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A further characteristic of abstract convexity structures on topological spaces ☆

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Abstract

In this paper, we give a characteristic of abstract convexity structures on topological spaces with selection property. We show that if a convexity structure $\mathcal{C}$ defined on a topological space has the weak selection property then $\mathcal{C}$ satisfies H_0-condition. Moreover, in a compact convex subset of a topological space with convexity structure, the weak selection property implies the fixed point property.

Keywords: Abstract convexity; Weak selection property; Fixed point property

1. Introduction

The convexity of space plays a very important role in fixed point theory and continuous selection theory. For example, the question of whether Brouwer fixed point theorem and Michael selection theorem can be generalized to non-locally-convex topological spaces is the one of the greatest unsolved problems in these areas. The study of abstract convexity structures on topological spaces originated in works of Michael [9], van de Vel [14], Horvath [5]. There are many works which deal with various kinds of generalized, topological, or axiomatically defined convexities (see [1–8,11–16]). Most of them establish various fixed point theorems and selection theorems in topological spaces without linear structure such as some generalizations of Brouwer fixed point theorem, Fan–Browder fixed point theorem and Michael selection theorem.

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The aim of this paper is not to give some generalized convex structure but to analyze the relationships among abstract convexity structures, selection property and fixed point property. We consider whether the various convexity structures should have some common characteristic. Around this questions, we prove that X satisfies H_0-condition if X is of weak selection property with respect to any standard simplex Δ_N, and we show that a compact convex subset in a topological space with convexity structure, the weak selection property implies the fixed point property.

2. Preliminaries

Given a topological (or uniform) space Y, van de Vel introduces the class of "convex" sets as a class of subsets of Y closed under intersections (see [14,15]). Horvath defines "convex hulls" of finite subsets of Y (see [5]). Michael [9], on the other hand, considers an analogue of convex combination functions of vector spaces.

We consider a generalized convexity structure as follows. We relax van de Vel's conditions in such a way that Horvath 's and Michael's "convex" sets are included.

Definition 2.1. A pair $(Y, \mathcal{C})$, where $\mathcal{C}$ is a family of subsets of Y, is called a convex structure if

(1) the empty set $\emptyset$ is in $\mathcal{C}$;
(2) $\mathcal{C}$ is stable for intersections, that is, if $\mathcal{D} \subset \mathcal{C}$ is nonempty, then $\bigcap_{A\in\mathcal{D}} A$ is in $\mathcal{C}$.

Let $\mathcal{C}$ be a convexity structure of Y. The convex hull *conv* is defined as

$$\mathrm{conv}(A) = \bigcap\{D \in \mathcal{C}\colon A \subset D\}, \quad A \subset Y.$$

A subset C of Y is said to be a convex subset if $C \in \mathcal{C}$. It is clear that C is convex if and only if $\mathrm{conv}(C) = C$, and it is easy to check that this convexity structure includes various abstract convexity structures mentioned above. For example, in Horvath's H-spaces, the class of "convex" set

$$\mathcal{C} = \{C \subset Y\colon \Gamma_A \subset C \text{ for any finite subset } A \subset C\},$$

where $\{\Gamma_A\}$ is a family of contractible subsets of Y indexed by all finite subsets of Y such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B$ (see [4] and [5]). A metric space (Y, d) with a convexity structure $\mathcal{C}$ is called an *l.c.* space if $\{y \in Y\colon d(y, E) < \varepsilon\} \in \mathcal{C}$ for any $\varepsilon > 0$ and any $E \in \mathcal{C}$.

A topological space X (with a convexity structure $\mathcal{C}$) is said to be of selection property with respect to S if every multivalued mapping $F : S \mapsto 2^X$ admits a singlevalued continuous selection whenever F is lower semicontinuous and nonempty closed convex valued. $(X, \mathcal{C})$ is said to be of weak selection property with respect to S if $F : S \mapsto 2^X$ admits a singlevalued continuous selection whenever F is a multivalued mapping with nonempty convex images and preimages relatively open in X (i.e., $F(x)$ is convex for each $x \in S$ and $F^{-1}(y)$ is open in S). X is said to be of fixed point property if every continuous selfmap T on X has a fixed point in X.

Let $N = \{0, 1, 2, \dots, n\}$, $\Delta_N = e^0e^1 \cdots e^n$ be the standard simplex of dimension n, where $\{e^0, e^1, \dots, e^n\}$ is the canonical basis of R^{n+1}, and for $J \subset N$, let $\Delta_J = \mathrm{co}\{e^j\colon j \in J\}$ be a face of Δ_N. For each $x \in e^0e^1 \cdots e^n$, there is a unique set of numbers $t_0, \dots, t_n$ with $\sum_{i=0}^n t_i = 1$,

$t_i \geqslant 0$, $i \in N$, such that $x = \sum_{i=0}^{n} t_i e^i$. The coefficients $t_0, \ldots, t_n$ are called the barycentric coordinates of x. Let

$$\chi(v) = \left\{ i\colon\ v = \sum_{i=0}^{n} t_i e^i,\ t_i > 0 \right\}.$$

Definition 2.2. Let $\{T_i\colon\ i \in I\}$ be some simplicial subdivision of standard simplex $\Delta_N = e^0 \cdots e^n$, $\mathcal{V}$ denote the collection of all vertices of all subsimplexes in the subdivision. A function $\lambda : \mathcal{V} \to \{0, \ldots, n\}$ satisfying

$$\lambda(v) \in \chi(v), \quad \forall v \in \mathcal{V},$$

is called a normal labeling of this subdivision. Moreover, T_i is called a completely labeled subsimplex or completely labeled lattice if T_i must have vertices with the complete set of labels: $0, \ldots, n$.

Sperner's Lemma. *(See [16].) Let $\{T_i\colon\ i \in I\}$ be any simplicial subdivision of Δ_N and normally labeled by a function λ. Then there exist odd numbers of completely labeled subsimplexes or lattices in the subdivision with respect to the labeling function λ.*

Horvath proved the following result, which is the basic tool for obtaining selection theorems and fixed point theorems in spaces with abstract convexity.

Horvath's Lemma. *(See [5,6].) Let Y be a topological space. For each $J \subset N$, let Γ_J be a nonempty contractible subset of Y. If $\emptyset \neq J \subset J' \subset N$ implies $\Gamma_J \subset \Gamma_{J'}$, then there exists a continuous mapping $f : \Delta_N \mapsto Y$ such that $f(\Delta_J) \subset \Gamma_J$ for each nonempty subset $J \subset N$.*

3. Main results

According to Horvath's Lemma, we call that a pair $(Y, \mathcal{C})$ satisfies H_0-condition if the convexity structure $\mathcal{C}$ has the following property:

(H0) For each finite subset $\{y_0, y_1, \ldots, y_n\} \subset Y$, there exists a continuous mapping $f : \Delta_N \mapsto \operatorname{conv}\{y_0, y_1, \ldots, y_n\}$ such that $f(\Delta_J) \subset \operatorname{conv}\{y_j\colon\ j \in J\}$ for each nonempty subset $J \subset N$.

Now, we first prove the crucial result of this section as below.

Theorem 3.1. *If a pair $(Y, \mathcal{C})$ is of weak selection property with respect to any standard simplex, then $(Y, \mathcal{C})$ satisfies H_0-condition.*

Proof. Let $A = \{y_0, y_1, \ldots, y_n\}$ be any finite subset of Y, $\Delta_N = e^0 e^1 \cdots e^n$ the standard simplex of dimension n. For each $J \subset N$ and each face Δ_J of Δ_N, let

$$\Delta_J^o = \{ v \in \Delta_J\colon\ \chi(v) = J \}$$

denote the interior of Δ_J.

Define $T : \Delta_N \mapsto 2^Y$ as follows:

$$T(x) = \operatorname{conv}\{ y_j\colon\ j \in \chi(x) \}, \quad x \in \Delta_N.$$

It is routinely to check that T is with nonempty convex images and preimages relatively open in Δ_N. In fact, for each $y \in Y$ and each $x \in T^{-1}(y)$, there is only one face Δ_J, $J = \chi(x)$ such that $x \in \Delta_J^o$. So $x \notin \Delta_{J'}$ for any face $\Delta_{J'}$ not containing Δ_J. For any $\Delta'_J \not\supset \Delta_J$, there exists a neighborhood $O(x) \subset \Delta_N$ of x such that $O(x) \cap \Delta_{J'} = \emptyset$ as every face $\Delta_{J'}$ is closed and the number of faces of Δ_N is finite. Therefore, for any $z \in O(x)$, any face $\Delta_{J'}$ contains z only if $\Delta_J \subset \Delta_{J'}$. Then for each $z \in O(x)$, $z \in \Delta_{\chi(z)}$ implies $\Delta_{\chi(z)} \supset \Delta_J$, so that $\chi(z) \supset J = \chi(x)$. It follows that $T(z) \supset T(x)$ for all $z \in O(x)$, and so $y \in T(x) \subset T(z)$, i.e., $z \in T^{-1}(y)$ for all $z \in O(x)$. Hence $T^{-1}(y)$ is relatively open in Δ_N.

In addition, it is obvious that T is nonempty closed and convex. Since Y is of selection property with respect to any standard simplex, there exists a singlevalued continuous mapping $f : \Delta_N \mapsto Y$ such that $f(x) \in T(x)$ for all $x \in \Delta_N$. The definition of T implies that $f(\Delta_J) \subset \operatorname{conv}\{y_j\colon j \in J\}$ for each nonempty subset $J \subset N$, which complete the proof. □

Corollary 3.1. *If a pair $(Y, \mathcal{C})$ is of weak selection property with respect to any compact topological space, then $(Y, \mathcal{C})$ satisfies H_0-condition.*

Proof. It is immediate from Theorem 3.1. □

Let $(Y, \mathcal{C})$ be a pair, X a subset of Y. A multivalued mapping $F : X \mapsto 2^Y$ is called a KKM mapping if $\operatorname{conv} A \subset \bigcup_{x \in A} F(x)$ for each finite subset $A \subset X$.

Theorem 3.2. *Let X be a topological space, $(Y, \mathcal{C})$ a pair satisfying H_0-condition and $F : Y \mapsto 2^X$ a KKM-mapping. If F is closed-valued, then the family $\{F(y)\colon y \in Y\}$ has the finite intersection property.*

Proof. Let $\{y_0, y_1, \ldots, y_n\}$ be arbitrary finite subset of X. Since $(Y, \mathcal{C})$ satisfies H_0-condition, there exists a singlevalued continuous mapping $f : \Delta_N \mapsto \operatorname{conv}\{y_0, y_1, \ldots, y_n\}$ such that $f(\Delta_J) \subset \operatorname{conv}\{y_j\colon j \in J\}$ for each nonempty subset $J \subset N$.

For each $k \in \{1, 2, \ldots\}$ and each $\varepsilon_k = 1/k > 0$, let $\{T_i^k\colon i \in I_k\}$ be some simplicial subdivision of Δ_N such that the mesh of the subdivision less than $1/2^k$. And let $\mathcal{V}^k$ be the set of vertices of all subsimplexes in this subdivision.

For each $v \in \mathcal{V}^k$, let

$$\lambda^k(v) = \min\bigl\{j \in \chi(v)\colon f(v) \in F(y_j)\bigr\}.$$

Then $\lambda^k(v)$ is nonempty, since $v \in \operatorname{conv}\{e^j\colon j \in \chi(v)\}$ and

$$f(v) \in f\bigl(\operatorname{conv}\bigl\{e^j\colon j \in \chi(v)\bigr\}\bigr) \subset \operatorname{conv}\bigl\{y_j\colon j \in \chi(v)\bigr\} \subset \bigcup_{j \in \chi(v)} F(y_j)$$

by the hypothesis. It is easy to see that λ^k is a normal label function of the subdivision.

So for each $k = 1, 2, \ldots$, there must exist a subsimplex T_{i_k} with complete labels by Sperner's Lemma. Let $z_0^k, \ldots, z_n^k$ be all vertices of subsimplex T_{i_k}, and

$$\lambda\bigl(z_0^k\bigr) = 0, \quad \lambda\bigl(z_1^k\bigr) = 1, \quad \ldots, \quad \lambda\bigl(z_n^k\bigr) = n.$$

By the definition of λ, we have

$$f\bigl(z_0^k\bigr) \in F(y_0), \quad f\bigl(z_1^k\bigr) \in F(y_1), \quad \ldots, \quad f\bigl(z_n^k\bigr) \in F(y_n).$$

Note that $z_0^k, \dots, z_n^k$ are some vertices of subsimplex T_{i_k}, so that $d(z_i^k, z_j^k) \leqslant \frac{1}{2^k}$, $i, j \in \{0, 1, \dots, n\}$. Since Δ_N is compact, we may assume that there is $y^* \in \Delta_N$ such that $z_i^k \to y^*$, $i = 0, 1, \dots, n$. Then $f(z_i^k) \to f(y^*)$. It follows from the closeness of each $F(y_i)$ that $f(y^*) \in F(y_i)$, $i = 0, 1, \dots, n$, and $\bigcap_{i \in N} F(y_i) \neq \emptyset$. This completes the proof. □

Theorem 3.3. *Let $(Y, \mathcal{C})$ be a pair satisfying H_0-condition, X a convex compact subset of $(Y, \mathcal{C})$, and $F : X \mapsto 2^X$ a multivalued mapping with nonempty convex images and preimages relatively open in X. Then F has a fixed point.*

Proof. Since X is compact and $X = \bigcup_{x \in X} F^{-1}(x)$, there exists a finite subset $\{x_0, x_1, \dots, x_n\}$ of X such that $X = \bigcup_{i=0}^n F^{-1}(x_i)$. Then $\bigcap_{i=0}^n [X \setminus F^{-1}(x_i)] = \emptyset$. Let

$$G(x) = \left[X \setminus F^{-1}(x)\right], \quad \forall x \in X.$$

With Theorem 3.2, we know that G is not a KKM-mapping, so that there exists a finite subset $\{y_0, y_1, \dots, y_m\}$ such that

$$\operatorname{conv}\{y_0, y_1, \dots, y_m\} \not\subset \bigcup_{i=0}^m G(y_i).$$

Then there is some $y^* \in \operatorname{conv}\{y_0, y_1, \dots, y_m\}$ such that $y^* \notin G(y_i)$ for all $i = 0, 1, \dots, m$, that is

$$y^* \in F^{-1}(y_i), \quad \forall i = 0, 1, \dots, m.$$

Consequently

$$y_i \in F(y^*), \quad \forall i = 0, 1, \dots, m.$$

Therefore

$$y^* \in \operatorname{conv}\{y_0, y_1, \dots, y_m\} \subset F(y^*),$$

which complete the proof. □

Theorem 3.4. *Let X be a compact topological space, $(Y, \mathcal{C})$ a pair satisfying H_0-condition, and $F : X \mapsto 2^Y$ a multivalued mapping with nonempty convex images and preimages relatively open in X. Then F has a continuous selection.*

Proof. Since X is compact and $X = \bigcup_{y \in Y} F^{-1}(y)$, there exists a finite subset $\{y_0, y_1, \dots, y_n\}$ of X such that $X = \bigcup_{i=0}^n F^{-1}(y_i)$. Now let $\{p_i\colon i = 0, 1, \dots, n\}$ be a partition of unity subordinate to the finite covering $\{F^{-1}(y_i)\colon i = 0, 1, \dots, n\}$. Define a mapping $\phi : X \mapsto \Delta_N$ by

$$\phi(x) = \sum_{i=0}^n p_i(x) e^i, \quad \forall x \in X.$$

On the other hand, since $(Y, \mathcal{C})$ satisfies H_0-condition, there exists a singlevalued continuous mapping $f : \Delta_N \mapsto \operatorname{conv}\{y_0, y_1, \dots, y_n\}$ such that $f(\Delta_J) \subset \operatorname{conv}\{y_j\colon j \in J\}$ for each nonempty subset $J \subset N$.

Now our desired mapping g is given by

$$g = f \circ \phi.$$

In fact, it is easy to verify that $\phi(x) \in \Delta_{J(x)}$ for each $x \in X$, where $J(x) = \{i \in N\colon\ p_i(x) \neq 0\}$. By the convexity of $F(x)$, we do have that $\operatorname{conv}\{y_j\colon\ j \in J(x)\} \subset F(x)$ and thus

$$g(x) = f\big(\phi(x)\big) \subset f(\Delta_{J(x)}) \subset \operatorname{conv}\{y_j\colon\ j \in J\} \subset \operatorname{conv}\big\{y_j\colon\ p_j(x) \neq 0\big\} \\ \subset \operatorname{conv}\big\{y_j\colon\ y_j \in F(x)\big\} \subset F(x).$$

This complete the proof. □

Combining Theorems 3.1 and 3.4, we have the following theorems.

Theorem 3.5. *Let $(Y, \mathcal{C})$ be a pair. Then Y has weak selection property respect to any compact topological space if and only if $(Y, \mathcal{C})$ satisfies H_0-condition.*

With Theorems 3.1 and 3.3, we have some relationships between the weak selection property and the fixed point property as follows.

Theorem 3.6. *Let $(Y, \mathcal{C})$ be an l.c. metric space such that every single point set $\{x\}$ is convex and X a convex compact subset of $(Y, \mathcal{C})$. If $(Y, \mathcal{C})$ has the weak selection property, then X has the fixed point property.*

Proof. Assume $f : X \mapsto X$ is arbitrary continuous mapping. Let $B_\varepsilon(y) = \{y' \in Y\colon\ d(y, y') < \varepsilon\}$ and $\varepsilon_n = \frac{1}{2^n}$, $n = 1, 2, \ldots.$ Define $F_n : X \mapsto 2^X$ by

$$F_n(x) = B_{\varepsilon_n}\big(f(x)\big), \quad \forall x \in X,\ n = 1, 2, \ldots.$$

It is easy to check that $F_n : X \mapsto 2^X$ is a multivalued mapping with nonempty convex images and preimages relatively open in X. It follows from Theorem 3.1 and the selection property of $(Y, \mathcal{C})$ that $(Y, \mathcal{C})$ satisfies H_0-condition. Theorem 3.3 implies F_n has a fixed point $x_n^* \in X$, that is $x_n^* \in F_n(x_n^*) = B_{\varepsilon_n}(f(x_n^*))$. Since X is compact, we may assume $x_n^* \to x^* \in X$. We claim that x^* is a fixed point of f. □

Lemma 3.1. *Let X be a compact topological space, $(Y, \mathcal{C})$ be an l.c. metric space satisfying H_0-condition and $F : X \mapsto 2^Y$ a lower semicontinuous multivalued mapping with nonempty convex images. Then for any $\varepsilon > 0$ there is a continuous function $g : X \mapsto Y$ such that*

$$d\big(g(x), F(x)\big) < \varepsilon, \quad \forall x \in X.$$

Proof. Let

$$G(x) = \big\{y \in Y\colon\ F(x) \cap B_\varepsilon(y) \neq \emptyset\big\} = \big\{y \in Y\colon\ d\big(y, F(x)\big) < \varepsilon\big\}.$$

Since $(Y, \mathcal{C})$ is *l.c.* space, $G(x)$ is convex and $G^{-1}(y)$ is open since F is lower semicontinuous. By Theorem 3.4, G has a continuous selection g, which is our desired mapping. □

Lemma 3.2. *Let X be a compact topological space, $(Y, \mathcal{C})$ be an l.c. complete metric space satisfying H_0-condition and $G : X \mapsto 2^Y$ a lower semicontinuous multivalued mapping with nonempty closed convex images. Then G has a continuous selection.*

Proof. Let $\varepsilon_n = \frac{1}{2^n}$, $n = 1, 2, \ldots.$ By induction, we construct a sequence $\{g_n\}$ of continuous mapping $g_n : X \mapsto Y$ with the following two properties:

$$d\big(g_n(x), F(x)\big) < \varepsilon_n, \quad n = 1, 2, \ldots, x \in X; \tag{1}$$

$$d\big(g_n(x), g_{n-1}(x)\big) < \varepsilon_{n-2}, \quad n = 2, 3, \ldots, x \in X. \tag{2}$$

For $n = 1$, it follows from Lemma 3.1 by setting $G = F$, $\varepsilon = \varepsilon_1$, and $g_1 = g$. Assume we have constructed $g_1, \ldots, g_{n-1}$ satisfying (1) and (2). Consider the mapping

$$G(x) = \big[B_{\varepsilon_{n-1}}\big(g_{n-1}(x)\big)\big] \cap F(x), \quad \forall x \in X.$$

By (1), $G(x)$ is nonempty. Further observe that g_{n-1} is continuous, $(Y, \mathcal{C})$ is *l.c.* and F is lower semicontinuous. It follows from Theorem 7.3.10 of [10] that $G : X \mapsto 2^Y$ is lower semicontinuous with nonempty convex images. Now Lemma 3.1 tell us that there is a continuous mapping $g_n : X \mapsto 2^Y$ such that

$$d\big(g_n(x), G(x)\big) < \varepsilon_n, \quad \forall x \in X.$$

By construction of G, g_n satisfies (1) and (2).

Since the series $\sum \varepsilon_n$ converges, the sequence $\{g_n\}$ converges uniformly on X to a continuous mapping $f : X \mapsto Y$ by (2). Furthermore, this is a continuous selection of F by (1). Note that the set $F(x)$ is closed by hypothesis. □

Theorem 3.7. *Let $(Y, \mathcal{C})$ be an l.c. complete metric space. If Y has weak selection property respect to any compact topological space, then Y has selection property respect to any compact topological space.*

It is clear from the constructions of the convexity directly that Horvath's H-convexity structure (see [5]) and convexity structure defined on topological semilattice space (see [8]) satisfy H_0-condition. In fact, for each finite subset $\{y_i\colon i \in N\} \subset Y$, there is a continuous mapping $f : \Delta_N \mapsto Y$ such that $f(\Delta_J) \subset \Gamma_J \subset \operatorname{conv} A$ (respectively, $f(\Delta_J) \subset \Delta(A) \subset \operatorname{conv} A$) for each $A = \{y_j\colon j \in J\}$ and $J \subset N$ (see [5,8]).

On the other hand, observe that the weak selection property has been established on various convexity structures such as convexity structures in Michael [9], van de Vel [14], and others (see [1–3,5–8,11–13,15,16]). Again with Theorem 3.1, We conclude the following property.

Property 3.1. *Let $(Y, \mathcal{C})$ be a pair. Then convexity structure $\mathcal{C}$ satisfies H_0-condition whenever $\mathcal{C}$ is one of following convexity structures*:

(i) *Horvath's H-convexity structure* (*see* [5]);
(ii) *the convexity structure defined on topological semilattice space* (*see* [8]);
(iii) *G-convexity structure* (*see* [11,12]);
(iv) *van de Vel's uniform convexity structure* (*see* [14]);
(v) *Michael's convexity structure* (*see* [9]);
(vi) *B-convexity structure* (*see* [3]).

References

[1] C. Bardaro, R. Ceppitelli, Some further generalizations of Knaster–Kuratowski–Mazurkiewicz theorem and minimax inequalities, J. Math. Anal. Appl. 132 (1988) 484–490.

[2] C. Bardaro, R. Ceppitelli, Applications of the generalized Knaster–Kuratowski–Mazurkiewicz theorem to variational inequalities, J. Math. Anal. Appl. 137 (1989) 46–58.

[3] W. Briec, C.D. Horvath, B-convexity, Optimization 53 (2) (2004) 103–127.

[4] X.P. Ding, W.A. Kim, K.K. Tan, A new minimax inequality on H-spaces with applications, Bull. Austral. Math. Soc. 41 (1990) 457–473.
[5] C.D. Horvath, Some results on multivalued mappings and inequalities without convexity, in: B.L. Lin, S. Simons (Eds.), Nonlinear and Convex Analysis, in: Lect. Notes Pure Appl. Math., Marcel Dekker, 1987, pp. 99–106.
[6] C.D. Horvath, Contractibility and generalized convexity, J. Math. Anal. Appl. 156 (1991) 341–357.
[7] C.D. Horvath, Extension and selection theorems in topological spaces with a generalized convexity structure, Ann. Fac. Sci. Toulouse Math. (6) 2 (1993) 253–269.
[8] C.D. Horvath, J.V. Llinares Ciscar, Minimal elements and fixed point for binary relations on topological ordered spaces, J. Math. Econom. 25 (1996) 291–306.
[9] E. Michael, Convex structures and continuous selections, Canad. J. Math. 11 (1959) 556–575.
[10] E. Klein, A.C. Thompson, Theory of Correspondences, John Wiley & Sons, New York, 1984, pp. 83–89.
[11] S. Park, H. Kim, Admissible classes of multifunctions on generalized convex spaces, Proc. Nat. Sci. SNU 18 (1993) 1–21.
[12] S. Park, H. Kim, Coincidence theorems for admissible multifunctions on generalized convex spaces, J. Math. Anal. Appl. 197 (1996) 173–187.
[13] P. Saveliev, Fixed points and selections of set-valued maps on spaces with convexity, Int. J. Math. Math. Sci. 24 (2000) 595–612.
[14] M. van de Vel, A selection theorem for topological convex structures, Trans. Amer. Math. Soc. 336 (1993) 463–496.
[15] M. van de Vel, Theory of Convex Structures, North-Holland Math. Library, vol. 50, North-Holland Publishing Co., Amsterdam, 1993.
[16] G.X.Z. Yuan, KKM Theory and Applications in Nonlinear Analysis, A Series of Monographs and Textbooks in Pure and Appl. Math., Marcel Dekker, New York, 1999, pp. 42–43.