On the asymptotic behavior of solutions of a nonlinear difference-differential equation

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Abstract

Sufficient conditions are established on the asymptotic behavior of solutions of the nonlinear delay differential equation

\[ x'(t) + p(t)f(x(t - \tau)) = 0, \quad t \geq 0, \]

where \( \tau \in (0, \infty) \), \( p \in C([0, \infty), [0, \infty)) \), \( f \in C(\mathbb{R}, \mathbb{R}) \) and \( xf(x) > 0 \) if \( x \neq 0 \). Some applications are given.

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Consider the nonlinear delay differential equation

\[ x'(t) + p(t)f(x(t - \tau)) = 0, \quad t \geq 0, \] (1)

where \( \tau \in (0, +\infty) \), \( p \in C([0, \infty), [0, \infty)) \), \( f \in C(\mathbb{R}, \mathbb{R}) \) and \( xf(x) > 0 \) if \( x \neq 0 \).

For the corresponding linear equation

\[ x'(t) + p(t)x(t - \tau) = 0, \quad t \geq 0, \] (2)
by constructing an effective Liapunov functional, Gopalsamy [1] and Shen and Yu [2] proved that if
\[ \int_0^\infty p(s)ds = \infty \] (3)
and
\[ \limsup_{t \to \infty} \int_{t-\tau}^{t+\tau} p(s+\tau)ds < 2, \] (4)
then every solution of Eq. (2) tends to zero as \( t \to \infty \).

Employing a non-Liapunov approach, Yoneyama [3] and So et al. [4] showed that
\[ \limsup_{t \to \infty} \int_{t-\tau}^{t+\tau} p(s)ds < \frac{3}{2}, \] (5)
together with (3), implies that every solution of Eq. (2) tends to zero as \( t \to \infty \). Note that conditions (4) and (5) are independent as illustrated by examples at the end of the paper.

Recently, the nonlinear Eq. (1) is considered under some restriction on \( f \) and the following result is obtained [4,5].

**Theorem 1.** Assume that (3) and (5) hold, and that
\[ |f(x)| \leq |x|. \] (6)
Then every solution of Eq. (1) tends to zero as \( t \to \infty \).

It is also shown in [5] that the upper bound \( 3/2 \) in (5) is the best possible by giving an example. Motivated by the work in [4,5], in this paper, we try to extend (4) to the nonlinear Eq. (1) under the restriction condition (6). And, we will apply the obtained result to the equation
\[ y'(t) + p(t)[a - y(t)][b + y(t)]y(t - \tau) = 0, \quad t \geq 0, \] (7)
and deduce a new result which is different from those in [5,6], where \( a, b, \tau \in (0, \infty) \), \( p \) is the same as in (1).

First, we establish the following theorem.

**Theorem 2.** Assume that (3), (4) and (6) hold. Then every solution of Eq. (1) tends to zero as \( t \to \infty \).

**Proof.** Let \( 0 < \lambda < 2 \) and \( T > 0 \) such that
\[ \int_{t-\tau}^{t+\tau} p(s+\tau)ds \leq \lambda, \quad t \geq T. \] (8)
Let \( x(t) \) be a solution of (1) and define
\[ V_1(t) = \left[ x(t) - \int_{t-\tau}^{t} p(s+\tau)f(x(s))ds \right]^2 \]
and
\[ V_2(t) = \int_{t-\tau}^{t} p(s+2\tau) \int_{s}^{t} p(u+\tau)f^2(x(u))du ds. \]
Then
\[ V'(t) = -2 \left[ x(t) - \int_{t-\tau}^{t} p(s+\tau)f(x(s))ds \right] p(t+\tau)f(x(t)) \]
\[
\begin{align*}
&\leq -p(t + \tau) \left[ 2x(t) f(x(t)) - f^2(x(t)) \int_{t-\tau}^{t} p(s + \tau)ds \\
&\qquad - \int_{t-\tau}^{t} p(s + \tau) f^2(x(s))ds \right]
\end{align*}
\]

and
\[
\begin{align*}
V'(t) &= -p(t + \tau) \int_{t-\tau}^{t} p(s + \tau) f^2(x(s))ds \\
&\quad + p(t + \tau) f^2(x(t)) \int_{t-\tau}^{t} p(s + 2\tau)ds.
\end{align*}
\]

Set \(V(t) = V_1(t) + V_2(t)\). Using (6) and the condition \(xf(x) \geq 0\), one can obtain the following estimate:
\[
\begin{align*}
V'(t) &\leq -p(t + \tau) \left[ 2x(t) f(x(t)) - f^2(x(t)) \int_{t-\tau}^{t} p(s + \tau)ds \\
&\qquad - f^2(x(t)) \int_{t-\tau}^{t} p(s + 2\tau)ds \right] \\
&\quad - \int_{t-\tau}^{t} p(s + \tau) f^2(x(s))ds \\
&\quad + f^2(x(t)) \int_{t-\tau}^{t} p(s + \tau)ds - f^2(x(t)) \int_{t-\tau}^{t} p(s + 2\tau)ds \\
&\quad \leq -p(t + \tau) f^2(x(t)) \left[ 2 - \int_{t-\tau}^{t} p(s + \tau)ds - \int_{t-\tau}^{t} p(s + 2\tau)ds \right] \\
&\quad \leq -p(t + \tau) f^2(x(t)) \left[ 2 - \int_{t-\tau}^{t+\tau} p(s + \tau)ds \right] \\
&\quad \leq -(2 - \lambda) p(t + \tau) f^2(x(t)).
\end{align*}
\]

This shows that \(V(t)\) is nonincreasing, and so the limit \(v = \lim_{t \to \infty} V(t)\) exists and \(v \geq 0\). Integrating the above, we have
\[
\int_{t}^{\infty} p(s + \tau) f^2(x(s))ds \leq \frac{V(T)}{2 - \lambda}. \tag{9}
\]

By this and (3), we obtain
\[
\lim_{t \to \infty} \int_{t-\tau}^{t} p(s + \tau) f^2(x(s))ds = 0 \tag{10}
\]

and
\[
\lim_{t \to \infty} f^2(x(t)) = 0. \tag{11}
\]

Note that
\[
\begin{align*}
V_1(t) &= \left[ x(t) - \int_{t-\tau}^{t} p(s + \tau) f(x(s))ds \right]^2 \\
&\geq x^2(t) - 2x(t) \int_{t-\tau}^{t} p(s + \tau) f(x(s))ds \\
&\geq x^2(t) - \int_{t-\tau}^{t} p(s + \tau) \left[ \frac{1}{4} x^2(t) + 4 f^2(x(s)) \right]ds
\end{align*}
\]
\[
V(t) \geq \frac{1}{2}x^2(t) - 4 \int_{t-\tau}^t p(s + \tau) f^2(x(s))ds.
\]

It follows that
\[
V(t) \geq \frac{1}{2}x^2(t) - 4 \int_{t-\tau}^t p(s + \tau) f^2(x(s))ds.
\]

Take the superior limit in the above as \( t \to \infty \), we have
\[
\limsup_{t \to \infty} |x(t)| \leq \sqrt{2v} < \infty. \tag{12}
\]

Combining (11) and (12) and using the fact that \( f \) is continuous and \( xf(x) > 0 \) for \( x \neq 0 \), it follows that
\[
\liminf_{t \to \infty} x^2(t) = 0. \tag{13}
\]

From (8) and (10), we see that
\[
0 \leq \lim_{t \to \infty} V_2(t) \leq \lim_{t \to \infty} \int_{t-\tau}^t p(s + 2\tau)ds \int_{t-\tau}^t p(u + \tau) f^2(x(u))du = 0,
\]
which implies that \( \lim_{t \to \infty} V_2(t) = 0 \). Note that
\[
V_1(t) = \left[ x(t) - \int_{t-\tau}^t p(s + \tau) f(x(s))ds \right]^2
\leq 2x^2(t) + 2 \left( \int_{t-\tau}^t p(s + \tau) f(x(s))ds \right)^2
\leq 2x^2(t) + 2 \int_{t-\tau}^t p(s + \tau)ds \int_{t-\tau}^t p(s + \tau) f^2(x(s))ds
\leq 2x^2(t) + 4 \int_{t-\tau}^t p(s + \tau) f^2(x(s))ds.
\]

It follows that
\[
v = \lim_{t \to \infty} V(t) = \lim_{t \to \infty} V_1(t) \leq 2 \liminf_{t \to \infty} x^2(t) = 0.
\]

On the other hand, from (12), we have
\[
\limsup_{t \to \infty} x^2(t) \leq 2v = 0.
\]

Thus \( \lim_{t \to \infty} x^2(t) = 0 \). The proof is complete. \( \square \)

Next, we apply Theorem 2 to Eq. (7). For (7), we consider the corresponding initial value condition
\[
y(t) = \varphi(t), \quad \varphi \in C([t-\tau, 0], [-b, a]), \quad -b < y(0) < a. \tag{14}
\]

It is easy to prove that every solution of (7) with (14) exists in the future and satisfies
\[
-b < y(t) < a, \quad t \geq 0. \tag{15}
\]
By the change of variables
\[ x(t) = \frac{1}{a + b} \ln \frac{1 + y(t)/b}{1 - y(t)/a}. \]
We can transform (7) into
\[ x'(t) + \left( \frac{a + b}{2} \right)^2 p(t) f(x(t - \tau)) = 0, \quad t \geq 0 \tag{16} \]
where
\[ f(x) = \left( \frac{2}{a + b} \right)^2 \frac{e^{ax} - e^{-bx}}{e^{ax}/a + e^{-bx}/b}. \tag{17} \]
It is easily seen that \( f'(0) = 0 \) and
\[ f'(x) \leq \frac{4e^{(a-b)x}}{ab(e^{ax}/a + e^{-bx}/b)^2}, \quad x \in (-\infty, \infty). \]
Hence, \( f \) is increasing. We also have
\[ f''(x) \leq \frac{4e^{(a-b)x}}{ab \left( 2\sqrt{e^{(a-b)x}/(ab)} \right)^2} = 1, \quad x \in (-\infty, \infty), \]
which implies that
\[ |f(x)| \leq |x|, \quad x \in (-\infty, \infty). \]

Thus, the function \( f \) satisfies the condition (6). Notice that the solution \( y(t) \) of (7) with (14) tends to zero as \( t \to \infty \), if \( x(t) \) tends to zero as \( t \to \infty \). Thus, we have the following result by Theorem 2.

**Corollary 1.** Assume that (3) holds, and that
\[ \limsup_{t \to \infty} \int_{t-\tau}^{t+\tau} p(s + \tau)ds < 2 \left( \frac{2}{a + b} \right)^2. \tag{18} \]
Then every solution of (7) with (14) tends to zero as \( t \to \infty \).

On the other hand, for Eq. (7), condition (5) in Theorem 1 becomes (see [5]):
\[ \limsup_{t \to \infty} \int_{t-\tau}^{t} p(s)ds < \frac{3}{2} \left( \frac{2}{a + b} \right)^2. \tag{19} \]
Hence, the conclusion of Corollary 1 still holds if (18) is replaced by (19). In [6], it is shown that (19) can be replaced by the following better condition
\[ \int_{t-\tau}^{t} p(s)ds \leq \frac{3}{2ab}, \text{ for all large } t. \quad \tag{20} \]

Finally, we compare the two conditions (4) and (5) by some examples. We will illustrate that these conditions are independent in the sense that neither of them implies the other, and therefore, Theorems 1 and 2 are complementary.

**Example 1.** Let \( p(t) = p \) be constant in Eq. (1). Then (4) becomes \( p \tau < 1 \), while (5) reduces to \( p \tau < 3/2 \). In this case, condition (5) is weaker than (4).
Example 2. Take \( \tau = \pi \) and let \( p(t) = k(1 + \cos t) \) in Eq. (1). By a simple calculation, we obtain

\[
\int_{t-\tau}^{t+\tau} p(s + \tau)\,ds = k \int_{t-\pi}^{t+\pi} [1 + \cos(s + \pi)]\,ds = 2\pi k,
\]

and

\[
\int_{t-\tau}^{t} p(s)\,ds = k \int_{t-\pi}^{t} (1 + \cos s)\,ds = k(\pi + 2\sin t).
\]

Therefore, (4) and (5) reduce to, respectively,

\[ k < \frac{1}{\pi} \approx 0.31, \quad (21) \]

and

\[ k < \frac{3}{2(\pi + 2)} \approx 0.29. \quad (22) \]

Clearly, (4) is better than (5) in this case. To further compare conditions (18) and (20), we give the following example.

Example 3. In Eq. (7), let \( a = b = 1 \) and let \( \tau \) and \( p(t) \) be the same as in Example 2. Then, (18) and (20) also reduce to, respectively, (21) and (22). Hence, in this case, condition (18) is better than (20).

Remark 1. Some related results on the asymptotic behavior of solutions of Eq. (1) can be found in [7–11].

References