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Note

Some inequalities about connected domination number

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Abstract

Let $G = (V, E)$ be a graph. In this note, γ_c, ir, $\gamma, i, \beta_0, \Gamma$, IR denote the connected domination number, the irredundance number, the domination number, the independent domination number, the independence number, the upper domination number and the upper irredundance number, respectively. We prove that $\gamma_c \leqslant 3\,\mathrm{ir} - 2$ for a connected graph G. Thus, an open problem in Hedetniemi and Laskar (1984) discuss further some relations between γ_c and γ, β_0, Γ, IR, respectively.

1. Introduction

All graphs under consideration are finite, undirected, and loopless without multiple lines.

Let $G = (V, E)$ be a graph with vertex set V and edge set E.

The open neighborhood of a vertex u, denoted by $N(u)$, is the set of vertices adjacent to u. The closed neighborhood of a vertex u, denoted by $N[u]$, is $N(u) \cup \{u\}$. The open neighborhood of a set S of vertices, denoted by $N(S)$, is the set of vertices adjacent to a vertex in S. The closed neighborhood of a set S of vertices, denoted by $N[S]$, is $N(S) \cup \{S\}$.

For a set $D \subseteq V$, $\langle D \rangle$ denotes the subgraph induced by D.

A set $D \subseteq V$ is a dominating set if every vertex in $V - D$ is adjacent to at least one vertex in D. A dominating set D is independent if no two vertices in $\langle D \rangle$ are adjacent. A dominating set D is a connected dominating set if $\langle D \rangle$ is connected. The domination (independent domination, connected domination) number, denoted by γ (i, γ_c), is the minimum number of vertices in a dominating (independent dominating, connected dominating) set. The upper domination number, denoted by Γ, is the maximum number of vertices in a minimal dominating set. The independence number, denoted

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by β_0, is the maximum cardinality of an independent set of vertices. A set $S \subset V$ is an irredundant set if for every $x \in S$, $N[x] \not\subseteq \bigcup_{y \in S-\{x\}} N[y]$. The irredundance (upper irredundance) number, denoted by ir (IR), is the minimum (maximum) cardinality of a maximal irredundant set of vertices.

For a set $S \subseteq V$, $|S|$ denotes the cardinality of S. We denote a set S as an ir-set if S is a maximal irredundant set with $|S| = \mathrm{ir}$.

Any definitions not given here can be found in [4].

The inequality $\gamma \leqslant 2\,\mathrm{ir} - 1$ was obtained independently in [2,3]. Does a similar result hold if γ_c is substituted for γ? This stands in [8] as an open problem.

In this paper, we prove that $\gamma_c \leqslant 3\,\mathrm{ir} - 2$ for a connected graph G and this result is best possible.

The parameters $\mathrm{ir}, \gamma, i, \beta_0, \Gamma$ and IR are related by the following inequalities:

Theorem 1.1 (Cockayne and Hedet [5,6]). *For any graph G, $\mathrm{ir} \leqslant \gamma \leqslant i \leqslant \beta_0 \leqslant \Gamma \leqslant \mathrm{IR}$.*

A natural question is: What relationships exist between γ_c and $\gamma, i, \beta_0, \Gamma, \mathrm{IR}$ respectively? We demonstrate some relationships and show that these results are best possible.

2. Main results

Lemma 2.1 (Duchet and Meyniel [7]). *For any connected graph, $\gamma_c \leqslant 3\gamma - 2$.*

Lemma 2.2 (Allan et al. [8]). *If S is an* ir*-set of graph G, and S is independent, then $\mathrm{ir} = \gamma = i$.*

Theorem 2.3. *If S is an* ir*-set of a connected graph G, and S is independent, then $\gamma_c \leqslant 3\,\mathrm{ir} - 2$.*

Proof. It follows from Lemmas 2.1 and 2.2. □

Theorem 2.4. *If a graph G is connected, then $\gamma_c \leqslant 3\,\mathrm{ir} - 2$.*

Proof. Let G be a connected graph and let $S = \{v_1, \ldots, v_m\}$ be an ir-set. All components of $\langle S \rangle$ are denoted by $S_1, \ldots, S_n$, $1 \leqslant n \leqslant m = \mathrm{ir}$. Suppose that there are t isolated vertices $v_1, \ldots, v_t$ in $\langle S \rangle$, $0 \leqslant t \leqslant n$, where $v_1, \ldots, v_t$ belong to components $S_1, \ldots, S_t$, respectively, but each of the other $n - t$ components contain atleast two vertices. Hence,

$$2(n-t) + t \leqslant \mathrm{ir}, \quad \text{i.e., } 2n - t \leqslant \mathrm{ir}. \tag{1}$$

We need to consider only two cases as follows.

Case 1: $t = n$, i.e., S is independent. Then from Theorem 2.3, we have $\gamma_c \leqslant 3\,\mathrm{ir} - 2$.

Case 2: $t < n$, Since S is an irredundant set, $N[v_i] \nsubseteq \bigcup_{j \neq i} N[v_j]$ for any $v_i \in S$. Assume that

$$N_i = N[v_i] - \bigcup_{j \neq i} N[v_j] \quad \text{for } i = 1, \ldots, m. \tag{2}$$

Since $N_i \neq \emptyset$, we may choose one vertex $u_i \in N_i$ for $i = 1, \ldots, m$. Suppose that $S' = S \cup \{u_{t+1}, \ldots, u_m\}$. It is clear that

$$|S'| = \mathrm{ir} + \mathrm{ir} - t = 2\,\mathrm{ir} - t. \tag{3}$$

We claim that S' is a dominating set.

If S is a dominating set of G, then it is clear that S' is also a dominating set.

If S is not a dominating set of G, let v be an arbitrary vertex in $V - N[S]$. we discuss the following two subcases.

Subcase 1.1: There exists i, $t < i \leqslant m$, such that $N_i \subset N[v]$.

Subcase 1.2: $N_i \nsubseteq N[v]$ for any i, $t < i \leqslant m$. In this subcase, we discuss:

Subcase 2.1: When $t = 0$, then $N_i \nsubseteq N[v]$ for $1 \leqslant i \leqslant m$.

Subcase 2.2: When $t > 0$, since $v_1, \ldots, v_t$ are isolated vertices in $\langle S \rangle$, thus we obtain $v_i \in N_i$ for $1 \leqslant i \leqslant t$ from (2).

Since $v_i \notin N[v]$ by the choice of vertex v, hence $N_i \nsubseteq N[v]$ for $1 \leqslant i \leqslant t$.

So $N_i \nsubseteq N[v]$ for $1 \leqslant i \leqslant m$.

Therefore, according to Subcases 2.1 and 2.2, we have that

$$N_i \nsubseteq N[v] \quad \text{for } 1 \leqslant i \leqslant m. \tag{4}$$

Since $v \in N[v]$, but $v \notin N[S]$, therefore

$$N[v] \nsubseteq \bigcup_{i=1}^{m} N[v_i]. \tag{5}$$

By (2), for any vertex $v_i \in S$,

$$N_i \subseteq N[v_i]. \tag{6}$$

$$N_i \cap \left(\bigcup_{j \neq i} N[v_j] \right) = \emptyset. \tag{7}$$

By (4),

$$\text{there exists at least one vertex } u \in N_i - N[v]. \tag{8}$$

Hence, by (6),

$$u \in N[v_i]. \tag{9}$$

But, from (8),

$$u \notin N[v]. \tag{10}$$

So, by (7) and (8),

$$u \notin \bigcup_{j \neq i} N[v_j]. \tag{11}$$

Therefore, due to (9), (10) and (11),

$$N[v_i] \nsubseteq \left(\bigcup_{j \neq i} N[v_j] \right) \cup N[v]. \tag{12}$$

Then, according to (5) and (12), $S \cup \{v\}$ is an irredundant set. But this contradicts the maximality of S.

So Subcase 2 cannot occur, and only Subcase 1 can occur, i.e., there exists $i, t \leqslant i \leqslant m$, such that $N_i \subseteq N[v]$. Then since $u_i \in N_i$, $v \in \bigcup_{i=t+1}^{m} N[u_i]$.

Note that v is an arbitrary vertex belonging to $V - N[S]$. So S' is a dominating set.

Assume that all components of $\langle S' \rangle$ are denoted by $S'_1, \ldots, S'_q$, hence $1 \leqslant q \leqslant n \leqslant \mathrm{ir}$. Let $Y_i = \{v \mid v \in N[s'_i]\}$, $G_i = \langle Y_i \rangle$, $i = 1, \ldots, q$. Then G_i is a connected subgraph of G. Any vertex in G belongs to some G_i.

If $q = 1$, then $\langle S' \rangle$ is connected:

$$\begin{aligned} \gamma_c \leqslant |S'| &= 2\,\mathrm{ir} - t && \text{(by (3))} \\ &\leqslant 2\,\mathrm{ir} - t + (\mathrm{ir} + t - 2) && \text{(since } \mathrm{ir} + t \geqslant 2n \text{ and } n \geqslant 1). \end{aligned}$$

So $\gamma_c \leqslant 3\,\mathrm{ir} - 2$.

If $q \geqslant 2$, since G is connected, there exists one vertex $y_1 \in G_1$, and y_1 is adjacent to one vertex $z_1 \in \bigcup_{i=2}^{q} G_i$. Without loss of generality, suppose that $z_1 \in G_2$. Similarly, there exists one vertex $y_2 \in G_1 \cup G_2$, and y_2 is adjacent to one vertex $z_2 \in \bigcup_{i=3}^{q} G_i$. Without loss of generality, suppose that $z_2 \in G_3$, and so on. We will make a set $Y = \{y_1, \ldots, y_{s-1}, z_1, \ldots, z_{k-1}\}$, where $s \leqslant q \leqslant n$, $k \leqslant q \leqslant n$. It is clear that $\langle S' \cup Y \rangle$ is a connected subgraph of G.

So

$$\begin{aligned} \gamma_c &\leqslant |\langle S' \cup Y \rangle| \leqslant 2\,\mathrm{ir} - t + s - 1 + k - 1 \\ &\leqslant 2\,\mathrm{ir} - t + 2n - 2 \leqslant 3\,\mathrm{ir} - 2 \ \text{(by (1))}. \qquad \square \end{aligned}$$

According to Theorem 1.1, the following corollary is obtained.

Corollary 2.5. *If G is connected, then $\gamma_c \leqslant 3i - 2$.*

Example 1. (a) For any positive integer n, consider the graph C_{3n}, which is a cycle with $3n$ vertices. $\gamma_c = 3n - 2$, $\mathrm{ir} = \gamma = i = n$.

(b) For any positive integers s, t, consider the graph H obtained by identifying one vertex from each of C_{3s}, C_{3t}; then the number of vertices in H is $3(s+t) - 1$, $\gamma_c = 3(s+t) - 1 - 4 = 3(s+t) - 5$, $\mathrm{ir} = \gamma = i = s + t - 1$.

(c) For any positive integers s, t, k, consider the graph H obtained by identifying one vertex from each of C_{3s}, C_{3t}, C_{3k}; then the number of vertices in H is $3(s+t+k) - 2$, $\gamma_c = 3(s+t+k) - 2 - 6 = 3(s+t+k) - 8$, $\mathrm{ir} = \gamma = i = s + t + k - 2$.

These examples show that the results of Theorem 2.4, Corollary 2.5 and Lemma 2.1 are best possible.

3. Conclusion

The fact that $\gamma_c \leqslant 2\beta_0 - 1$ is proved in [7]. From Theorem 1.1, we know that $\gamma_c \leqslant 2\Gamma - 1$, $\gamma_c \leqslant 2\mathrm{IR} - 1$.

We will show that the bounds are best possible by the following example.

Example 2. Consider the graph C_5, $\beta_0 = \Gamma = \mathrm{IR} = 2$, $\gamma_c = 3$.

We draw a conclusion that $\gamma_c \leqslant 3\,\mathrm{ir} - 2$, $\gamma_c \leqslant 3\gamma - 2$, $\gamma_c \leqslant 3i - 2$, $\gamma_c \leqslant 2\beta_0 - 1$, $\gamma_c \leqslant 2\Gamma - 1$, $\gamma_c \leqslant 2\mathrm{IR} - 1$ for a connected graph G and these results are best possible.

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