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Doubly Nonlinear Parabolic Equation with Nonlinear Boundary Conditions[1]

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This paper deals with the existence and nonexistence of global positive solutions of the doubly nonlinear parabolic equation with nonlinear boundary conditions. Necessary and sufficient conditions in order that all positive solutions exist globally are obtained by using the upper and lower solutions method.

Key Words: doubly nonlinear parabolic equation; nonlinear boundary conditions; global solutions; finite time blow-up; upper and lower solutions method.

1. INTRODUCTION

In this paper, we study the existence and nonexistence of global solutions of the problem

$$
\begin{aligned}
&(|u|^{m-1}u)_t = \left(\left(|u_x|^2 + \epsilon\right)^{(p-1)/2} u_x\right)_x, \qquad 0 < x < 1, \quad t > 0,\\
&u_x|_{x=0} = 0, \qquad u_x|_{x=1} = u^\alpha|_{x=1}, \qquad t > 0, \\
&u(x,0) = u_0(x) \geq \delta > 0, \qquad 0 \leq x \leq 1,
\end{aligned}
\tag{1}
$$

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where $\alpha \geq 0$, $m, p > 0$ are all constants; ϵ and δ are any given positive constants.

Throughout the paper we assume that $u_0(x) \in C^{2+\mu}([0,1])$ for some $0 < \mu < 1$, $u_0(x) \geq \delta > 0$ on $[0,1]$ and satisfies the compatibility conditions

$$u_{0x}(0) = 0, \qquad u_{0x}(1) = u_0^\alpha(1).$$

Our main result reads as follows:

THEOREM. *All positive solutions of* (1) *exist globally if and only if*

$$\alpha \leq \min\left\{ \frac{m}{p}, \frac{m+1}{p+1} \right\}. \tag{2}$$

In recent years questions like blow-up and global solvability for semilinear parabolic equation or systems with nonlinear boundary conditions have been intensively studied; see [2–6, 8–11, 14–18] and references therein. The Dirichlet or Cauchy problem for doubly nonlinear parabolic equation has also been studied in that extent; see [1, 7, 13, 19] and references therein.

Filo and Kacur [11] considered the local existence of a more general version of (1) and gave some sufficient conditions on the global existence of weak solutions under some assumptions of general nonlinear terms. Chipot and Filo [5] considered the interesting model problem

$$\begin{aligned} &u_t = \big(a(u_x)\big)_x, \qquad 0 < x < 1, \qquad t > 0, \\ &u_x|_{x=0} = 0, \qquad a(u_x)|_{x=1} = u^\alpha|_{x=1}, \qquad t > 0, \\ &u(x,0) = u_0(x), \qquad 0 \leq x \leq 1, \end{aligned} \tag{3}$$

where $a(\xi) \in C^4(R)$ satisfies $a(\xi) = |\xi|^{p-2}\xi$ if $|\xi| \geq \eta > 0$, $a'(\xi) > 0$, and $|a(\xi)| \leq |\xi|^{p-1}$ for all $\xi \in R$; $0 < \eta \ll 1$ is any given positive constant. They show that the solutions of (3) exist globally if and only if $\alpha \leq \frac{2p-2}{p}$ for $1 < p < 2$ while $\alpha \leq 1$ for $p \geq 2$ by using the methods of [10] and also derive some blow-up rate estimates near blow-up time.

In this paper we will consider the global-existence and global-nonexistence of solutions to (1). Our main purpose is to show that how the diffusion coefficient, nonlinearly depending on the solution and the gradient of a solution, affects the value of the critical exponent for global solvability. Necessary and sufficient conditions on the global existence of all positive (classical) solutions are obtained by using upper and lower solutions method, which is different from that of [5, 6, 10].

From a physical point of view, the differential equations in (1) have been suggested as some models; see [1–3, 6–11, 13–16] and the references therein. For example, the gas flow equation through a porous medium falls in the class of equations we consider. The nonlinear boundary conditions in (1) can be physically interpreted as a nonlinear radiation law, which here is actually an absorption law; see [3, 10, 15, 16].

By the results of [12], we know that there exists a maximal time $T : 0 < T \leq +\infty$ such that (1) has a unique solution $u(x,t)$ satisfying $u \geq \delta > 0$ on $[0,1] \times [0,T)$. Furthermore, if $T < +\infty$, then $\lim_{t \to T-} \max_{0 \leq x \leq 1} u(x,t) = +\infty$. It is also obvious that the classical comparison principle holds for (1).

2. PROOF OF THE THEOREM

We will construct various upper and lower solutions and compare them with the solutions of (1).

We will divide the proof of the theorem into two lemmas.

LEMMA 1. *Assume that* (2) *holds. Then the solution* $u(x,t)$ *of* (1) *exists globally.*

Proof.

Case 1. $m < p$ and $p \geq 1$. Take

$$w(x,t) = \left[a(x^2+1) + e^{l(t+1)}\right]^{p/(p-m)} = y^{p/(p-m)},$$

where $a = \frac{p-m}{2p}$ and $l = \max\{\log(2a), \frac{4ap}{m}(1 + \frac{2am}{p-m})((\frac{2ap}{p-m})^2 + \epsilon)^{(p-1)/2}, \frac{p-m}{p}\log(\max u_0(x))\}$.

By the direct computations we have, for $(x,t) \in (0,1) \times [0,+\infty)$,

$$1 \leq y \leq 2e^{l(t+1)},$$

$$(w^m)_t = \frac{pml}{p-m} y^{pm/(p-m)-1} e^{l(t+1)}, \qquad w_x = \frac{2ap}{p-m} y^{m/(p-m)} x,$$

$$w_{xx} = \frac{2ap}{p-m}\left\{y^{m/(p-m)} + \frac{2am}{p-m} y^{m/(p-m)-1} x^2\right\},$$

$$\begin{aligned}
&\left(\left(w_x^2+\epsilon\right)^{(p-1)/2} w_x\right)_x \\
&\quad = \left(w_x^2+\epsilon\right)^{(p-1)/2-1}\left(pw_x^2+\epsilon\right) w_{xx} \leq p\left(w_x^2+\epsilon\right)^{(p-1)/2} w_{xx} \\
&\quad = p\left(\left(\frac{2ap}{p-m}\right)^2 y^{2m/(p-m)} x^2 + \epsilon\right)^{(p-1)/2}
\end{aligned}$$

$$
\begin{aligned}
&\times \frac{2ap}{p-m}\left\{ y^{m/(p-m)} + \frac{2am}{p-m} y^{m/(p-m)-1} x^2 \right\} \\
&\le p\left(\left(\frac{2ap}{p-m}\right)^2 + \epsilon\right)^{(p-1)/2} \frac{2ap}{p-m} y^{mp/(p-m)}\left(1 + \frac{2am}{p-m}\right) \\
&\le 2p\left(\left(\frac{2ap}{p-m}\right)^2 + \epsilon\right)^{(p-1)/2} \frac{2ap}{p-m}\left(1 + \frac{2am}{p-m}\right) \\
&\quad \times y^{mp/(p-m)-1} e^{l(t+1)} \\
&\le \frac{mpl}{p-m} y^{pm/(p-m)-1} e^{l(t+1)} = (w^m)_t.
\end{aligned}
$$

Obviously

$$
w_x|_{x=0} = 0, \qquad t > 0.
$$

By using (2) and the choice of a we have

$$
\begin{aligned}
w_x|_{x=1} &= \frac{2ap}{p-m}\left(2as + e^{l(t+1)}\right)^{m/(p-m)} \\
&= w^{\alpha}|_{x=1}\left(2a + e^{l(t+1)}\right)^{(m-\alpha p)/(p-m)} \\
&\ge w^{\alpha}|_{x=1}, \qquad t > 0
\end{aligned}
$$

and

$$
w(x,0) = \left(a(x^2+1) + e^l\right)^{p/(p-m)} \ge e^{lp/(p-m)} \ge \max_{0 \le x \le 1} u_0(x) \ge u_0(x).
$$

These show that w is an upper solution of (1). By the comparison principle we have $u \le w$. It is obvious that w exists globally, and hence u exists globally.

Case 2. $m < p < 1$. Take

$$
w(x,t) = \left[a(x^{1+1/p} + 1) + e^{l(t+1)}\right]^{p/(p-m)} = y^{p/(p-m)},
$$

where

$$a = \frac{p-m}{p+1} \qquad \text{and}$$

$$l = \max\left\{ \log(2a), \frac{2(p-m)}{mp^2}\left(\frac{a(p+1)}{p-m}\right)^p \left(1 + \frac{am(p+1)}{p-m}\right), \right.$$
$$\left. \frac{p-m}{p}\log(\max u_0(x)) \right\}.$$

By the direct computations we have, for $(x,t) \in (0,1) \times [0, +\infty)$,

$$1 \le y \le 2e^{l(t+1)},$$

$$(w^m)_t = \frac{pml}{p-m} y^{pm/(p-m)-1} e^{l(t+1)},$$

$$w_x = \frac{a(p+1)}{p-m} y^{m/(p-m)} x^{1/p},$$

$$w_{xx} = \frac{a(p+1)}{p(p-m)}\left\{ y^{m/(p-m)} x^{1/p-1} + \frac{am(p+1)}{p-m} y^{m/(p-m)-1} x^{2/p} \right\},$$

$$\begin{aligned}
&\left(\left(w_x^2+\epsilon\right)^{(p-1)/2} w_x\right)_x \\
&\quad = \left(w_x^2+\epsilon\right)^{(p-1)/2-1}\left(pw_x^2+\epsilon\right) w_{xx} \\
&\quad \le \left(w_x^2+\epsilon\right)^{(p-1)/2} w_{xx} \\
&\quad = \left(\left(\frac{a(p+1)}{p-m}\right)^2 y^{2m/(p-m)} x^{2/p} + \epsilon\right)^{(p-1)/2} \frac{a(p+1)}{p(p-m)} \\
&\qquad \times y^{m/(p-m)}\left(x^{1/p-1} + \frac{am(p+1)}{p-m} y^{-1} x^{2/p}\right) \\
&\quad \le \frac{1}{p}\left(\frac{a(p+1)}{p-m}\right)^p y^{mp/(p-m)}\left(1 + \frac{am(p+1)}{p-m}\right) \\
&\quad \le \frac{2}{p}\left(\frac{a(p+1)}{p-m}\right)^p y^{mp/(p-m)-1}\left(1 + \frac{am(p+1)}{p-m}\right) e^{l(t+1)} \\
&\quad \le \frac{pml}{p-m} y^{pm/(p-m)-1} e^{l(t+1)} = (w^m)_t.
\end{aligned}$$

Obviously

$$w_x|_{x=0} = 0, \qquad t > 0.$$

By using (2) and the choice of a we have

$$\begin{aligned} w_x|_{x=1} &= \frac{a(p+1)}{p-m}\left(2a + e^{l(t+1)}\right)^{m/(p-m)} \\ &= w^{\alpha}|_{x=1}\left(2a + e^{l(t+1)}\right)^{(m-\alpha p)/(p-m)} \\ &\geq w^{\alpha}|x = 1, \qquad t > 0 \end{aligned}$$

and

$$\begin{aligned} w(x,0) &= \left(a(x^{1+1/p}+1) + e^{l}\right)^{p/(p-m)} \geq e^{lp/(p-m)} \geq \max_{0 \leq x \leq 1} u_0(x) \\ &\geq u_0(x). \end{aligned}$$

These show that w is an upper solution of (1). By the comparison principle we have $u \leq w$. It is obvious that w exists globally, and hence u exists globally.

Case 3. $m = p \geq 1$. Take $w(x,t) = (x^2+1)e^{l(t+1)}$. As in Case 1, it is easy to verify that there exists $l > 0$ such that w is an upper solution of (1). Hence u exists globally.

Case 4. $m = p < 1$. Take $w(x,t) = (x^{1+1/p}+1)e^{l(t+1)}$. As in Case 2, it is easy to verify that there exists $l > 0$ such that w is an upper solution of (1). Hence u exists globally.

Case 5. $m > p \geq 1$. It suffices to prove that for any $T > 0$ there exists $C(T) > 0$ such that

$$u(x,t) \leq C(T) < +\infty, \qquad (x,t) \in [0,1] \times [0,T]. \tag{4}$$

To this aim, denote

$$\tau = \frac{p}{p+1}, \qquad \theta = \frac{p}{m-p},$$

$$k = \max\left\{1, \frac{4^{\alpha\theta+1}}{\theta}\right\}, \qquad M = \frac{2^{2\theta+1}k^2(\theta+3)p}{m}\left(\left(4^{\theta}k\theta\right)^2 + \epsilon\right)^{(p-1)/2}.$$

It is obvious that for any given $T: 0 < T < +\infty$ there exists a natural number $N = N(T)$ such that $\frac{N\log 2}{M} \geq T \geq \frac{(N-1)\log 2}{M}$.

For any fixed i, set $\epsilon_i = a(\frac{1}{4})^i$, $i = 1, \dots, N$, where

$$a = \min\left\{4^i, 4^i\left(\frac{k}{2}\right)^{1/\tau}, 4(\max u_0(x))^{-1/\theta}\right\}.$$

Take

$$w_i(x,t) = \left\{\epsilon_i\left[e^{-M(t-T_{i-1})} - \tfrac{1}{4}(1-\phi(x))^{k/\epsilon_i^\tau}\right]\right\}^{-\theta} = \epsilon_i^{-\theta} y^{-\theta},$$

where $T_0 = 0$, $T_i = \frac{i \log 2}{M}$, $i = 1, \dots, N$, $\phi(x) = \frac{1}{2}(1-x^2)$.

It is also obvious that for any given $i = 1, \dots, N$, $y(x,t)$ is well defined on $Q_i := [0,1] \times [T_{i-1}, T_i] \cap [0,1] \times [0,T]$ and $\frac{1}{4} \le y \le 1$.

By the direct calculations we have for $(x,t) \in Q_i$

$$(w_i^m)_t = M\theta m \epsilon_i^{-m\theta} y^{-m\theta-1} e^{-M(t-T_{i-1})} \ge \frac{M\theta m}{2}\epsilon_i^{-m\theta},$$

$$w_{ix} = \frac{k\theta}{4}\epsilon_i^{-(\theta+\tau)} y^{-\theta-1}(1-\phi(x))^{k/\epsilon_i^\tau - 1} x,$$

$$\begin{aligned}
w_{ixx} = \frac{k\theta}{4}\epsilon_i^{-(\theta+\tau)}\Bigg\{ & y^{-\theta-1}(1-\phi(x))^{k/\epsilon_i^\tau-1} \\
& + \left(\frac{k}{\epsilon_i^\tau} - 1\right) y^{-\theta-1}(1-\phi(x))^{k/\epsilon_i^\tau-2} x^2 \\
& + \frac{k(\theta+1)}{4\epsilon_i^\tau} y^{-\theta-2}(1-\phi(x))^{2(k/\epsilon_i^\tau-1)} x^2 \Bigg\} \\
\le k\theta 4^\theta \epsilon_i^{-(\theta+\tau)} & \left\{1 + \frac{k}{\epsilon_i^\tau} + \frac{k(\theta+1)}{\epsilon_i^\tau}\right\},
\end{aligned}$$

$$\begin{aligned}
&\left((w_{ix}^2 + \epsilon)^{(p-1)/2} w_{ix}\right)_x \\
&\quad \le p(w_{ix}^2+\epsilon)^{(p-1)/2} w_{ixx} \\
&\quad \le 4^\theta k^2 \theta p\left((4^\theta k\theta)^2 + \epsilon\right)^{(p-1)/2} \epsilon^{-(p(\theta+\tau)+\tau)}(\epsilon_i^\tau + \theta + 2) \\
&\quad \le 4^\theta k^2 \theta p\left((4^\theta k\theta)^2 + \epsilon\right)^{(p-1)/2} \epsilon^{-(p(\theta+\tau)+\tau)}(\theta + 3) \\
&\quad = 4^\theta k^2 \theta(\theta+3) p\left((4^\theta k\theta)^2 + \epsilon\right)^{(p-1)/2} \epsilon_i^{-m\theta} \\
&\quad = \frac{M\theta m}{2}\epsilon_i^{-m\theta} \le (w^m)_t.
\end{aligned}$$

Obviously

$$w_{ix}|_{x=0} = 0, \qquad t > 0.$$

By (2) and the choice of k we have

$$\begin{aligned} w_{ix}|_{x=1} &= \epsilon_i^{-(\theta+\tau)} \frac{k\theta}{4} y^{-\theta-1}|_{x=1} \\ &= w_i^{\alpha}|_{x=1} \epsilon_i^{-(\theta+\tau)+\alpha\theta} \frac{k\theta}{4} \left(e^{-M(t-T_{i-1})} - \frac{1}{4} \right)^{(\alpha-1)\theta-1} \\ &\geq w_i^{\alpha}|_{x=1} \epsilon_i^{-(\theta+\tau)+\alpha\theta} \frac{k\theta}{4} \left(\frac{1}{4} \right)^{\alpha\theta} \\ &= w_i^{\alpha}|_{x=1} \epsilon^{p[(p+1)\alpha-(m+1)]/(m-p)(p+1)} \frac{k\theta}{4} \left(\frac{1}{4} \right)^{\alpha\theta} \\ &\geq w_i^{\alpha}|_{x=1} \frac{k\theta}{4} \left(\frac{1}{4} \right)^{\alpha\theta} \geq w_i^{\alpha}|_{x=1}, \qquad t > 0 \end{aligned}$$

and

$$\begin{aligned} w_1(x,0) &= \left\{ \epsilon_1 \left[1 - \frac{1}{4}(1-\phi)^{k/\epsilon_1^{\tau}} \right] \right\}^{-\theta} \geq \epsilon_1^{-\theta} = \left(\frac{a}{4} \right)^{\theta} \\ &\geq \max_{0 \leq x \leq 1} u_0(x) \geq u_0(x), \qquad 0 \leq x \leq 1. \end{aligned}$$

For $i = 2, \ldots, N$, we have

$$\begin{aligned} w_{i-1}(x, T_{i-1}) &= \left\{ \epsilon_{i-1} \left[e^{-M(T_{i-1}-T_{i-2})} - \tfrac{1}{4}(1-\phi)^{k/\epsilon_{i-1}^{\tau}} \right] \right\}^{-\theta} \\ &\leq \left(\epsilon_{i-1} \tfrac{1}{4} \right)^{-\theta} = \epsilon_i^{-\theta} \leq w_i(x, T_{i-1}), \qquad 0 \leq x \leq 1. \end{aligned}$$

These show that w_i is an upper solution of (1) on Q_i. By the comparison principle we have $u \leq w_i$ on Q_i, $i = 1, \ldots, N$. It is obvious that there exists $C(T): 0 < C(T) < +\infty$ such that (4) holds. Hence the solutions of (1) exist globally.

Case 6. $m > p$ and $p < 1$. As in Case 5, it suffices to prove that for any $T > 0$ there exists $C(T) > 0$ such that (4) holds. To this aim, denote $\tau = \frac{p}{p+1}$, $\theta = \frac{p}{m-p}$, $k = \max\{1, 4^{\alpha\theta+1}/(p+1)\theta\}$, $M = \theta^{p-1}k^{p+1}(2\tau + p(\theta+2))/8^p\tau^{p+1}m$, $N = N(T), T_i$ as above. Take

$$w_i(x,t) = \left\{ \epsilon_i \left[e^{-M(t-T_{i-1})} - \tfrac{1}{4}(1-\phi(x))^{k/\epsilon_i^{\tau}} \right] \right\}^{-\theta} = \epsilon_i^{-\theta} y^{-\theta},$$

where $\phi(x) = \frac{1}{2}(1 - x^{1+1/p})$, $\epsilon_i = a(\frac{1}{4})^i$, $i = 1, \dots, N$, and a is a constant to be determined later. As in Case 5, it is easy to prove that, for any fixed i, there exists $a > 0$ such that w_i is an upper solution of (1) on Q_i. By the comparison principle we have $u \le w_i$ on Q_i, $i = 1, \dots, N$. It is obvious that there exists $C(T): 0 < C(T) < +\infty$ such that (4) holds. Hence the solutions of (1) exist globally.

This completes the proof of Lemma 1.

LEMMA 2. *If* $\alpha > \min\{\frac{m}{p}, \frac{m+1}{p+1}\}$, *then the solution* u *of* (1) *blows up in a finite time*.

Proof. First, denote $\Theta(z) = z^{(p-1)\alpha - m + 1}(1 + \alpha z^{\alpha - 1})$; then it is easy to verify that there exists $z_1 > 0$ such that $\Theta(z)$ is monotone for $z \ge z_1$. Choose z_0 such that $0 < z_0 < \min\{z_1, \delta\} \le u_0(x)$; then there exist positive constants C_1 and C_2 such that $C_1 \le \Theta(z) \le C_2$ for $z_0 \le z \le z_1$. Let $z(s)$ be a solution of the following problem:

$$z'(s) = z^{\alpha}(s), \qquad z(0) = z_0. \tag{5}$$

Denote the maximal existence time of solution of (5) by s_∞. Evidently, $s_\infty \le +\infty$, $z(s) \to \infty$ as $s \to s_\infty$ and there exists $0 < \lambda \le 1$ such that $z_0 < z(\lambda) \le \delta \le u_0(x)$ and $\lambda < s_\infty$. Denote

$$C_3 = \min\left\{\left(\frac{\lambda}{2}\right)^{p+1}, \frac{\lambda^2 p}{4}\left(1 + \epsilon z_0^{-2\alpha}\right)^{(p-1)/2}\right\},$$

$$C_4 = \min\left\{\frac{C_1 C_3}{m C_2}, \frac{C_3 \Theta(z_1)}{m C_2}, \frac{C_3}{m}\right\}, \qquad \text{and}$$

$$C_5 = \min\left\{\frac{C_1 C_3}{m C_2}, \frac{C_1 C_3}{m \Theta(z_1)}, \frac{C_3}{m}\right\}.$$

If $\Theta(z)$ is nondecreasing for $z \ge z_1$, then let $\theta(t)$ be a solution of

$$\theta'(t) = C_4 \Theta(z(\theta(t))), \qquad t > 0, \quad \theta(0) = 0. \tag{6}$$

If $\Theta(z)$ is nonincreasing for $z \ge z_1$, then let $\theta(t)$ be a solution of

$$\theta'(t) = C_5 \Theta(z(\theta(t) + \lambda)), \qquad t > 0, \quad \theta(0) = 0.$$

Since $\alpha > \min\{\frac{m}{p}, \frac{m+1}{p+1}\}$, the direct calculations yield

$$\int^{+\infty} \frac{dz}{z^{\theta}\Theta(z)} = \int^{+\infty} \frac{dz}{z^{p\alpha + 1 - m}(1 + \alpha z^{\alpha - 1})} < +\infty.$$

Therefore, there exists $T_0 : 0 < T_0 < +\infty$ such that

$$\lim_{t\to T_0^-} z(\theta(t)) = \infty \qquad \text{or} \qquad \lim_{t\to T_0^-} z(\theta(t)+\lambda) = +\infty.$$

Set $w(x,t) = z(\theta(t) + \eta(x))$, where $\eta(x) = \frac{\lambda}{4}(x+1)^2$; then there exists $T_1 : 0 < T_1 \le T_0 < \infty$ such that $w(x,t)$ is well defined on $[0,1]\times[0,T_1)$ and $\lim_{t\to T_1^-}\|w(\cdot,t)\|_\infty = +\infty$.

Now we show that $w(x,t)$ is a lower solution of (1).

By the direct calculations we have for $0 < x < 1$ and $0 < t < T_1$

$$w_x = \frac{\lambda}{2}(x+1)z^\alpha \ge \frac{\lambda}{2}z^\alpha,$$

$$w_{xx} = \frac{\lambda}{2}z^\alpha + \frac{\alpha\lambda^2}{4}z^{2\alpha-1}(x+1)^2 \ge \frac{\lambda^2}{4}z^\alpha(1+\alpha z^{\alpha-1}).$$

In the following we verify that

$$\left(\left(w_x^2+\epsilon\right)^{(p-1)/2}w_x\right)_x \ge (w^m)_t. \tag{7}$$

First we show that

$$\left(\left(w_x^2+\epsilon\right)^{(p-1)/2}w_x\right)_x \ge C_3 z^{m-1+\alpha}\Theta(z). \tag{8}$$

In fact, for $p \ge 1$, we have

$$\begin{aligned}\left(\left(w_x^2+\epsilon\right)^{(p-1)/2}w_x\right)_x &= \left(w_x^2+\epsilon\right)^{(p-1)/2-1}\left(pw_x^2+\epsilon\right)w_{xx}\\ &\ge w_x^{p-1}w_{xx} = \left(\frac{\lambda}{2}\right)^{p-1} z^{\alpha p}\left(\frac{\lambda}{2}+\frac{\alpha\lambda^2}{4}z^{\alpha-1}\right)\\ &\ge \left(\frac{\lambda}{2}\right)^{p+1} z^{\alpha p}(1+\alpha z^{\alpha-1})\\ &= \left(\frac{\lambda}{2}\right)^{p+1} z^{m-1+\alpha}\Theta(z) \ge C_3 z^{m-1+\alpha}\Theta(z).\end{aligned}$$

For $p < 1$, we have

$$\begin{aligned}
&\left(\left(w_x^2 + \epsilon\right)^{(p-1)/2} w_x\right)_x \\
&\quad = \left(w_x^2 + \epsilon\right)^{(p-1)/2-1}\left(p w_x^2 + \epsilon\right) w_{xx} \\
&\quad \geq p\left(w_x^2 + \epsilon\right)^{(p-1)/2} w_{xx} \geq \frac{p\lambda^2}{4}\left(\left(\lambda z^{\alpha}\right)^2 + \epsilon\right)^{(p-1)/2} z^{\alpha}\left(1 + \alpha z^{\alpha-1}\right) \\
&\quad \geq \frac{p\lambda^2}{4} z^{p\alpha}\left(\lambda^2 + \epsilon z_0^{-2\alpha}\right)^{(p-1)/2}\left(1 + \alpha z^{\alpha-1}\right) \\
&\quad = \frac{p\lambda^2}{4}\left(\lambda^2 + \epsilon z_0^{-2\alpha}\right)^{(p-1)/2} z^{m-1+\alpha}\Theta(z) \geq C_3 z^{m-1+\alpha}\Theta(z).
\end{aligned}$$

This shows that (8) holds.

Next, we consider the following two cases respectively.

Case 1. $\Theta(z)$ is nondecreasing for $z \geq z_1$.

If $z = z(\theta(t) + \eta(x)) \in [z_0, z_1]$, then $z(\theta(t)) \in [z_0, z_1]$. Hence we have

$$\Theta(z(\theta(t))), \Theta(z(\theta(t) + \eta(x))) \in [C_1, C_2]. \tag{9}$$

By (6), (8), (9), and the choice of C_4 we have

$$\begin{aligned}
(w^m)_t &= m z^{m-1} z' \theta'(t) = m z^{m-1+\alpha} C_4 \Theta(z(\theta(t))) \\
&\leq m z^{m-1+\alpha} C_4 C_2 \leq z^{m-1+\alpha} C_1 C_3 \leq C_3 z^{m-1+\alpha}\Theta(z) \\
&\leq \left(\left(w_x^2 + \epsilon\right)^{(p-1)/2} w_x\right)_x.
\end{aligned}$$

If $z = z(\theta(t) + \eta(x)) \geq z_1 \geq z(\theta(t))$, then we have

$$\Theta(z(\theta(t))) \leq C_2, \qquad \Theta(z(\theta(t) + \eta(x))) \geq \Theta(z_1). \tag{10}$$

By (6), (8), (10), and the choice of C_4 we have

$$\begin{aligned}
(w^m)_t &= m z^{m-1} z' \theta'(t) = m z^{m-1+\alpha} C_4 \Theta(z(\theta(t))) \\
&\leq m z^{m-1+\alpha} C_4 C_2 \leq z^{m-1+\alpha} C_3 \Theta(z_1) \\
&\leq C_3 z^{m-1+\alpha}\Theta(z) \leq \left(\left(w_x^2 + \epsilon\right)^{(p-1)/2} w_x\right)_x.
\end{aligned}$$

If $z = z(\theta(t) + \eta(x)) \geq z(\theta(t)) \geq z_1$, then we have

$$\begin{aligned}
(w^m)_t &= m z^{m-1} z' \theta'(t) = m z^{m-1+\alpha} C_4 \Theta(z(\theta(t))) \\
&\leq z^{m-1+\alpha} C_3 \Theta(z) \leq \left(\left(w_x^2 + \epsilon\right)^{(p-1)/2} w_x\right)_x.
\end{aligned}$$

Case 2. $\Theta(z)$ is nonincreasing for $z \geq z_1$.

The verification of (7) is similar to that of Case 1.

Obviously

$$w_x|_{x=0} \geq 0, \qquad t > 0. \tag{11}$$

$$w_x|_{x=1} = \lambda z^\alpha|_{x=1} \leq z^\alpha|_{x=1} = w^\alpha|_{x=1}, \qquad t > 0. \tag{12}$$

$$w(x,0) = z(\eta(x)) \leq z(\lambda) \leq \delta \leq u_0(x), \qquad x \in [0,1]. \tag{13}$$

From (7), (11)–(13) we see that w is a lower solution of (1). By the comparison principle we have $w \leq u$. Obviously, w blows up in a finite time. And hence u blows up in a finite time.

The proof of Lemma 2 is completed.

By Lemmas 1 and 2 we get our theorem.

Remark. The results of the paper may be extended to the N-dimensional form of problem (1),

$$\begin{aligned} (|u|^{m-1}u)_t &= \Delta_{p,\epsilon} u, \qquad x \in \Omega \subset R^N, \qquad t > 0, \\ \tfrac{\partial u}{\partial \eta} &= u^\alpha, \qquad x \in \partial\Omega, \qquad t > 0, \\ u(x,0) &= u_0(x) \geq \delta > 0, \qquad x \in \overline{\Omega}, \end{aligned}$$

where

$$\Delta_{p,\epsilon} = \sum_{i=1}^N \left(\left(|u_{x_i}|^2 + \epsilon \right)^{(p-1)/2} u_{x_i} \right)_{x_i} \quad \text{or}$$

$$\Delta_{p,\epsilon} = \sum_{i=1}^N \left(\left(|\nabla u|^2 + \epsilon \right)^{(p-1)/2} u_{x_i} \right)_{x_i}.$$

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