W^s_{\varepsilon_0}(x, d), \\
    d(f^{-n}(x), f^{-n}(y)) &\leq \mu^n d(x, y) & \text{if } y \in W^u_{\varepsilon_0}(x, d),
\end{align*}
\]

(iii) \( (X, f) \) is a Smale space.

As a strict generalization of Smale’s Axiom A diffeomorphisms, the concept of a Smale space was introduced by Ruelle with regards to the thermodynamic formalism. For the definitions and results for Smale spaces see [15, Chapter 7].

In this paper, first of all we shall derive the following assertions (H.3) and (H.4) from the condition (i) besides (H.1) and (H.2) by choosing a compatible metric.

(H.3) There is a constant \( A > 0 \) such that if \( r(x, y) \) is as in (H.1), then \( d(r(x, y), x) \leq Ad(x, y) \) and \( d(r(x, y), y) \leq Ad(x, y) \).

(H.4) \( f \) is a Lipschitz homeomorphism.

Hence we can include (H.3) and (H.4) in condition (ii). More precisely, we prove the following theorem. For convenience, we say that \( f : (X, d) \to (X, d) \) is \( L \)-hyperbolic if all the assertions (H.1), (H.2), (H.3) and (H.4) are satisfied with respect to \( d \).

**Theorem 1.** Let \( f \) be a homeomorphism on a compact metric space \( X \). Then the following conditions are mutually equivalent:

(1) \( f \) is expansive and has the shadowing property,

(2) there is a compatible metric \( d \) for \( X \) such that \( f \) is \( L \)-hyperbolic,

(3) \( (X, f) \) is a Smale space.

Every Anosov diffeomorphism and every Axiom A diffeomorphism restricted to the non-wandering set are \( L \)-hyperbolic with respect to a Riemannian distance (see [2]).
Remark. In [15, p. 133], Ruelle proved Corollary 7.12 by assuming the assertion (H.3) for Smale spaces (see also [15, Remark 7.11]). However, by Theorem 1 the corollary is always true for every Smale space (with respect to some metric).

Clearly, every $\mathcal{L}$-hyperbolic homeomorphism has the shadowing property. In this paper, other two shadowing properties are considered for $\mathcal{L}$-hyperbolic homeomorphisms.

Let $(X, d)$ be as before, and let $f : X \to X$ be a homeomorphism. We say that $f$ has the Lipschitz shadowing property (see [13]) if there are positive constants $L$ and $\epsilon_0$ such that for any $0 < \epsilon < \epsilon_0$ and any $\epsilon$-pseudo-orbit $\{x_i\}_{i \in \mathbb{Z}}$ of $f$, there is $y \in X$ such that $d(f^i(y), x_i) < L\epsilon$ for all $i \in \mathbb{Z}$. The Lipschitz shadowing property is in generally stronger than the shadowing property. This property is proved for an Axiom A diffeomorphism and is used in the stability theory of random dynamical systems (see [9]).

The average shadowing property which is also discussed in the context of random dynamical systems in [2,3] is defined as follows. For $\delta > 0$, a sequence $\{x_i\}_{i \in \mathbb{Z}}$ of points in $X$ is called a $\delta$-average-pseudo-orbit of $f$ if there is a number $N = N(\delta) > 0$ such that for all $n \geq N$ and $k \in \mathbb{Z}$,

$$\frac{1}{n} \sum_{i=1}^{n} d(f(x_{i+k}), x_{i+k+1}) < \delta.$$ 

The notion of average-pseudo-orbits is a certain generalization of the notion of pseudo-orbits and is arising naturally in the realizations of independent Gaussian random perturbations with zero mean etc. [2, p. 368]. We say that $f$ has the average shadowing property if for every $\epsilon > 0$, there is $\delta > 0$ such that every $\delta$-average-pseudo-orbit $\{x_i\}_{i \in \mathbb{Z}}$ is $\epsilon$-shadowed in average by some $y \in X$; that is,

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} d(f^i(y), x_i) < \epsilon.$$ 

It is known that every Axiom A diffeomorphism restricted to a basic set has the average shadowing property (see [2,3]). These properties are depending on the metric for $X$.

Theorem 2. Let $f : X \to X$ be a homeomorphism of a compact metric space $(X, d)$. If $f$ is $\mathcal{L}$-hyperbolic, then $f$ has the Lipschitz shadowing property.

As we stated above, the Lipschitz shadowing property is in generally stronger than the shadowing property. Indeed, consider $S^1$ as $\mathbb{R}/\mathbb{Z}$ and let $d$ denote the standard metric on $S^1$. Let $f : S^1 \to S^1$ be a homeomorphism such that the set of all periodic points of $f$ is non-empty, finite and every element is topologically hyperbolic (for the definition see [17, Definition 6]). We assume further that there is a fixed point $p = 0 \in S^1$ of $f$ such that the local expression of $f$ (with respect to the canonical projection $\pi : \mathbb{R} \to S^1$) in a small neighborhood $U(p)$ of $p$ is $f(x) = x + x^2 \text{sgn}(x)$ for $x \in U(p)$. Here $\text{sgn}(x)$ is the sign of $x$. It can be easily seen that $f$ does not have the Lipschitz shadowing property (in $U(p)$, see [13]) with respect to $d$. On the other hand, it is stated in [17, Proof of Theorem 1] that
$f: S^1 \to S^1$ is topologically conjugate to some Morse–Smale diffeomorphism so that $f$ has the shadowing property.

However, for an expansive homeomorphism, we can find a suitable metric such that the shadowing property and the Lipschitz shadowing property are equivalent. The following is a direct consequence of Theorems 1 and 2.

**Corollary 1.** Let $f: X \to X$ be an expansive homeomorphism of a compact metric space. Then the following conditions are equivalent:

1. $f$ has the shadowing property,
2. there is a compatible metric $d$ such that $f$ has the Lipschitz shadowing property with respect to $d$.

Recall that a homeomorphism $f$ on $X$ is said to be topologically transitive if there is a dense orbit. The average shadowing property is closely related to a topological transitivity for $L$-hyperbolic homeomorphisms.

**Theorem 3.** Let $f: (X,d) \to (X,d)$ be an $L$-hyperbolic homeomorphism of a compact metric space. Then the following conditions are equivalent:

1. $f$ has the average shadowing property,
2. $f$ is topologically transitive.

If $f: X \to X$ is expansive and has the shadowing property, then the non-wandering set $\Omega(f)$ of $f$ has the spectral decomposition $\Omega(f) = \Lambda_1 \cup \cdots \cup \Lambda_\ell$ (see [1, 12]). The following is obtained from Theorems 1 and 3.

**Corollary 2.** Let $f: X \to X$ be an expansive homeomorphism having the shadowing property on a compact metric space. Then there is a compatible metric $d$ such that $f|_{\Lambda_i}$ has the average shadowing property with respect to $d$ for $1 \leq i \leq \ell$.

It is easy to see that if $f$ has the Lipschitz (respectively, the average) shadowing property, then so does $f^n$ ($n > 0$), respectively. Conversely, if $f$ is a Lipschitz homeomorphism and $f^n$ ($n > 0$) has the Lipschitz (respectively, the average) shadowing property, then $f$ so does, respectively. These shadowing properties are invariant under a topological conjugacy $h$ if both $h$ and $h^{-1}$ are Lipschitz.

It seems that a new metric constructed in Theorem 1 is abstract. In Section 4, we give three examples of expansive homeomorphisms having the shadowing property such that each of them is $L$-hyperbolic with respect to a familiar metric.

**1. Proof of Theorem 1**

To get the conclusion it is enough to show that if a homeomorphism $f$ on a compact metric space $X$ is expansive and has the shadowing property, then there exists a metric $D$ for $X$ (throughout this paper, this term means that $D$ is a metric compatible with the
topology of $X$) such that $f$ is $\mathcal{L}$-hyperbolic with respect to $D$. The following is proved by using a result stated in [14, Proof of Proposition].

**Lemma.** Let $f : (X, d) \to (X, d)$ be an expansive homeomorphism on a compact metric space. Then there are a metric $D$ for $X$ and constants $\varepsilon_0 > 0$, $0 < \mu < 1$ such that both $f$ and $f^{-1}$ are Lipschitz and

$$D(f^n(x), f^n(y)) \leq \mu^n D(x, y) \quad \text{if } y \in W^x_{\varepsilon_0} (x, D),$$

$$D(f^{-n}(x), f^{-n}(y)) \leq \mu^n D(x, y) \quad \text{if } y \in W^x_{\varepsilon_0} (x, D)$$

for all $x \in X$ and $n \geq 0$.

**Proof.** Let $\varepsilon > 0$ be an expansive constant and define a sequence of closed neighborhoods of the diagonal, $\Delta$ (in $X \times X$), as follows. Set $W_0 = X \times X$ and define

$$W_n = \{(x, y) \in X \times X : d(f^j(x), f^j(y)) \leq \varepsilon \text{ for } -n < j < n \} \quad \text{for } n \geq 1.$$

Then $\bigcap_{n=0}^{\infty} W_n = \Delta$ (see [14, Lemma 1]). Let $N > 1$ be as in [14, p. 207], and define a new sequence $\{V_k\}_{k=0}^{\infty}$ by $V_0 = W_0$, $V_k = W_{1+(k-1)N}$ for $k \geq 1$. By [14, Lemma 3], there is a metric $D$ for $X$ such that

$$V_k \subset \{(x, y) \in X \times X : D(x, y) < 1/2^k\} \subset V_{k-1} \quad \text{for } k \geq 1.$$

Let $L = \max\{1, \text{diam}_D(X)\}$ and put $K^N = 2^4L$, where $\text{diam}_D(X) = \sup\{D(x, y) : x, y \in X\}$. If $(x, y) \notin V_3$, then $\max\{D(f^N(x), f^N(y)), D(f^{-N}(x), f^{-N}(y))\} \leq L \leq K^N D(x, y)$ since $D(x, y) \geq 1/2^4$. If $(x, y) \in V_k \setminus V_{k+1}$ for $k \geq 3$, then $D(x, y) \geq 1/2^{k+2}$. Since $(x, y) \in V_k = W_{1+(k-1)N}$, we have $(f^N(x), f^N(y)), (f^{-N}(x), f^{-N}(y)) \in V_{k-1}$. Thus

$$\max\{D(f^N(x), f^N(y)), D(f^{-N}(x), f^{-N}(y))\} \leq \frac{1}{2^{k-1}} \leq 2^4D(x, y).$$

Therefore, $\max\{D(f^N(x), f^N(y)), D(f^{-N}(x), f^{-N}(y))\} \leq K^N D(x, y)$ for all $x, y \in X$.

If we choose $\nu > 0$ as in [14, p. 208], then

$$\begin{cases}
D(f^{3N}(x), f^{3N}(y)) \leq \frac{1}{2}D(x, y) & \text{if } y \in W^x_{\varepsilon_0} (x, D),
D(f^{-3N}(x), f^{-3N}(y)) \leq \frac{1}{2}D(x, y) & \text{if } y \in W^x_{\varepsilon_0} (x, D)
\end{cases}$$

for all $x \in X$ (see [14, Proposition]).

By using the same procedure stated in [16, Proof of Theorem] twice, the conclusions will be proved. Define a metric $\rho$ for $X$ by

$$\rho(x, y) = \sum_{i=0}^{N-1} \frac{1}{K^i} D\left(f^i(x), f^i(y)\right) \quad \text{for } \forall x, y \in X.$$

Then $\rho(f(x), f(y)) \leq K\rho(x, y)$ for all $x, y \in X$ since $D(f^N(x), f^N(y)) \leq K^N D(x, y)$. Furthermore, since $D(f^{-N}(x), f^{-N}(y)) \leq K^N D(x, y)$ for all $x, y \in X$, $\rho(f^{-N}(x), f^{-N}(y)) \leq K^N \rho(x, y)$ for all $x, y \in X$. Next, define

$$D'(x, y) = \sum_{i=0}^{N-1} \frac{1}{K^i} \rho(f^{-i}(x), f^{-i}(y)) \quad \text{for } \forall x, y \in X.$$
Then $D'(f(x), f(y)) \leq KD'(x, y)$. Since $\rho(f^{-N}(x), f^{-N}(y)) \leq K^N \rho(x, y)$, we can check that $D'(f^{-N}(x), f^{-N}(y)) \leq KD'(x, y)$, and so both $f$ and $f^{-1}$ are Lipschitz. If we fix $\delta > 0$ small enough so that $D'(x, y) < \delta$ implies $D'(f^i(x), f^i(y)) < \nu$ for $-N \leq i \leq N$ (notice that both $f$ and $f^{-1}$ are uniformly continuous), then $D'(f^{3N}(x), f^{3N}(y)) \leq \frac{1}{2} D'(x, y)$ if $y \in W_\delta^s(x, D')$ and $D'(f^{3N}(x), f^{-3N}(y)) \leq \frac{1}{2} D'(x, y)$ if $y \in W_\delta^u(x, D')$ for all $x \in X$ by (*).

We are in a position to construct a metric $D''$ for $X$ what we want. Put $\mu^{3N} = \frac{1}{2}$ and define a metric $\rho'$ for $X$ by

$$\rho'(x, y) = \sum_{i=0}^{3N-1} \frac{1}{\mu^i} D'(f^i(x), f^i(y)) \quad \text{for } \forall x, y \in X.$$ 

It is easy to see that both $f$ and $f^{-1}$ are Lipschitz with respect to $\rho'$ and $\rho'(f(x), f(y)) \leq \mu \rho'(x, y)$ ($\forall y \in W_\delta^s(x, D')$) for all $x \in X$. Take $0 < \varepsilon_0 < \delta$ such that $D'(x, y) < \varepsilon_0$ implies $D'(f^i(x), f^i(y)) \leq \delta$ for $-3N \leq i \leq 3N$. Then $\rho'(f^{3N}(x), f^{-3N}(y)) \leq \frac{1}{2} \rho'(x, y)$ ($\forall y \in W_\varepsilon^s(x, D')$) for all $x \in X$. Finally, define a metric $D''$ for $X$ by

$$D''(x, y) = \sum_{i=0}^{3N-1} \frac{1}{\mu^i} \rho'(f^{-i}(x), f^{-i}(y)) \quad \text{for } \forall x, y \in X.$$ 

Then both $f$ and $f^{-1}$ are Lipschitz with respect to $D''$ and $D''(f^{-1}(x), f^{-1}(y)) \leq \mu D''(x, y)$ ($\forall y \in W_\varepsilon^u(x, D')$) for all $x \in X$ since $\varepsilon_0 < \delta$. Furthermore, by the choice of $\varepsilon_0$, $D''(f(x), f(y)) \leq \mu D''(x, y)$ ($\forall y \in W_\varepsilon^u(x, D')$) for all $x \in X$. Clearly, if $D''(x, y) < \varepsilon_0$ (x, y \in X), then $D'(x, y) < \varepsilon_0$. Thus $W_\varepsilon^s(x, D'') \subset W_\varepsilon^u(x, D')$ for all $x$ and $\sigma = s,u$. Denote $D''$ by $D$ and the lemma is proved.

**Proof of Theorem 1.** Let $f : X \to X$ be an expansive homeomorphism having the shadowing property, and let $D$, $0 < \varepsilon_0 < \mu < 1$ be given by the above lemma. We may suppose that $K > 1$ is a Lipschitz constant for both $f$ and $f^{-1}$ with respect to $D$ and that $\varepsilon_0$ is an expansive constant. For every $0 < \varepsilon \leq \varepsilon_0/2$, there is $0 < \delta \leq \varepsilon$ such that if $D(x, y) < \delta$ ($x, y \in X$), then there exists $r(x, y) = W_\varepsilon^u(x, D) \cap W_\varepsilon^s(y, D)$. It only remains to show the assertion (H.3) since for all $x \in X$ and $n \geq 0$,

$$
D(f^n(x), f^n(y)) \leq \mu^n D(x, y) \quad \text{if } y \in W_\varepsilon^s(x, D), \\
D(f^{-n}(x), f^{-n}(y)) \leq \mu^n D(x, y) \quad \text{if } y \in W_\varepsilon^u(x, D).
$$

Let $\delta = \delta(\varepsilon_0/2) > 0$ be as above and take $0 < \delta_1 \leq \delta$ such that $D(x, y) \leq \delta_1$ ($x, y \in X$) implies $r(x, y) \in f^{-1}(W_\varepsilon^u(f(x), D)) \cap f(W_\varepsilon^s(f^{-1}(y), D))$. Fix any $x, y \in X$ with $D(x, y) < \delta_1$.

**Case 1.** $D(x, r(x, y)) \geq D(y, r(x, y)).$

By the choice of $\delta_1$,

$$
\frac{1}{\mu} D(x, r(x, y)) \leq D(f(x), f(r(x, y))) \quad \text{and} \\
D(f(y), f(r(x, y))) \leq \mu D(y, r(x, y)).
$$
Thus
\[ \frac{1}{\mu} D(x, r(x, y)) \leq D(f(x), f(y)) + \mu D(y, r(x, y)) \]
since \( D(f(x), f(r(x, y))) \leq D(f(x), f(y)) + D(f(y), f(r(x, y))) \). Since \( D(x, r(x, y)) \geq D(y, r(x, y)) \), we have \( \left( \frac{1}{\mu} - \mu \right) D(x, r(x, y)) \leq D(f(x), f(y)) \) so that
\[ D(x, r(x, y)) \leq \frac{\mu}{1 - \mu^2} D(f(x), f(y)). \]

Hence
\[ \max\{D(x, r(x, y)), D(y, r(x, y))\} \leq \frac{K \mu}{1 - \mu^2} D(x, y). \]

Case 2. \( D(x, r(x, y)) \leq D(y, r(x, y)). \)

Clearly, \( D(f^{-1}(y), f^{-1}(r(x, y))) \leq D(f^{-1}(x), f^{-1}(y)) + D(f^{-1}(x), f^{-1}(r(x, y))) \). From this, we have \( \left( \frac{1}{\mu} - \mu \right) D(y, r(x, y)) \leq D(f^{-1}(x), f^{-1}(r(x, y))) \) and
\[ D(f^{-1}(x), f^{-1}(r(x, y))) \leq \mu D(x, r(x, y)). \]
Therefore
\[ \max\{D(x, r(x, y)), D(y, r(x, y))\} \leq \frac{K \mu}{1 - \mu^2} D(x, y). \]

\[ \square \]

2. Proof of Theorem 2

Let \((X, d)\) be a compact metric space and let \( f : X \to X \) be an expansive homeomorphism. We say that \( f \) has Lipschitz canonical coordinates if there are constants \( L > 0 \) and \( \epsilon_0 > 0 \) such that for every \( 0 < \epsilon \leq \epsilon_0 \), \( d(x, y) < \epsilon \ (x, y \in X) \) implies \( W^u_{\epsilon_0}(x, d) \cap W^s_{\epsilon_0}(y, d) \neq \emptyset \). We say that Lipschitz canonical coordinates are hyperbolic if there are constants \( 0 < \epsilon_0, \mu < 1 \) such that for all \( x \in X \) and \( n > 0 \),
\[
\begin{aligned}
&d(f^n(x), f^n(y)) \leq \mu^n d(x, y) \quad \text{if } y \in W^u_{\epsilon_0}(x, d), \\
&d(f^{-n}(x), f^{-n}(y)) \leq \mu^n d(x, y) \quad \text{if } y \in W^s_{\epsilon_0}(x, d).
\end{aligned}
\]

Obviously, every \( L \)-hyperbolic homeomorphism \( f \) has hyperbolic Lipschitz canonical coordinates. Thus, to prove Theorem 2 it is enough to show the following proposition since \( f \) is Lipschitz. The same result was proved in [13] for a hyperbolic set of a diffeomorphism defined in a Euclidean space (see also [9]).

**Proposition.** Let \( f : (X, d) \to (X, d) \) be an expansive homeomorphism on a compact metric space and let \( f \) be Lipschitz. If \( f \) has hyperbolic Lipschitz canonical coordinates, then \( f \) has the Lipschitz shadowing property.
Proof. Let $L$ and $\epsilon_0$ be numbers as in the definition of Lipschitz canonical coordinates of $f$. We may assume $L \geq 1$. For every $0 < \epsilon \leq \epsilon_0$, if $d(x, y) < \epsilon$ ($x, y \in X$), then there is
\[ r(x, y) = W^u_{L\epsilon}(x, d) \cap W^s_{L\epsilon}(y, d). \]

Take $n > 0$ such that $\mu^n L < 1$, and recall that if $f^n$ has the Lipschitz shadowing property, then $f$ so does since $f$ is Lipschitz.

Now, we show the property for $f^n$. To simplify the notations, denote $W^u_{\epsilon}(x, d)$, $W^s_{\epsilon}(x, d)$, $\mu^n$ and $f^n$ by $W^u_{\epsilon}(x)$, $W^s_{\epsilon}(x)$, $\mu$ and $f$, respectively.

Let $\{x_i\}_{i=0}^k$ be an $\epsilon$-pseudo-orbit of $f$ ($0 < \epsilon < (1 - \mu L) \epsilon_0$) and put
\[ L' = \frac{L}{(1 - \mu L)(1 - \mu)} + \frac{1}{1 - \mu L}. \]

For simplicity, we will try to find $y \in X$ such that $d(f^i(y), x_i) < L' \epsilon$ for $0 \leq i < k$ when $k = 4$.

Put $y_0 = x_4$, and denote $\sum_{i=0}^n (\mu L)^i$ by $v_n$ for convenience. Then
\[ L' = L \left( \sum_{i=0}^\infty (\mu L)^i \right) v_\infty + v_\infty, \quad \text{where } v_\infty = \lim_{n \to \infty} v_n. \]

Since $d(f(x_3), y_0) < \epsilon$, there exists $r(f(x_3), y_0) \in W^u_{L\epsilon}(f(x_3)) \cap W^s_{L\epsilon}(y_0)$. Thus $y_1 = f^{-1}(r(f(x_3), y_0)) \in W^u_{\mu L\epsilon}(x_3)$ so that $d(f(x_2), y_1) < v_1 \epsilon < \epsilon_0$ (since $d(f(x_2), x_3) < \epsilon$). Pick $r(f(x_2), y_1) \in W^u_{\mu L\epsilon}(f(x_2)) \cap W^s_{\mu L\epsilon}(y_1)$. Then $y_2 = f^{-1}(r(f(x_2), y_1)) \in W^u_{\mu L\epsilon}(x_2)$. Since $d(f(x_1), x_2) < \epsilon$, we have $d(f(x_1), y_2) < v_2 \epsilon < \epsilon_0$. Take
\[ r(f(x_1), y_2) \in W^u_{\mu L\epsilon}(f(x_1)) \cap W^s_{\mu L\epsilon}(y_2) \]
and let $y_3 = f^{-1}(r(f(x_1), y_2)) \in W^u_{\mu L\epsilon}(x_1)$. Then, since $d(f(x_0), x_1) < \epsilon$, we see $d(f(x_0), y_3) < v_3 \epsilon < \epsilon_0$. Hence $r(f(x_0), y_3) \in W^u_{\mu L\epsilon}(f(x_0)) \cap W^s_{\mu L\epsilon}(y_3)$. Clearly $y_4 = f^{-1}(r(f(x_0), y_3)) \in W^u_{\mu L\epsilon}(x_0)$.

Now, denote $y_4$ by $y$. Then $d(y, x_0) \leq v_4 \epsilon \leq L \epsilon_0 \epsilon$. Since $f(y) = r(f(x_0), y_3) \in W^s_{\mu L\epsilon}(y_3)$ and
\[ d(f(y), x_1) \leq d(f(y), y_3) + d(y_3, x_1) = d(f(x_0), y_3) + d(x_1, y_3), \]
we have $d(f(y), x_1) < (L v_3 + \mu v_2) \epsilon \leq (L v_\infty + v_\infty) \epsilon$. Since
\[ d(f^2(y), x_2) \leq d(f^2(y), f(y)) + d(f(y), x_2) \]
and $f^2(y) \in W^s_{\mu L\epsilon}(f(y_3))$, we see
\[ d(f^2(y), x_2) \leq \{ L(\mu v_3 + v_2) + \mu v_1 \} \epsilon \leq \{ L(\mu + 1) v_\infty + v_\infty \} \epsilon. \]
By the same way, since $f^2(y) \in W^s_{L(\mu v_3 + v_2)\epsilon}(y_2)$
\[ d(f^3(y), x_3) \leq d(f^3(y), f(y_2)) + d(f(y_2), y_1) + d(y_1, x_3), \]
we have
\[ d(f^3(y), x_3) \leq \{ L(\mu^2 v_3 + \mu v_2 + v_1) + \mu L \} \epsilon \leq \{ L(\mu^2 + \mu + 1) v_\infty + v_\infty \} \epsilon. \]
Next, since \( f^3(y) \in W^s_{L(\mu_3^2 v_3 + \mu_2 v_2 + v_1)}(y_1) \) and
\[
 d\left(f^4(y), x_4\right) \leq d\left(f^4(y), f(y_1)\right) + d\left(f(y_1), y_0\right) + d(y_0, x_4),
\]
d\( (f^4(y), x_4) < L(\mu_3^3 v_3 + \mu_2^2 v_2 + \mu v_1 + 1) \varepsilon \leq L(\mu_3^3 + \mu_2^2 + \mu + 1) v_0 \varepsilon \). Therefore
\[
d(f^i(y_k), x_i) < L' \varepsilon \text{ for } 0 \leq i \leq 4.
\]
Inductively, it is possible to find \( y_k \in X \) satisfying \( d(f^i(y_k), x_i) < L' \varepsilon \) for \( 0 \leq i \leq k \).
Since \( k \) is arbitrary and \( X \) is compact, the conclusion will be proved by using a diagonal method. □

3. Proof of Theorem 3

Let \( f : (X, d) \to (X, d) \) be a homeomorphism of a compact metric space. The global stable set and global unstable set of \( x \in X \) are defined by
\[
 W^s(x, d) = \{ y \in X: d(f^n(x), d^n(y)) \to 0 \text{ as } n \to \infty \},
\]
\[
 W^u(x, d) = \{ y \in X: d(f^{-n}(x), d^{-n}(y)) \to 0 \text{ as } n \to \infty \}.
\]
A compact \( f \)-invariant set \( A \) is called isolated if there is a neighborhood \( U \) of \( A \) such that
\[
 \bigcap_{n \in \mathbb{Z}} f^n(U) = A.
\]
Before starting the proof, we collect some well known dynamical properties of an expansive homeomorphism having the shadowing property. For a moment, let \( f : (X, d) \to (X, d) \) be an expansive homeomorphism having the shadowing property. It is easy to see that if \( e > 0 \) is an expansive constant, then
\[
 W^s(x, d) = \bigcup_{n \geq 0} f^{-n}(W^s_e\left(f^n(x), d\right)) \quad \text{and} \quad W^u(x, d) = \bigcup_{n \geq 0} f^n\left(W^u_e\left(f^{-n}(x), d\right)\right).
\]
Let \( \Omega(f) \) be the non-wandering set of \( f \). Then it is proved in [1] that
(3.1) \( f|\Omega(f) : \Omega(f) \to \Omega(f) \) has the shadowing property,
(3.2) the set of all periodic points, \( P(f) \), of \( f \) is dense in \( \Omega(f) \),
(3.3) \( \Omega(f) \) is decomposed into a finite disjoint union of closed \( f \)-invariant sets \( \{A_1, \ldots, A_\ell\} \), that is, \( \Omega(f) = A_1 \cup \cdots \cup A_\ell \) such that \( f|_{A_i} \) is topologically transitive for \( 1 \leq i \leq \ell \).

Such a set \( A_i \) is called a basic set (see also [12, Theorem 21]). Moreover it is proved in [12, Proposition 9] that
(3.4) \( A_i \) is an isolated set for \( 1 \leq i \leq \ell \).

A cycle for the above family \( \{A_1, \ldots, A_\ell\} \) is a subsequence \( \{A_{i_1}, \ldots, A_{i_k}\} \) such that
\[
 A_{i_1} = A_{i_k} \quad \text{and} \quad W^s(\sigma(A_{i_j}, d)) \cap W^u(\sigma(A_{i_{j+1}}, d)) \setminus (A_{i_j} \cup A_{i_{j+1}}) \neq \emptyset \quad \text{for } 1 \leq j \leq k.
\]
Here \( W^\sigma(A, d) = \bigcup_{x \in A} W^\sigma(x, d) \) for \( \sigma = s, u \). We can check that
(3.5) there are no-cycles in \( \{A_1, \ldots, A_\ell\} \) and especially \( W^s(A_i, d) \cap W^u(A_i, d) = A_i \) for \( 1 \leq i \leq \ell \).

For, suppose that there is a cycle \( \{A_{i_1}, \ldots, A_{i_k}\} \) for some \( k \geq 1 \). By using topological transitivity of \( f|_{A_{i_1}} \), for each \( \delta > 0 \), we can construct a cyclic \( \delta \)-pseudo-orbit \( \{x_n\}^{n_{\infty}}_{n=0} \) such that \( x_0 = x_n \notin \Omega(f) \) for some \( n = n(\delta) > 0 \). Let \( e > 0 \) be an expansive constant of \( f \) and
take $0 < \varepsilon \leq \varepsilon/2$ such that $B_\varepsilon(x_0) \cap \Omega(f) = \emptyset$. Here $B_\varepsilon(x) = \{y \in X: d(x, y) \leq \varepsilon\}$. Fix $\delta = \delta(\varepsilon) > 0$ as in the definition of the shadowing property. Then, for the cyclic $\delta$-pseudo-orbit constructed above, there exists $p \in X$ sufficiently near $x_0$ such that $f^n(p) = p$. This is a contradiction because $p \in B_\varepsilon(x_0) \subset \Omega(f)$.

Since $\Omega(f) = A_1 \cup \cdots \cup A_\ell$,

(3.6) $X = \bigcup_{i=1}^{\ell} W^s(A_i, d) = \bigcup_{i=1}^{\ell} W^u(A_i, d)$ (cf. [10, Corollary 1.6]).

(3.7) there exists a filtration for $\{A_1, \ldots, A_\ell\}$; that is, there are compact sets $\emptyset = X_0 \subset X_1 \subset \cdots \subset X_\ell = X$ such that $f(X_i) \subset \text{int} X_i$ and $\bigcap_{n\in\mathbb{Z}} f^n(X_i \setminus X_{i-1}) = A_i$ for all $1 \leq i \leq \ell$ (this fact follows from (3.4), (3.5) and [10, p. 186, Filtration lemma]).

(3.8) There is a Markov partition of $X$ with arbitrarily small diameter (see [4,7] for the definitions and its proof).

For a proof of Theorem 3, we prepare two lemmas. In the following lemmas, let $f : X \to X$ be $\mathcal{L}$-hyperbolic (with respect to $d$).

**Lemma 3.1** (Local product structure). There are constants $0 < \varepsilon_0 < 1 < \lambda$ and $B > 0$ such that for each $0 < \varepsilon \leq \varepsilon_0$, there is $\delta > 0$ with the property that if $d(x, y) < \delta$, then there exists $r(x, y) \in X$ satisfying

1. $r(x, y) \in W^u_{\varepsilon}(x, d) \cap W^s_\varepsilon(y, d)$,
2. $d(r(x, y), x) + d(r(x, y), y) \leq Bd(x, y)$,
3. $d(f^n(z), f^n(y)) \leq \lambda^{-n} d(z, y)$ for any $z \in W^u_\varepsilon(y, d)$ for all $n \geq 0$,
4. $d(f^{-n}(z), f^{-n}(y)) \leq \lambda^{-n} d(z, x)$ for any $z \in W^u_\varepsilon(x, d)$ for all $n \geq 0$.

**Proof.** Since $f$ is $\mathcal{L}$-hyperbolic, the assertions are obtained quickly from the definition. \(\square\)

A proof of the next lemma is almost the same as that of [2, Lemma 4.1] and [3, Lemma 6.3.2]. However, in generally, two sets $W^u(x, d)$ and $W^s(y, d)$ are not manifolds, so we need to choose $r(x, y)$ carefully.

**Lemma 3.2** (Global product structure). We assume further that $f$ is topologically transitive. Then there are constants $B, C > 1$ and $\lambda > 1$ such that for each pair $x, y \in X$, there exists $r(x, y) \in X$ satisfying

1. $r(x, y) \in W^u(x, d) \cap W^s(y, d)$,
2. $d(r(x, y), x) + d(r(x, y), y) \leq Bd(x, y)$,
3. $d(f^n(r(x, y)), f^n(y)) \leq C\lambda^{-n} d(x, y)$ for all $n \geq 0$,
4. $d(f^{-n}(r(x, y)), f^{-n}(x)) \leq C\lambda^{-n} d(x, y)$ for all $n \geq 0$.

**Proof.** Since $f$ is $\mathcal{L}$-hyperbolic, there are constants $0 < \varepsilon_0 < 1 < \mu < 1$, $A > 0$ and $\delta > 0$ such that if $d(x, y) < \delta$, then there exists $r(x, y) \in W^u_{\varepsilon_0}(x, d) \cap W^s_{\varepsilon_0}(y, d)$ satisfying $d(r(x, y), x) \leq Ad(x, y)$ and $d(r(x, y), y) \leq Ad(x, y)$. We may suppose that $\varepsilon_0$ is an expansive constant of $f$. Since $f$ is expansive, $r(x, y) \in W^s(x, d) \cap W^u(y, d)$. Thus, for the case when $d(x, y) < \delta$, the conclusions are obtained from Lemma 3.1.

We may suppose further that $K > 1$ is a Lipschitz constant of $f$ and that $\delta \leq \varepsilon_0$. Let $0 < \delta_1 = \delta_1(\delta) \leq \delta$ be as in the definition of the shadowing property of $f$, and fix a Markov
partition $\mathcal{R} = \{ R_1, \ldots, R_m \}$ with $\max_{1 \leq i \leq m} \text{diam}_d(R_i) \leq \delta_1$ (see (3.8)). Let $A$ be a $m \times m$-transition matrix of the Markov partition. Then, since $f$ is topologically transitive, there is an integer $n_0 > 0$ such that the matrix $A^{n_0}$ is strictly positive (see [6]). Thus, for the case when $d(x, y) \geq \delta (x, y \in X)$, $f^{n_0}(R(f^{-n_0}(x))) \cap R(y) \neq \emptyset$. Here $R(x)$ is an element of $\mathcal{R}$ containing $x$. Pick $z \in f^{n_0}(R(f^{-n_0}(x))) \cap R(y)$. Then a sequence
\[
\{ \ldots, f^{-n_0-2}(x), f^{-n_0-1}(x), f^{-n_0}(z), \ldots, f^{-1}(z), y, f(y), \ldots \}
\]
is a $\delta_1$-pseudo-orbit of $f$. By using the shadowing property we can find $r(x, y) \in X$ such that
\[
r(x, y) \in W^s_{\delta}(y, d) \quad \text{and} \quad f^{-n_0}(r(x, y)) \in W^u(t^{-n_0}(x), d).
\]
Since $d(f^{-n_0}(r(x, y)), f^{-n_0}(x)) < \delta$ and $f$ is Lipschitz with constant $K$,
\[
d(r(x, y), x) = d(f^{n_0}(f^{-n_0}(r(x, y))), f^{n_0}(f^{-n_0}(x))) < K^{n_0} \delta \leq K^{n_0}d(x, y).
\]
Furthermore, since $d(f^{-n_0-j}(r(x, y)), f^{-n_0-j}(x)) \leq \mu^j d(f^{-n_0}(r(x, y)), f^{-n_0}(x))$ for all $j \geq 0$,
\[
d(f^{-n_0-j}(r(x, y)), f^{-n_0-j}(x)) < \mu^j \delta \quad (\forall j \geq 0).
\]
Thus $d(f^{-j}(r(x, y)), f^{-j}(x)) \leq K^{n_0}d(f^{-n_0-j}(r(x, y)), f^{-n_0-j}(x)) < K^{n_0} \mu^j d(x, y)$.

On the other hand, since $d(f^j(r(x, y)), f^j(y)) < \delta$ for all $j \geq 0$,
\[
d(f^j(r(x, y)), f^j(y)) < \mu^j \delta \leq \mu^j d(x, y) \quad (\forall j \geq 0).
\]
Especially, since $d(r(x, y), y) < d(x, y)$,
\[
d(r(x, y), x) + d(r(x, y), y) \leq (K^{n_0} + 1)d(x, y).
\]
Finally, we set $\lambda^{-1} = \mu$. The proof of the lemma is completed.$\square$

**Proof of Theorem 3.** Let $f : X \to X$ be $\mathcal{L}$-hyperbolic. By Lemma 3.2, if $f$ is topologically transitive, then the average shadowing property follows from [2, pp. 375–377, Proof of Theorem 4] (see also [3, Proof of Theorem 6.3.1]). The proof is the same as the original one.

To show the converse, suppose $f$ has the average shadowing property. Since $f$ is expansive and has the shadowing property, there exists a filtration $\emptyset = X_0 \subset X_1 \subset \cdots \subset X_\ell = X$ for the decomposition $\Omega(f) = A_1 \cup A_2 \cup \cdots \cup A_\ell$ (see (3.3) and (3.7)).

**Claim.** Under the above notations, we have $\ell = 1$.

If this claim is true, then $X = \Omega(f) = A_1$ so that $f$ is topologically transitive. Actually, $X = W^s(A_1, d) \cap W^u(A_1, d) = A_1$ by (3.5) and (3.6). Thus Theorem 3 is proved.$\square$

To prove the claim, by assuming that $\ell \geq 2$ we derive a contradiction. For simplicity, suppose $\ell = 2$. Take $\varepsilon > 0$ small enough and fix integers $n_1, n_2 \geq 5$ such that $(n_1 - 1)\varepsilon < d(X_1, A_2) \leq n_1 \varepsilon$ and $(n_2 - 1)\varepsilon < d(A_1, A_2) \leq n_2 \varepsilon$, respectively. Here $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$ for $A, B \subset X$. Since $f$ has the average shadowing property,
there is $0 < \delta = \delta(\varepsilon) < \varepsilon$ such that every $\delta$-average-pseudo-orbit $\{x_i\}_{i \in \mathbb{Z}}$ is $\varepsilon$-shadowed in average by some point in $X$. Finally, let us fix $n_3 \geq 3$ such that $(n_3 - 1)\delta < d(A_1, A_2) \leq n_3\delta$. Take $x \in A_1$, $y \in A_2$ with $d(x, y) = d(A_1, A_2)$. Since $\Omega(f) = P(f)$ by (3.2), there are $p \in A_1 \cap P(f)$ and $q \in A_2 \cap P(f)$ such that

$$\max\{d(x, p), d(y, q), d(f(x), f(p)), d(f(y), f(q))\} < \delta.$$ 

Let $\ell_1, \ell_2 > 0$ be the (minimum) periods of $p, q$ respectively; that is, $f^{\ell_1}(p) = p, f^{\ell_2}(q) = q$. Fix $\ell_3 > 0$ such that $\ell_i\ell_3 > n_3$ for $i = 1, 2$, and denote a cyclic sequence

$$\{\ldots, y, f(q), f^2(q), \ldots, f^{\ell_1\ell_2\ell_3-1}(q), x, f(p), f^2(p), \ldots, f^{\ell_1\ell_2\ell_3-1}(p), y, f(q), \ldots\}$$

(composed of two points $\{x, y\}$ and two periodic orbits) by $\{z_i\}_{i \in \mathbb{Z}}$ ($z_0 = y$). It is easy to see that this is a $\delta$-average-pseudo-orbit. Indeed, for every $m > 2\ell_1\ell_2\ell_3$ and $k \in \mathbb{Z}$, we have

$$\frac{1}{m} \sum_{i=1}^{m} d(f(z_{i+k}), z_{i+k+1}) < \delta.$$ 

Pick $w \in X$ such that $\varepsilon$-shadows $\{z_i\}_{i \in \mathbb{Z}}$ in average. If $w \in A_2$, then $f^i(w) \in A_2$ for all $i \geq 0$. Hence, for a sufficiently large $m > 3\ell_1\ell_2\ell_3$, we have

$$\frac{1}{m} \sum_{i=1}^{m} d(f^i(w), z_i) > \frac{(n_2 - 1}\varepsilon}{3} > \varepsilon.$$ 

This is a contradiction. If $w \notin A_2$, then there exists a neighborhood $U_2$ of $A_2$ with $w \notin U_2$. By using a filtration property, we can show that there is $m' > 0$ satisfying $f^i(w) \in X_1$ for all $i > m'$. Thus

$$\frac{1}{m} \sum_{i=1}^{m} d(f^i(w), z_i)$$

$$= \frac{1}{m} \left( \sum_{i=1}^{m'} d(f^i(w), z_i) + \sum_{j=1}^{m-m'} d(f^{m'+j}(w), z_{m'+j}) \right) > \frac{(n_1 - 1)\varepsilon}{3} > \varepsilon$$

if we take $m$ ($> m'$) large enough. This is also a contradiction.

4. Examples of $\mathcal{L}$-hyperbolic homeomorphisms

In this section, we give three examples of $\mathcal{L}$-hyperbolic homeomorphisms with respect to familiar metrics.

Example 4.1. Let $f : (X, d) \to (X, d)$ be Ruelle’s expanding map on a compact metric space $\Omega(f) = P(f)$ by (3.2), that is, $f$ is a continuous onto map, and there are constants $\delta_0 > 0$ and $\lambda > 1$ such that for $x, y \in X$,

(a) $d(f(x), f(y)) \geq \lambda d(x, y)$ whenever $d(x, y) < \delta_0$,

(b) $B_{\delta_0}(x) \cap f^{-1}(y) = \{\text{a single point}\}$ whenever $d(f(x), y) < \lambda\delta_0$. 
We assume further that $f$ is a Lipschitz map with constant $K > 1$. Every expanding differentiable map on a closed $\mathcal{C}^\infty$ manifold satisfies the above conditions with respect to a Riemannian distance. Let

$$X_f = \{ x = \{ x_i \}_{i=0}^{-\infty} : x_i \in X \text{ and } f(x_i) = x_{i+1} \text{ for all } i \geq 0 \},$$

and define $f(x) = \{ f(x_i) \}_{i=0}^{-\infty}$ and $f^{-1}(x) = \{ x_{i-1} \}_{i=0}^{\infty}$. Then $X_f$ is a compact with respect to a metric

$$d(x, y) = \sum_{i=0}^{-\infty} \frac{d(x_i, y_i)}{\lambda^{|i|}} \quad \text{for } x = \{ x_i \}_{i=0}^{-\infty}, \ y = \{ y_i \}_{i=0}^{-\infty} \in X_f.$$

It is easy to see that $f$ is an expansive homeomorphism having the shadowing property (see [15, p. 144]).

We show that $f : X_f \rightarrow X_f$ is $\mathcal{L}$-hyperbolic with respect to $d$. For $0 < \varepsilon \leq \varepsilon_0$, if $d(x, y) < \varepsilon$, then by the definition of an expanding map, there exists $z = \{ z_i \}_{i=0}^{-\infty} \in X_f$ such that $z_0 = y_0$ and $d(x_i, z_i) \leq \lambda^i d(x_0, y_0)$ for all $i \leq 0$. Denote $z$ by $r(x, y)$. Then, since

$$d(r(x, y), y) \leq \sum_{i=0}^{-\infty} \frac{d(x_i, y_i)}{\lambda^{|i|}} + \sum_{i=0}^{-\infty} \frac{d(x_i, z_i)}{\lambda^{|i|}},$$

and

$$\sum_{i=0}^{-\infty} \frac{d(x_i, z_i)}{\lambda^{|i|}} \leq \sum_{i=0}^{\infty} \frac{1}{\lambda^{|i|}} \cdot d(x_0, y_0)$$

(because $z_0 = y_0$), we see

$$d(r(x, y), y) \leq 2\lambda^2 - 1 - \frac{d(x, y)}{\lambda^2 - 1}.$$

And also we have

$$d(r(x, y), x) \leq \frac{\lambda^2}{\lambda^2 - 1} d(x, y)$$

since

$$d(z, x) = \sum_{i=0}^{\infty} \frac{1}{\lambda^{|i|}} \cdot d(x_0, z_0) = \frac{\lambda^2}{\lambda^2 - 1} \cdot d(x_0, y_0).$$

Furthermore, it is not hard to show that for all $j \geq 0$,

$$d(f^j(r(x, y)), f^j(x)) \leq \frac{1}{\lambda^j} d(r(x, y), x),$$

$$d(f^{-j}(r(x, y)), f^{-j}(y)) \leq \frac{1}{\lambda^j} d(r(x, y), y).$$

Thus $r(x, y) = W^u_\delta(x, d) \cap W^s_\delta(y, d)$, where $\delta = \frac{\lambda^2}{\lambda^2 - 1} \varepsilon$. By the same way, we see that for all $x \in X_f$ and $j \geq 0$,

$$d(f^j(x), f^j(y)) \leq \lambda^{-j} d(x, y) \quad \text{if } y \in W^s_{\varepsilon_0}(x, d),$$

$$d(f^{-j}(x), f^{-j}(y)) \leq \lambda^{-j} d(x, y) \quad \text{if } y \in W^u_{\varepsilon_0}(x, d).$$
Finally, \( f \) is Lipschitz since \( f \) is Lipschitz with constant \( K \). Actually, for any \( x, y \in X_f \),
\[
d(f(x), f(y)) \leq \frac{\lambda K + 1}{\lambda} d(x, y).
\]

**Example 4.2.** Let \( k \) be a natural number and let \( C = \{0, 1, \ldots, k-1\} \) endowed with the discrete topology. Consider the product space \( \Sigma = \prod_{i=1}^\infty C \) and define a shift homeomorphism \( \sigma : \Sigma \rightarrow \Sigma \) by \( \sigma(x) = y = (y_i)_{i \in \mathbb{Z}} \) and \( y_i = x_{i+1} \) for all \( i \in \mathbb{Z} \), where \( x = (x_i)_{i \in \mathbb{Z}} \in \Sigma \). A metric on \( \Sigma \) is defined by \( d(x, y) = 2^{-n} \) if \( n \) is the largest natural number with \( x_i = y_i \) for \( |i| < n \), and \( d(x, y) = 1 \) if \( x_0 \neq y_0 \) for \( x, y \in \Sigma \). Let \( A = (a_{ij})_{i,j=0}^{k-1} \) be a \( k \times k \)-matrix whose entries \( a_{ij} \) are either 0 or 1, and set \( S_A = \{ x \in \Sigma : a_{x_{i+1}x_{i+1}} = 1 \text{ for all } i \in \mathbb{Z} \} \). Clearly \( \sigma(S_A) = S_A \). The restriction \( \sigma_A = \sigma|_{S_A} \) is called a subshift of finite type (see [6]). It is well known that \( \sigma_A : S_A \rightarrow S_A \) is expansive and has the shadowing property.

We see that \( \sigma : \Sigma \rightarrow \Sigma \) is Lipschitz. For every \( x = (x_i)_{i \in \mathbb{Z}}, \ y = (y_i)_{i \in \mathbb{Z}} \in \Sigma \), if \( d(x, y) = 1/2^n \) for some \( n > 0 \), then \( x_i = y_i \) for all \( |i| < n \). Thus \( d(\sigma(x), \sigma(y)) = d([(x_{i+1})_{i \in \mathbb{Z}}, (y_{i+1})_{i \in \mathbb{Z}}]) = 1/2^{n-1} \) so that \( d(\sigma(x), \sigma(y)) = 2d(x, y) \). If \( d(x, y) = 1 \), then \( x_0 \neq y_0 \). Thus \( d(\sigma(x), \sigma(y)) \leq 1 = d(x, y) \).

If \( (S_A, \sigma_A) \) is a subshift of finite type, then \( \sigma_A \) is \( \mathcal{C} \)-hyperbolic. Actually, for every \( x, y \in S_A \), if \( d(x, y) < 1/2 \), then there exists \( N > 1 \) such that \( 1/2^{N+1} \leq d(x, y) < 1/2^N \). Since \( d(x, y) < 1/2^N \) \( (x = (x_i)_{i \in \mathbb{Z}}, y = (y_i)_{i \in \mathbb{Z}} \in S_A) \), then \( x_i = y_i \) for \( -N \leq i \leq N \).

Put
\[
r(x, y) = \{ x_{-N-1}, x_{-N}, \ldots, x_0, \ldots, x_N, y_{N+1}, y_{N+2}, \ldots \}.
\]
Then \( r(x, y) \in W^u_{1/2^N}(x, d) \cap W^s_{1/2^N}(y, d) \cap S_A \). Since \( 1/2^{N+1} \leq d(x, y) \), we see that
\[
d(r(x, y), x) < 2d(x, y) \quad \text{and} \quad d(r(x, y), y) < 2d(x, y).
\]
Moreover, it can be easily checked that for all \( x \in S_A \) and \( j \geq 0 \),
\[
\begin{align*}
&d(\sigma_A^j(x), \sigma_A^j(y)) \leq 2^{-j} d(x, y) \quad \text{if } y \in W^u_{1/2}(x, d), \\
&d(\sigma_A^{-j}(x), \sigma_A^{-j}(y)) \leq 2^{-j} d(x, y) \quad \text{if } y \in W^s_{1/2}(x, d).
\end{align*}
\]

**Example 4.3.** Let \( M^2 \) be a closed surface and let \( f : M^2 \rightarrow M^2 \) be an expansive homeomorphism having the shadowing property. Then, it is proved in [8] that \( M^2 \) is a 2-torus, \( T^2 \), and there are a hyperbolic linear automorphism \( g : T^2 \rightarrow T^2 \) and a homeomorphism \( h : T^2 \rightarrow T^2 \) such that \( h \circ f = g \circ h \). Note that \( g \) is an Anosov diffeomorphism so that it is \( \mathcal{C} \)-hyperbolic with respect to a usual metric \( d \) on \( T^2 \).

Define a metric \( D \) on \( T^2 \) by a formula \( D(x, y) = d(h(x), h(y)) \) for \( x, y \in T^2 \). Then it is easy to see that \( f \) is \( \mathcal{C} \)-hyperbolic with respect to \( D \).

**References**


