

## Note

## Comments on generalized Heron polynomials and Robbins' conjectures

Robert Connelly

Department of Mathematics, Malott Hall, Cornell University, Ithaca, NY 14853, United States

## ARTICLE INFO

## Article history:

Received 21 October 2008

Received in revised form 29 October 2008

Accepted 29 October 2008

Available online 10 January 2009

## Keywords:

Heron's formula

Robbins' conjecture

Area

Circumradius

Brahmagupta's formula

Cyclic polygon

## ABSTRACT

Heron's formula for a triangle gives a polynomial for the square of its area in terms of the lengths of its three sides. There is a very similar formula, due to Brahmagupta, for the area of a cyclic quadrilateral in terms of the lengths of its four sides. (A polygon is cyclic if its vertices lie on a circle.) In both cases if  $A$  is the area of the polygon,  $(4A)^2$  is a polynomial function of the square in the lengths of its edges. David Robbins in [D.P. Robbins, Areas of polygons inscribed in a circle, *Discrete Comput. Geom.* 12 (2) (1994) 223–236. MR 95g:51027; David P. Robbins, Areas of polygons inscribed in a circle, *Amer. Math. Monthly* 102 (6) (1995) 523–530. MR 96k:51024] showed that for any cyclic polygon with  $n$  edges,  $(4A)^2$  satisfies a polynomial whose coefficients are themselves polynomials in the edge lengths, and he calculated this polynomial for  $n = 5$  and  $n = 6$ . He conjectured the degree of this polynomial for all  $n$ , and recently Igor Pak and Maksym Fedorchuk [Maksym Fedorchuk, Igor Pak, Rigidity and polynomial invariants of convex polytopes, *Duke Math. J.* 129 (2) (2005) 371–404. MR 2006f:52015] have shown that this conjecture of Robbins is true. Robbins also conjectured that his polynomial is monic, and that was shown in [V.V. Varfolomeev, Inscribed polygons and Heron polynomials (Russian. Russian summary), *Mat. Sb.* 194 (3) (2003) 3–24. MR 2004d:51014]. A short independent proof will be shown here.

© 2008 Elsevier B.V. All rights reserved.

## 1. Introduction

If a triangle has side lengths  $a_1, a_2, a_3$  and area  $A$ , then Heron's formula says that

$$A^2 = s(s - a_1)(s - a_2)(s - a_3), \quad (1)$$

where  $s = (a_1 + a_2 + a_3)/2$  is the semiperimeter of the triangle. But this formula can be rewritten as

$$(4A)^2 = 2a_1^2a_2^2 + 2a_2^2a_3^2 + 2a_3^2a_1^2 - a_1^4 - a_2^4 - a_3^4. \quad (2)$$

In the seventh century A.D., Brahmagupta proved the following formula for the area of a cyclic quadrilateral with area  $A$ , and side lengths  $a_1, a_2, a_3, a_4$

$$A^2 = (s - a_1)(s - a_2)(s - a_3)(s - a_4), \quad (3)$$

where  $s = (a_1 + a_2 + a_3 + a_4)/2$  is the semiperimeter as before. Recall that a polygon is defined to be *cyclic* if its vertices lie on a circle. Similarly to (2), we rewrite (3) as

$$(4A)^2 = 2a_1^2a_2^2 + 2a_2^2a_3^2 + 2a_3^2a_4^2 + 2a_4^2a_1^2 + 2a_1^2a_3^2 + 2a_2^2a_4^2 - a_1^4 - a_2^4 - a_3^4 - a_4^4 + 8a_1a_2a_3a_4. \quad (4)$$

E-mail address: [connelly@math.cornell.edu](mailto:connelly@math.cornell.edu).

The area of a polygon can be defined, given an orientation of the polygon, regardless of whether it intersects itself or not. Here it is assumed that all the edges are non-negative numbers, but formulas (3) and (4) only hold in case the quadrilateral is convex. If it is not convex, in formula (4) the  $8a_1a_2a_3a_4$  term is replaced by  $-8a_1a_2a_3a_4$ . In all cases, one can see that  $(4A)^2$  satisfies a polynomial whose leading term is 1, i.e. a *monic* polynomial, whose coefficients are themselves polynomials in  $a_i^2$ , for  $i = 1, \dots, 4$ .

In [10] and [11], formulas (2) and (4) are generalized in the following way:

**Theorem 1.1** ([10,11]). *For any  $n = 1, 2, 3, \dots$  there is a unique (up to sign) irreducible homogeneous polynomial  $f$  with integer coefficients such that when  $(a_1, \dots, a_n)$  are the side lengths of a cyclic  $n$ -gon and  $A$  is its area, then*

$$f(16A^2, a_1^2, \dots, a_n^2) = 0. \tag{5}$$

Robbins gave an explicit description of  $f$  when  $n = 5$  and  $n = 6$ , and, for all  $n$ , gave a lower bound for the degree of its first variable, conjecturing that this lower bound was the degree of  $f$ . This conjecture was shown to be correct in [4] and then later in [7], who calculate a relatively explicit version of the formula for  $n = 7$ . Also see [8] and [9, Chapter 34, appendix] for a discussion. The polynomial  $f$  can be quite complicated. Even for  $n = 5$ , where although the degree of  $f$  is only 7, there are 153 terms. See also [13].

Robbins also conjectured that  $f$  was a monic polynomial in its first argument. Apparently unaware of [10] and [11] in [14] it is shown that  $f$  is monic as well as many other interesting results. Here we provide a short simple proof that  $f$  is monic different from [14], but in the spirit of [2], where the theory of places is introduced for this sort of problem.

**Theorem 1.2.** *For any  $n = 1, 2, 3, \dots$ , Robbins' polynomial  $f$  is monic in the first variable.*

An explicit description of  $f$  seems quite unwieldy, or at least unwise, so we will use indirect techniques to show [Theorem 1.2](#). The techniques in [4] are also not direct, and their techniques do not seem to be able to show that  $f$  is monic.

In the next section we will first write the structural equations that Robbins used to derive his polynomial  $f$ . Then will introduce the notion of places that are a natural tool to show that a polynomial is monic. Next, we apply the theory of places to Robbins' system of polynomial equations to show [Theorem 1.2](#). Finally, we make some remarks relating the result here to other problems, in particular the “Bellows Problem” in [2].

Sadly, in 2003, David Robbins died of cancer. We are all sorry not to have him here to discuss this very interesting problem.

## 2. Robbins' polynomial equations

Suppose that the vertices of the polygon are  $v_1, \dots, v_n$  on a circle of radius  $R \neq 0$  centered at the origin in the complex plane. With  $v_{n+1} = v_1$  define  $q_j = v_{j+1}/v_j, j = 1, \dots, n$ . Then Robbins shows for  $j = 1, \dots, n$

$$a_j^2 = R^2(2 - q_j - q_j^{-1}) \tag{6}$$

$$-16A^2 = R^4(q_1 + \dots + q_n - q_1^{-1} - \dots - q_n^{-1})^2, \tag{7}$$

where  $A$  is the area bounded by the polygon as before.

It is also true that  $q_1q_2 \dots q_n = 1$ , but we will not need that here. Instead, we need the following formula for  $R$ , the circumradius of a triangle. This is the case when  $n = 3$ .

$$R^2 = \frac{a_1^2 a_2^2 a_3^2}{2a_1^2 a_2^2 + 2a_2^2 a_3^2 + 2a_3^2 a_1^2 - a_1^4 - a_2^4 - a_3^4}. \tag{8}$$

This can be found with a proof in many elementary geometry texts, for example [3].

## 3. Places

We remind the reader of some basic definitions from algebra. Let  $L$  be any field that contains a ring  $R$ . An element  $x \in L$  is defined to be *integral* over  $R$  if there are elements  $b_j \in R, j = 0 \dots m - 1$  such that

$$x^m + b_{m-1}x^{m-1} + \dots + b_1x + b_0 = 0.$$

A very useful tool to show integrality is concerned with places. Suppose  $L$  and  $F$  are fields. Let  $\varphi : L \rightarrow F \cup \{\infty\}$  be a function such that for all  $x, y \in L$ , the following holds:

- (i)  $\varphi(x + y) = \varphi(x) + \varphi(y)$
- (ii)  $\varphi(xy) = \varphi(x)\varphi(y)$  and
- (iii)  $\varphi(1) = 1$ ,

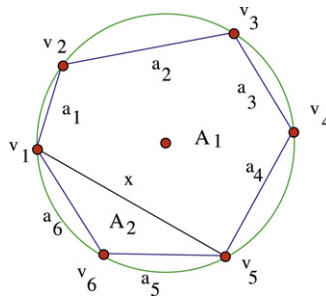


Fig. 1. Adding areas.

where it is understood that for  $a \in F$  (called a *finite a*)  $a \pm \infty = \infty \cdot \infty = 1/0 = \infty$ ,  $a/\infty = 0$ , and if  $a \neq 0$ , then  $a \cdot \infty = \infty$ . The expressions  $0/0$ ,  $\infty/\infty$ ,  $0 \cdot \infty$ ,  $\infty \pm \infty$  are *not* defined, and it is also understood that (i) and (ii) only hold when the right hand side is defined.

We call such a function  $\varphi$  a *place* for the field  $L$ . Our basic tool for integrality is the following from [5], page 12.

**Lemma 3.1.** *An element  $x$  in a field  $L$  containing the ring  $R$  is integral over  $R$  if and only if every place defined on  $L$  that is finite on  $R$  is finite on  $x$ .*

### 4. The integrality of area

With integrality in mind a big step in proving Theorem 1.2 is as follows.

**Proposition 4.1.** *If  $A$  is the area bounded by a cyclic polygon, then  $4A$  is integral over the ring generated by the squares of the edge lengths.*

**Proof.** We proceed by induction on  $n$  the number of edges in the cyclic polygon. We assume that the area of a cyclic polygon with  $n - 1$  or fewer edges is integral over the ring generated by those edge lengths, and we wish to show that the area of a cyclic polygon with  $n$  edges is integral over the ring generated by all  $n$  edge lengths. It is clear that the area bounded by a cyclic polygon is integral for  $n = 3$  and  $n = 4$  from Eqs. (2) and (4).

Suppose that  $\varphi : \mathbb{C} \rightarrow F \cup \{\infty\}$  is a place defined on  $\mathbb{C}$  that is finite on the ring generated by  $a_1^2, \dots, a_n^2$ , where the vertices of the cyclic polygon are  $v_1, v_2, \dots, v_n$  and  $\mathbb{C}$  is the field of complex numbers. We wish to show that  $\varphi(4A)$  is finite as well, where  $A$  is the area of the whole polygon. Let  $A_1$  be the area of the polygon  $v_1, v_2, \dots, v_{n-1}$ , and  $A_2$  the area of the triangle  $v_1, v_{n-1}, \dots, v_n$ . So  $A = A_1 + A_2$ .

Let  $x = |v_1 - v_{n-1}|$ , the length of the edge from  $v_1$  to  $v_{n-1}$ . See Fig. 1. If  $\varphi(x)$  is finite, then  $\varphi(4A_1)$  and  $\varphi(4A_2)$  are both finite by the induction hypothesis. So  $\varphi(4A_1) + \varphi(4A_2) = \varphi(4A_1 + 4A_2) = \varphi(4A)$  is finite as well, and we are done. So we assume that  $\varphi(x) = \infty$  from here on. We observe that Eq. (8) determines the radius  $R$  in terms of  $a_n, a_{n-1}, x$ . Dividing the numerator and denominator of the right hand side of Eq. (8) by  $x^2$  we get the following:

$$R^2 = \frac{a_n^2 a_{n-1}^2}{2a_n^2 a_{n-1}^2 / x^2 + 2a_{n-1}^2 + 2a_n^2 - a_n^4 / x^2 - a_{n-1}^4 / x^2 - x^2}. \tag{9}$$

Applying the place  $\varphi$  to both sides of Eq. (9) and using the properties of a place, we see that  $\varphi(R^2) = 0$ .

Next apply the place  $\varphi$  to both sides of Eq. (6) to get, after bringing  $R^2$  inside the parenthesis,

$$\varphi(a_j^2) = \varphi(2R^2) - \varphi(R^2 q_j) - \varphi(R^2 q_j^{-1}). \tag{10}$$

All the terms in (10) are finite, except at most one of  $\varphi(R^2 q_j)$  or  $\varphi(R^2 q_j^{-1})$ . But then both terms must be finite. Finally apply  $\varphi$  to both sides of (7) to get

$$\varphi(4A)^2 = -(\varphi(R^2 q_1) + \dots + \varphi(R^2 q_n) - \varphi(R^2 q_1^{-1}) - \dots - \varphi(R^2 q_n^{-1}))^2. \tag{11}$$

Each term on the right of (11) is finite, so  $\varphi(4A)$  must be finite as well. This is what was to be shown.  $\square$

To finish the proof of Theorem 1.2 we need to observe some basic algebraic facts about the situation at hand.

**Proof of Theorem 1.2.** We first observe that the quantities  $a_1, a_2, \dots, a_n$  can be taken to be algebraically independent over the rationals. This is because of the following. Take any configuration of  $n$  points on a circle of radius  $R$  so that all the edge lengths are non-zero with none of the lengths equal to the diameter of the circle. Vary all  $n$  of the lengths slightly, but otherwise arbitrarily, to some nearby algebraically independent lengths. Then lay out  $n - 1$  of the lengths on this circle.

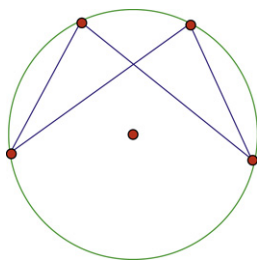


Fig. 2. A flexible cyclic quadrilateral.

Then vary  $R$  in such a way that the last length  $a_n$  is achieved as well. Then the integral domain  $D = \mathbb{Z}[a_1^2, a_2^2, \dots, a_n^2]$  is a unique factorization domain.

Let  $m(x)$  be any monic polynomial such that  $m(16A^2) = 0$  for  $A$  the area of cyclic polygon as guaranteed by Proposition 4.1. Let  $F$  be the field of fractions of  $D$ . We know that Robbins' polynomial (5)  $f(x) = f(x, a_1^2, a_2^2, \dots, a_n^2)$  is primitive; that is, its coefficients are relatively prime in  $D$ , as well as minimal over  $F$ . Since  $f$  is minimal  $m(x) = f(x)g(x)/c$ , where  $g(x)/c \in F[x]$ ,  $g(x) \in D[x]$ , and  $c \in D$  has no common divisor with all of the coefficients of  $g(x)$ . So  $cm(x) = f(x)g(x)$  in  $D[x]$ . The polynomial  $g(x)$  is primitive as well since any prime divisor of all of its coefficients does not divide  $c$ , and it cannot divide  $m(x)$  since  $m(x)$  is monic.

Since  $D$  is a unique factorization domain, the following Lemma of Gauss applies. It says that the product of two primitive polynomials is primitive, so  $cm(x) = f(x)g(x)$  must be primitive, and  $c$  must be a unit in  $D = \mathbb{Z}[a_1^2, a_2^2, \dots, a_n^2]$ . So  $c = \pm 1$ , and  $f$  must be monic (as well as  $g$ ) possibly after a change of sign.

This shows that when the quantities  $a_1, a_2, \dots, a_n$  are algebraically independent,  $(4A)^2$  satisfies Robbins' polynomial (5). But for an arbitrary cyclic configuration, it is clear that the area will still satisfy (5), for example, by taking a topological limit of configurations where the  $a_1, a_2, \dots, a_n$  are algebraically independent. This finishes the proof.  $\square$

### 5. Comments and related results

The area of a planar cyclic polygon is naturally related to the volume of a suspension or bipyramid over the polygon. This is a 2-dimensional triangulated polyhedral surface  $\Sigma$  obtained by taking two points  $N$  and  $S$  in 3-space and joining each of them to all the points  $v_1, \dots, v_n$  which lie on a circle in a plane. We assume that the center of the circle is at the origin in the  $x$ - $y$  plane, and  $N$  and  $S$  lie on the  $z$ -axis (usually not at the origin).

We allow a triangulated, orientable surface to intersect itself in any way we like, and regardless there is a well-defined notion of what is meant by the (signed) volume bounded by the surface, depending only on an orientation defined for the surface. The volume of any triangulated surface is known to be integral over the ring generated by the squares of its edge lengths. See [12] and [2] for proofs. Indeed, the proof of the integrality of the volume in [2] uses the theory of places as with Proposition 4.1 here. See also [9, end of chapter 34] for a description of the proof in [2]. One of the reasons for showing integrality is to prove the “bellows” property. That means that if there is a continuous motion, called a flex, of the vertices of any triangulated surface such that the edge lengths stay constant during the flex, then the volume bounded by the surface is constant during the flex. Since the Sabitov polynomial is monic, it is never the 0 polynomial, and since its coefficients are constant during the flex, and the volume bounded by the surface is a root of the polynomial, then the volume must be constant during the flex as well.

A similar situation occurs with Robbins' polynomial (5). Since Robbins' polynomial is monic, it is never the 0 polynomial, and since its coefficients are constant during the flex, and the area bounded by the cyclic polygon is a root of the polynomial, then the area must be constant during the flex.

But there is more. For any cyclic planar polygon  $P$ , we can associate the corresponding bipyramid  $\Sigma(P)$  as above, and any flex in the plane of  $P$  as a cyclic polygon defines a corresponding flex of  $\Sigma(P)$  as an oriented triangulated surface in  $\mathbb{E}^3$ . A special case of Theorems 2 and 1 of [1] says that if  $\Sigma(P)$  flexes with distance  $|N - S|$  varying, then the (generalized) volume of  $\Sigma(P)$  is 0, and the topological winding number of  $P$  about the origin is also 0. Since  $V(\Sigma(P)) = A(P)|N - S|/3$ , where  $V(\Sigma(P))$  is the volume of  $\Sigma(P)$  and  $A(P)$  is the area of  $P$ , we see that if  $P$  flexes with its circumradius varying, then not only is  $A(P)$  constant, but  $A(P) = 0$  during the flex.

But there is more. The structural equations of [1] imply that when  $\Sigma(P)$  flexes with distance  $|N - S|$  varying then the non-zero edges of  $P$  must pair off in such a way that they correspond to opposite directions along the circle. See [6] for a direct proof of this special case of suspensions in [1]. (Actually this was the first case that I considered.) Fig. 2 shows an example for  $n = 4$ . Then it is easy to see that  $V(\Sigma(P)) = 0 = A(P)$  during the flex. A consequence of the comments above is that if  $f_R(x, a_1^2, a_2^2, \dots, a_n^2)$  is a polynomial such that the radius  $R$  of a cyclic polygon satisfies  $f_R(R, a_1, a_2, \dots, a_n) = 0$ , then all the coefficients of  $f_R$  must vanish when the  $a_i$  are equal in pairs, and  $n$  is even. But conversely, it follows that if the lengths of the edges do not pair off, then the radius  $R$  should be algebraic over the ring generated by the lengths  $a_1^2, a_2^2, \dots, a_n^2$  and thus not all the coefficients of the polynomial  $f_R$  should vanish. This is Lemma 20 of [14]. Similarly the constant term of  $f_A$ , Robbins' polynomial for the area (or the  $S$ -polynomial of [14]) should vanish. This is a corollary of Lemma 19 of [14].

When  $n$  is even Robbins conjectured and [14] proved (apparently unaware of Robbins' work) that Robbins' polynomial  $f$  of the area factors over  $\mathbb{Z}[a_1, \dots, a_n]$  into two polynomials  $\beta(x)\beta^*(x)$ , where  $\beta(x)^*$  is obtained from  $\beta(x)$  by replacing any  $a_i$  by  $-a_i$ . The area of any cyclic polygon is a root of one of the polynomials  $\beta(x)$  or  $\beta^*(x)$  and you can tell which one, by the following process. If the polygon is convex, its area is a root of  $\beta(x)$ . Otherwise, assume that none of the edges have 0 length. Then let the vertices move continuously around the circle in such a way that that no three consecutive vertices cross at the same time. When two consecutive vertices do cross, a parity changes and we switch between  $\beta(x)$  and  $\beta^*(x)$  as to the relevant polynomial. For example, for  $n = 4$ , formula (4) describes the  $\beta(x)$ -polynomial, and the example in Fig. 2, will satisfy the  $\beta^*(x)$ -polynomial. This distinction is implicit in the discussion in [14], and was implicitly conjectured in [11].

## Acknowledgement

This research was supported in part by NSF Grant No. DMS-0209595.

## References

- [1] Robert Connelly, The rigidity of suspensions, *J. Differential Geom.* 13 (3) (1978) 399–408. MR 80j:51019.
- [2] R. Connelly, I. Sabitov, A. Walz, The bellows conjecture. (English. English summary), *Beiträge Algebra Geom.* 38 (1) (1997) 1–10. MR 98c:52026.
- [3] H.S.M. Coxeter, S.L. Greitzer, *Geometry Revisited*, The Mathematical Association of America, Washington, DC, 1967.
- [4] Maksym Fedorchuk, Igor Pak, Rigidity and polynomial invariants of convex polytopes, *Duke Math. J.* 129 (2) (2005) 371–404. MR 2006f:52015.
- [5] S. Lang, Serge Lang, *Introduction to algebraic geometry*, Interscience Publishers, Inc, New York-London, 1958, xi+260 pp. MR 49 #8983.
- [6] Hiroshi Maehara, On a special case of Connelly's suspension theorem, *Ryukyu Math. J.* 4 (1991) 35–45. MR 93d:52030.
- [7] F. Miller Maley, David P. Robbins, Julie Roskies, On the areas of cyclic and semicyclic polygons (English summary), *Adv. in Appl. Math.* 34 (4) (2005) 669–689. MR 2006b:51016.
- [8] Igor Pak, The area of cyclic polygons: recent progress on Robbins' conjectures, *Adv. in Appl. Math.* 34 (4) (2005) 690–696. MR 2006b:51017.
- [9] Igor Pak, *Lectures on Discrete and Polyhedral Geometry*, <http://www.math.umn.edu/pak/book.htm>.
- [10] D.P. Robbins, Areas of polygons inscribed in a circle, *Discrete Comput. Geom.* 12 (2) (1994) 223–236. MR 95g:51027.
- [11] David P. Robbins, Areas of polygons inscribed in a circle, *Amer. Math. Monthly* 102 (6) (1995) 523–530. MR 96k:51024.
- [12] I.Kh. Sabitov, The volume as a metric invariant of polyhedra, *Discrete Comput. Geom.* 20 (4) (1998) 405–425. MR 2000f:52010.
- [13] V.V. Varfolomeev, Galois groups of the Heron-Sabitov polynomials for pentagons inscribed in a circle (Russian. Russian summary), *Mat. Sb.* 195 (2) (2004) 3–16. translation in *Sb. Math.* 195 (2004), no. 1–2, 149–162 MR 2005b:12010.
- [14] V.V. Varfolomeev, Inscribed polygons and Heron polynomials (Russian. Russian summary), *Mat. Sb.* 194 (3) (2003) 3–24. MR 2004d:51014.