The second-order derivatives of matrices of eigenvalues and eigenvectors with an application to generalized $F$-statistic

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Abstract

In this paper we derive the second-order derivatives of an orthogonal matrix of eigenvectors and of a matrix of eigenvalues of a real symmetric matrix. Obtained expressions depend on the first-order derivatives of these matrices, which were presented in Linear Algebra Appl. 264 (1997) 489. These results we use to find the main term of the bias of a generalized $F$-statistic [Biometrical J. 38 (1996) 5] in the case of normal population. A simulation experiment is carried out, in which we compare the sample mean and unbiased estimator of $F$-statistic with its asymptotic mean, which was obtained in Linear Algebra Appl. 321 (2000) 27.

Keywords: Eigenvalues of symmetric matrix; Eigenvectors of symmetric matrix; Generalized $F$-statistics; Matrix derivative; Second-order matrix derivative

1. Introduction

In this paper we consider a generalized $F$-statistic proposed by Läuter [5], to test the hypothesis about the population mean vector $\mu$:

$H_0 : \mu = 0,$

against

$H_1 : \mu \neq 0.$

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Läuter et al. [6] showed that under the null-hypothesis this statistic is $F$-distributed if the population distribution is spherical. Later Läuter et al. [7] showed that the idea can be generalized to a wider class of similar statistics. In a special case $F$-statistic becomes Hotelling’s $T^2$-statistic. The exact non-null distributions of $T^2$ and Läuter’s generalized $F$-statistic have been examined under assumption of normality by Fang et al. [1]. They derived general formulae for the density of the $F$-statistic through zonal polynomials. However, the formulae are very complicated and hard to use. Fang et al. [2] derived simpler asymptotic normal distribution of the $F$-statistic for elliptical and also for the normal population. It comes out that the asymptotic mean of the $F$-statistic is biased. In this paper we examine the bias for the normal population using the Taylor expansion. In Section 2 the necessary matrix derivatives of matrices of eigenvalues and eigenvectors are derived. In Section 3 the main term of the bias is given in Theorem 2. In Section 4 a simulation experiment is described and the results of the simulation study are presented.

2. The second-order derivatives of matrices of eigenvalues and eigenvectors

Let us consider a real symmetric $p \times p$ matrix $M$ with eigenvalues $d_1 > d_2 > \cdots > d_p > 0$ and associated normalized eigenvectors $w_i$ ($i = 1, 2, \ldots, p$). Then

\begin{align*}
MW &= WD; \\
W'W &= I_p,
\end{align*}

where $D$ is the diagonal matrix of eigenvalues and $W = (w_1, \ldots, w_p)$.

Let us then consider a real symmetric random matrix $\hat{M}$ with eigenvalues $\hat{d}_i$ and normalized eigenvectors $\hat{w}_i$-estimators of $d_i$ and $w_i$, respectively. In matrix form we have equalities:

\begin{align*}
\hat{M}\hat{W} &= \hat{W}\hat{D}; \\
\hat{W}'\hat{W} &= I_p.
\end{align*}

From this we have

\[ \hat{D} = \hat{W}'\hat{M}\hat{W}. \]

In the following we are going to differentiate matrix expressions.

The matrix derivative $\frac{dY}{dX}$ of matrix $Y: r \times s$ by matrix $X: p \times q$ is $rs \times pq$-matrix [8]:

\[ \frac{dY}{dX} = \left( \left( \frac{\hat{\partial}}{\hat{\partial}x_{ij}} \right) \otimes \text{vec}Y \right), \]

where

\[ \frac{\hat{\partial}}{\hat{\partial}x_{ij}} = \left( \frac{\hat{\partial}}{\hat{\partial}x_{11}}, \cdots, \frac{\hat{\partial}}{\hat{\partial}x_{p1}}, \frac{\hat{\partial}}{\hat{\partial}x_{12}}, \cdots, \frac{\hat{\partial}}{\hat{\partial}x_{p2}}, \cdots, \frac{\hat{\partial}}{\hat{\partial}x_{1q}}, \cdots, \frac{\hat{\partial}}{\hat{\partial}x_{pq}} \right). \]
Higher order derivatives are defined recursively:
\[
\frac{d^k Y}{dX^k} = \frac{d}{dX} \frac{d^{k-1} Y}{dX^{k-1}}.
\]
For properties of the matrix derivative an interested reader is referred to Magnus and Neudecker [8], for example.

The first-order derivative \( \frac{d \hat{D}}{d \hat{M}} \) is a \( p^2 \times p^2 \)-matrix ([3] or [4]):
\[
\frac{d \hat{D}}{d \hat{M}} = (K_{p,p})_d (\hat{W}' \otimes \hat{W}'),
\]
where the commutation matrix
\[
(K_{p,q})_{(i,j)(g,h)} = \begin{cases} 
1, & g = j \text{ and } h = i, \\
0, & \text{otherwise}.
\end{cases}
\]
The diagonal elements of \( (K_{p,p})_d \) are given by \( K_{p,p} \) and the other equal 0.

The second-order derivative \( \frac{d^2 \hat{D}}{d \hat{M}^2} \) is a \( p^4 \times p^2 \)-matrix:
\[
\frac{d^2 \hat{D}}{d \hat{M}^2} = d \left( (K_{p,p})_d (\hat{W}' \otimes \hat{W}') \right) \frac{d \hat{W}'}{d \hat{M}} = (I_{p^2} \otimes (K_{p,p})_d (I_p \otimes K_{p,p} \otimes I_p)(I_{p^2} \otimes \text{vec}\hat{W}' + \text{vec}\hat{W}' \otimes I_{p^2}) \frac{d \hat{W}'}{d \hat{M}}.
\]
where Kollo and Neudecker [4] have shown
\[
\frac{d \hat{W}'}{d \hat{M}} = (I_p \otimes W)(\hat{D} \otimes I_p - I_p \otimes \hat{D}^+) (\hat{W}' \otimes \hat{W}').
\]
(1)

Here \( A^+ \) denotes the Moore–Penrose inverse of \( A \).

The second-order derivative of the matrix of eigenvectors we get from the following chain of equalities:
\[
\frac{d^2 \hat{W}}{d \hat{M}^2} = d ((I_p \otimes \hat{W})(\hat{D} \otimes I_p - I_p \otimes \hat{D}^+) (\hat{W}' \otimes \hat{W}')) \frac{d \hat{W}'}{d \hat{M}} = (\hat{W} \otimes \hat{W} \otimes I_{p^2}) \frac{d ((I_p \otimes \hat{W})(\hat{D} \otimes I_p - I_p \otimes \hat{D}^+) \hat{W}')}{d \hat{M}} + (I_{p^2} \otimes (I_p \otimes \hat{W})(\hat{D} \otimes I_p - I_p \otimes \hat{D}^+) \frac{d (\hat{W}' \otimes \hat{W}')}{d \hat{M}}.
\[ \begin{align*} &= (\hat{W} \otimes \hat{W} \otimes I_p^2)
         \left( \left( \hat{D} \otimes I_p - I_p \otimes \hat{D} \right)^+ \otimes I_p^2 \right) \frac{d(I_p \otimes \hat{W})}{d\hat{M}} \\
         &\quad + (I_p^2 \otimes I_p \otimes \hat{W}) \frac{d((\hat{D} \otimes I_p - I_p \otimes \hat{D})^+)}{d\hat{M}} \\
         &\quad + (I_p^2 \otimes (I_p \otimes \hat{W})(\hat{D} \otimes I_p - I_p \otimes \hat{D})^+) \frac{d(\hat{W}' \otimes \hat{W}')}{d\hat{M}} \\
   &= (\hat{W} \otimes \hat{W} \otimes I_p^2)
         \left( \left( \hat{D} \otimes I_p - I_p \otimes \hat{D} \right)^+ \otimes I_p^2 \right)(I_p \otimes K_{p,p} \otimes I_p) \\
         &\quad \times (\text{vec}(I_p \otimes I_p)^2) \frac{d\hat{W}}{d\hat{M}} + (I_p^2 \otimes I_p \otimes \hat{W}) \frac{d((\hat{D} \otimes I_p - I_p \otimes \hat{D})^+)}{d\hat{M}} \\
         &\quad + (I_p^2 \otimes (I_p \otimes \hat{W})(\hat{D} \otimes I_p - I_p \otimes \hat{D})^+) \frac{d(\hat{W}' \otimes \hat{W}')}{d\hat{M}}, \end{align*} \]

where
\[ \frac{d(\hat{W}' \otimes \hat{W}')} {d\hat{M}} = (I_p \otimes K_{p,p} \otimes I_p)(I_p^4 + K_{p^2,p^2})(I_p^2 \otimes \text{vec}(\hat{W}')) \frac{d\hat{W}}{d\hat{M}} \] (2)

and
\[ \frac{d((\hat{D} \otimes I_p - I_p \otimes \hat{D})^+)} {d\hat{M}} = -((\hat{D} \otimes I_p - I_p \otimes \hat{D})^+ \otimes (\hat{D} \otimes I_p - I_p \otimes \hat{D})^+) \]
\[ \times \frac{d((\hat{D} \otimes I_p - I_p \otimes \hat{D})^+)} {d\hat{M}}. \] (3)

Comparing (3) with the expression of the differential of general Moore–Penrose inverse in Theorem 5 in Magnus and Neudecker [8, p. 154], our equality is simpler because of diagonal structure of the matrices in (3). Finally
\[ \frac{d(\hat{D} \otimes I_p - I_p \otimes \hat{D})}{d\hat{M}} = (I_p \otimes K_{p,p} \otimes I_p)(I_p^4 - K_{p^2,p^2})(I_p^2 \otimes \text{vec}(I_p)) \frac{d\hat{D}}{d\hat{M}}. \]

The next theorem presents the obtained results.

**Theorem 1.** Let \( \hat{M} \) be a real symmetric matrix with the diagonal eigenvalue matrix \( \hat{D} \) and orthogonal matrix of eigenvectors \( \hat{W} \). Then the second-order derivatives of the matrices of eigenvalues and eigenvectors are
\[ \begin{align*} \frac{d^2 \hat{D}} {d\hat{M}^2} &= (I_p^2 \otimes (K_{p,p})_d)(I_p \otimes K_{p,p} \otimes I_p)(I_p^4 + K_{p^2,p^2}) \\
   &\quad \times (I_p^2 \otimes \text{vec}(\hat{W}')) K_{p,p} \frac{d\hat{W}}{d\hat{M}}. \end{align*} \]
and
\[ \frac{d^2 \hat{W}}{dM^2} = (\hat{W} \otimes \hat{W} \otimes I_p^2) \left( ((\hat{D} \otimes I_p - I_p \otimes \hat{D})^+ \otimes I_p^2)(I_p \otimes K_{p,p} \otimes I_p) \right. 
\times (\text{vec} I_p \otimes I_p^2) \frac{d \hat{W}}{dM} + (I_p^2 \otimes I_p \otimes \hat{W}) \frac{d((\hat{D} \otimes I_p - I_p \otimes \hat{D})^+)}{dM} \right) 
+ (I_p^2 \otimes (I_p \otimes \hat{W})(\hat{D} \otimes I_p - I_p \otimes \hat{D})^+) \frac{d(\hat{W}' \otimes \hat{W}'^\prime)}{dM}, \]
where \( \frac{d \hat{W}}{dM} \), \( \frac{d((\hat{D} \otimes I_p - I_p \otimes \hat{D})^+)}{dM} \) and \( \frac{d(\hat{W}' \otimes \hat{W}'^\prime)}{dM} \) are given by equalities (1), (2) and (3), respectively.

3. Generalized F-statistics

Let us consider a classical multivariate testing problem about the population mean vector \( \mu \):
\[ H_0 : \mu = 0, \]
against
\[ H_1 : \mu \neq 0. \]
As a test-statistic for this hypothesis Hotelling’s \( T^2 \)-statistic may be used:
\[ T^2 = n \bar{x}' S^{-1} \bar{x}, \]
where \( n \) is the sample size, \( \bar{x} \) and \( S \) are the sample mean and covariance matrix, respectively. If the population is normally distributed, then \( T^2 \sim F_{p,n-p} \) under the null-hypothesis. But it is a known fact that if the dimensionality \( p \) of the population distribution is close to the sample size \( n \), the performance of the \( T^2 \)-statistic is not good. To overcome the problem, a new test has been proposed by Läuter [5]. In this paper we consider a generalized F-statistic, proposed by him. The construction of this test-statistic is based on a weight matrix \( Q : q \times p \), which is assumed to be a unique function of the matrix \( XX' \), where \( X : p \times n \) is the matrix of i.i.d. observations. The data \( X \) will be transformed into the matrix \( Z : q \times n \) by \( Q : q \times p \):
\[ Z = QX. \]
For fixed \( Q \) assume that \( P(\text{rank}(XX') = p) = 1 \) and \( 1 \leq q \leq \min(p,n-1) \).
Define matrices
\[ H_Z = \frac{1}{n} I_n 1_n' Z', \]
\[ G_Z = Z \left( I_n - \frac{1}{n} 1_n 1_n' \right) Z'. \]
where $1_n$ denotes the column $n$-vector consisting of ones. Remark that we may write
\[ H_Z = n Q \bar{x} \bar{x}' Q', \]
\[ G_Z = QAQ', \]
where
\[ \bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i = \frac{1}{n} X1_n, \]
\[ A = (n - 1)S = \sum_{i=1}^{n} (x_i - \bar{x})(x_i - \bar{x})'. \]
Hence the matrices $H_Z$ and $G_Z$ are functions of $\bar{x}$ and $S$ only.

Läuter [5] proposed the generalized $F$-test statistic in the following form:
\[ F = \frac{n - q}{q} \text{tr}(H_Z G_Z^{-1}) \]
and Läuter et al. [6] have shown that under the null-hypothesis this $F$-statistic has exact $F$-distribution with $q$ and $n - q$ degrees of freedom for any transformation matrix $Q$ which is a unique function of $XX'$ and for which $P(\text{rank}(Q) = q) = 1$ holds. There can be several choices of $Q$. We use the idea of the principal component method for defining $Q$.

Let $Q' = W_q = (w_1, \ldots, w_q)$, where $w_i$ is the unit-length eigenvector of $XX'$, which corresponds to the eigenvalue $\lambda_i$ of $XX'$ and let $\lambda_1 > \lambda_2 > \cdots > \lambda_q$. To define the eigenvectors uniquely, we have to fix their directions: let the first coordinates be positive, i.e., the first column of $Q$ consists of positive elements. Fang et al. [1] have shown that $F$-statistic can be presented in the form:
\[ F = \frac{n - q}{q} \frac{\bar{x}'W_q A_q^{-1}W_q'\bar{x}}{1 - n\bar{x}'W_q A_q^{-1}W_q'\bar{x}}, \]
where the diagonal matrix $A_q$ has eigenvalues $(\lambda_1, \ldots, \lambda_q)$ on its main diagonal.

When $q = p$, this statistic $F$ differs from the Hotelling’s $T^2$-statistic just by a constant:
\[ F = \frac{n - p}{p} \bar{x}' S^{-1} \bar{x} = \frac{n - p}{np} T^2. \]

Through eigenvalues and eigenvectors of $\frac{1}{n} XX'$ the $F$-statistic can be presented as
\[ F = \frac{n - q}{q} \frac{\bar{x}'W_q A_q^{-1}W_q'\bar{x}}{1 - n\bar{x}'W_q A_q^{-1}W_q'\bar{x}}, \tag{4} \]
where the diagonal matrix $A_q$ is formed by the largest $q$ eigenvalues $\delta_i$ of $\frac{1}{n} XX'$ ($\delta_i = \frac{\lambda_i}{n}$).
Fang et al. [2] have shown that under assumption of ellipticity of the population the asymptotic non-null distribution of the statistic \( \frac{q_n}{n-q} F \) is the normal distribution with the mean

\[
\mu' W_q (A_0^q)^{-1} W_0' \mu / 1 - \mu' W_0^q (A_0^q)^{-1} W_0' \mu,
\]

where \( A_0^q \) is the diagonal matrix of the \( q \) largest eigenvalues of \( M_2 = \Sigma + \mu \mu' \) and \( W_0^q \) is the \( p \times q \) matrix of corresponding unit-length eigenvectors of \( M_2 \).

From the simulation experiments, described in the same paper, follows, that the asymptotic mean has a bias. In this paper we find the expression of the main term of the bias for the normal population.

We are interested in the difference

\[
E \left( \frac{q_n}{n-q} F - \frac{\mu' W_q (A_0^q)^{-1} W_0' \mu / 1 - \mu' W_0^q (A_0^q)^{-1} W_0' \mu}{1 - \mu' W_q A_q^{-1} W_q' \bar{x}} \right)
\]

The expectation in the last equality was approximated in two ways. Firstly the Taylor expansion was found:

\[
E \left( \frac{1}{1 - \bar{x}' W_q A_q^{-1} W_q' \bar{x}} \right)
\]

Unfortunately the estimated bias by this formula appeared to be very small in simulation experiments. Much better approximation was obtained in the following way:
The formula of the main term of the bias will be presented for the normal population $N(\mu, \Sigma)$. For finding expectation $E(\bar{x}'W_q^{-1}W_q'\bar{x})$ we are using an approximation once again:

$$E(\bar{x}'W_q^{-1}W_q'\bar{x}) \approx E(\bar{x}' \otimes \bar{x})E(\text{vec}(W_q^{-1}W_q')).$$

It is known ([2], for example) that

$$E(\bar{x}' \otimes \bar{x}) = \mu' \otimes \mu' + \frac{1}{n} \text{vec}^\top \Sigma.$$

Let us find $E(\text{vec}(W_q^{-1}W_q'))$. We use the Taylor expansion of $\text{vec}(W_q^{-1}W_q')$:

$$\text{vec}(W_q^{-1}W_q') = \text{vec}(W_0^q (A_0^q)^{-1} W_0^q') + \frac{1}{2} E \left\{ \left( \text{vec} \left( \frac{1}{n} XX' - M_2 \right) \right)' \otimes I_p^2 \right\} \times \text{vec} \left( \frac{1}{n} XX' - M_2 \right) + \cdots$$

Taking expectation from both sides gives us

$$E(\text{vec}(W_q^{-1}W_q')) = \text{vec}(W_0^q (A_0^q)^{-1} W_0^q') + \frac{1}{2} E \left\{ \left( \text{vec} \left( \frac{1}{n} XX' - M_2 \right) \right)' \otimes I_p^2 \right\} \times \frac{d^2(W_q^{-1}W_q')}{d(\frac{1}{n} XX')^2} \bigg|_{\frac{1}{n} XX' = M_2} \times \text{vec} \left( \frac{1}{n} XX' - M_2 \right) + \cdots$$

The second term on the right-hand side can be transformed

$$E \left\{ \left( \text{vec} \left( \frac{1}{n} XX' - M_2 \right) \right)' \otimes I_p^2 \right\} \frac{d^2(W_q^{-1}W_q')}{d(\frac{1}{n} XX')^2} \bigg|_{\frac{1}{n} XX' = M_2} \times \text{vec} \left( \frac{1}{n} XX' - M_2 \right)$$
Let us find the second-order derivative \( \frac{d^2 (W_q A_q^{-1} W_q')}{d\left(\frac{1}{n} XX'\right)^2} \mid_{\frac{1}{n} XX' = M_2} \).

We need the first-order derivative:

\[
\frac{d(W_q A_q^{-1} W_q')}{d\left(\frac{1}{n} XX'\right)} = (I_{p^2} + K_{p,p})(W_q A_q^{-1} \otimes I_p) \frac{dW_q}{d\left(\frac{1}{n} XX'\right)}
\]

\[
+ (W_q \otimes W_q) \frac{dA_q^{-1}}{d\left(\frac{1}{n} XX'\right)}.
\]

It depends on the first-order derivatives of the matrices of eigenvectors

\[
\frac{dW_q}{d\left(\frac{1}{n} XX'\right)} = (I_q \otimes W)(A_q \otimes I_p - I_q \otimes A)^{+} (W_q \otimes W)',
\]

and eigenvalues

\[
\frac{dA_q^{-1}}{d\left(\frac{1}{n} XX'\right)} = -(K_{q,q}d(A_q^{-1} \otimes A_q^{-1})(W_q' \otimes W_q').
\]

The second-order derivative

\[
\frac{d^2 (W_q A_q^{-1} W_q')}{d\left(\frac{1}{n} XX'\right)^2} = (I_{p^2} \otimes (I_{p^2} + K_{p,p})) \left( \left( \frac{dW_q}{d\left(\frac{1}{n} XX'\right)} \right)' \otimes I_{p^2} \right)
\]

\[
\times \frac{d(W_q A_q^{-1} \otimes I_p)}{d\left(\frac{1}{n} XX'\right)}
\]

\[
+ (I_{p^2} \otimes W_q A_q^{-1} \otimes I_p) \frac{d^2W_q}{d\left(\frac{1}{n} XX'\right)^2}
\]

\[
+ \left( \left( \frac{dA_q^{-1}}{d\left(\frac{1}{n} XX'\right)} \right)' \otimes I_{p^2} \right) \frac{d(W_q \otimes W_q)}{d\left(\frac{1}{n} XX'\right)}
\]

\[
+ (I_{p^2} \otimes W_q \otimes W_q) \frac{d^2A_q^{-1}}{d\left(\frac{1}{n} XX'\right)^2}.
\]
The obtained equality (5) includes the second-order derivatives of the matrices of eigenvalues

\[
\frac{d^2 A_q^{-1}}{d(\frac{1}{n}XX')^2} = - (I_q \otimes (K_{q,q})_d)(I_p \otimes K_{q,p} \otimes I_q) \left( I_{pq} \otimes \text{vec}(A_q^{-1}W'_q) \right)
+ \text{vec}(A_q^{-1}W'_q) \otimes I_{pq} \left( \frac{dA_q^{-1}W'_q}{d(\frac{1}{n}XX')} \right)
= - (I_q \otimes (K_{q,q})_d)(I_p \otimes K_{q,p} \otimes I_q)(I_{pq}^{-1} + K_{pq,pq})
\times (I_{pq} \otimes \text{vec}(A_q^{-1}W'_q))
\times \left( (W_q \otimes I_q) \frac{dA_q^{-1}}{d(\frac{1}{n}XX')} + (I_p \otimes A_q^{-1})K_{q,p} \frac{dW_q}{d(\frac{1}{n}XX')} \right)
\] (6)

and eigenvectors

\[
\frac{d^2 W_q}{d(\frac{1}{n}XX')^2} = (W_q \otimes W \otimes I_{pq}) \left( (A_q \otimes I_p - I_q \otimes A)^+ \otimes I_{pq} \right)
\times (I_q \otimes K_{q,p} \otimes I_p)(\text{vec}I_q \otimes I_{pq}) \frac{dW}{d(\frac{1}{n}XX')} \left( (I_q \otimes I_p - I_q \otimes A)^+ \right)
+ (I_{pq} \otimes I_q \otimes W) \frac{d((A_q \otimes I_p - I_q \otimes A)^+)}{d(\frac{1}{n}XX')} \times (I_p \otimes (I_q \otimes W)(A_q \otimes I_p - I_q \otimes A)^+) \frac{d(W_q \otimes W'_q)}{d(\frac{1}{n}XX')}.
\] (7)

where

\[
\frac{d((A_q \otimes I_p - I_q \otimes A)^+)}{d(\frac{1}{n}XX')} = - \left( A_q \otimes I_p - I_q \otimes A \right)^+
\otimes (A_q \otimes I_p - I_q \otimes A)^+
\times \frac{d(A_q \otimes I_p - I_q \otimes A)}{d(\frac{1}{n}XX')},
\] (8)

where

\[
\frac{d(A_q \otimes I_p - I_q \otimes A)}{d(\frac{1}{n}XX')} = (I_q \otimes K_{q,p} \otimes I_p) \left( I_{pq} \otimes \text{vec}I_p \right) \frac{dA_q}{d(\frac{1}{n}XX')}
- (\text{vec}I_q \otimes I_{pq}^2) \frac{dA}{d(\frac{1}{n}XX')}.
\] (9)
Now all terms for the expression of the bias have been found.

The next theorem summarizes the obtained results.

**Theorem 2.** Let the statistic $F$ be defined by (4). Then the expression of the main term of the bias of the statistic $q_n - q F$ for the normal population $N(\mu, \Sigma)$ equals

$$\frac{1}{1 - E \left( \bar{x}' W_q^{-1} W_q' \bar{x} \right)} - 1 - \frac{\mu' W_q^q (A_0^q)^{-1} W_q'^q \mu}{1 - \mu' W_q^q (A_0^q)^{-1} W_q'^q \mu},$$

where

$$E(\bar{x}' W_q^{-1} W_q' \bar{x}) = \left( \mu' \otimes \mu' + \frac{1}{n} \text{vec} \Sigma \right) \times \left[ \text{vec} \left( W_q^q (A_0^q)^{-1} W_q'^q \right) \right] + \frac{1}{2n} \left( \text{vec} (M_4 - \text{vec} M_2 (\text{vec} M_2')^\prime) \otimes I_p \right) \times \text{vec} \left. \frac{d^2(W_q A_q^{-1} W_q')}{d \left( \frac{1}{n} XX' \right)^2} \right|_{XX' = M_2},$$

and where $d^2(W_q A_q^{-1} W_q')$ is given by equalities (5)–(9).

### 4. Simulation

In the simulation experiment we compared the sample mean and the bias-corrected estimates of $F$-statistic with its asymptotic mean. The simulated random vector was normal, with mean vector $\mu$ and covariance matrix $\Sigma$, $x \sim N(\mu, \Sigma)$. The components of vector $x$ were assumed to be independent, so $\Sigma$ was a diagonal matrix. The parameters of the experiment were the following:

- $p$ – the dimension of the vector $x$,
- $q$ – the number of eigenvalues and eigenvectors, used for calculating $F$-statistic,
- $n$ – the sample size,
- the number of replications, i.e. the number of simulated values of $F$-statistic, was 200.

The results in Table 1 were received using parameter values $p = 4, q = 2, \mu_i = 0.332, i = 1, \ldots, 4, \text{diag}(\Sigma) = (17, 13, 9, 5)$.

The results in Table 2 were received using parameter values $p = 6, q = 2, \mu_i = 0.387, i = 1, \ldots, 6, \text{diag}(\Sigma) = (25, 21, 17, 13, 9, 5)$. 
The results in Table 3 were received using parameter values $p = 6$, $q = 4$, $\mu_i = 0.387$, $i = 1, \ldots, 6$, $\text{diag}(\Sigma) = (25, 21, 17, 13, 9, 5)$.

As we see, the asymptotic mean is biased and the bias-corrected estimator generally works better. From the tables one can see, that for a small sample size ($n = 20$) asymptotics does not work well and the correction term is too big. For sample sizes 50 and larger, the bias-corrected estimator is a good approximation to the asymptotic mean.
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References