Journal of Algebra 347 (2011) 53–59

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Groups generated by a finite Engel set

Alireza Abdollahi [a,b,1], Rolf Brandl [c], Antonio Tortora [d,*,2]

[a] Department of Mathematics, University of Isfahan, 81746-73441, Isfahan, Iran
[b] School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran
[c] Mathematisches Institut, Am Hubland 12, 97074 Würzburg, Germany
[d] Dipartimento di Matematica, Università di Salerno, Via Ponte don Melillo, 84084, Fisciano (SA), Italy

ARTICLE INFO

Article history:
Received 4 March 2011
Available online 5 October 2011
Communicated by E.I. Khukhro

MSC:
20F45
20F19

Keywords:
Engel set
Nilpotent group

ABSTRACT

A subset S of a group G is called an Engel set if, for all $x, y \in S$, there is a non-negative integer $n = n(x, y)$ such that $[x, _n y] = 1$. In this paper we are interested in finding conditions for a group generated by a finite Engel set to be nilpotent. In particular, we focus our investigation on groups generated by an Engel set of size two.

1. Introduction

A subset S of a group G is called an Engel set if, for all $x, y \in S$, there is a non-negative integer $n = n(x, y)$ such that $[x, _n y] = 1$. It is known that, for a group G satisfying Max-ab, a normal subset $S \subseteq G$ is an Engel set if and only if it is contained in the Fitting subgroup of G (see [7, Theorem 7.23]; see also [1]) and so, in this case, $\langle S \rangle$ is nilpotent whenever S is finite. However, a group generated by a finite Engel set is not necessarily nilpotent: Golod's examples show that there exist infinite non-nilpotent groups generated by an Engel set with three or more elements (see [5]). Furthermore, if S is an Engel set of size three, then an easier example of a non-nilpotent group generated by

* Corresponding author.
E-mail addresses: a.abdollahi@math.ui.ac.ir (A. Abdollahi), antortora@unisa.it (A. Tortora).

1 The author's research was in part supported by a grant from IPM (No. 90050219) and by the Center of Excellence for Mathematics, University of Isfahan.

2 The author is grateful to Professor A. Rhemtulla for interesting discussions and useful suggestions. He also wishes to thank the Department of Mathematical and Statistical Sciences at the University of Alberta for its fine hospitality while part of this work was being carried out.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2011.09.013

S is the wreath product of the alternating group of degree 5 with the cyclic group of order 3: it has a *presentation of type* (r,s,t) (see [3]), i.e. $S=\{a,b,c\}$ where $\langle a,b\rangle$ is nilpotent of class r, $\langle a,c\rangle$ is nilpotent of class s and $\langle b,c\rangle$ is nilpotent of class t. All these groups are not soluble, but the nilpotency does not hold even in the soluble case. In [3] it was shown that every group with a presentation of type $(1,2,2)$ is soluble of length at most 3 and that there are non-nilpotent groups of this type.

In this paper, we first get that any nilpotent-by-abelian group generated by a finite Engel set is nilpotent and then we focus on groups generated by an Engel set of size two. In particular, we prove that such a group is nilpotent whenever it is abelian-by-(nilpotent of class 2). This is the best possible result in the soluble case. In fact, we construct by GAP (see [4]) a non-nilpotent counterexample which is abelian-by-(nilpotent of class 3). On the other hand, some of the counterexamples in [3], mentioned above, are abelian-by-(nilpotent of class 2) and generated by an Engel set of size three.

2. Groups that are nilpotent-by-abelian

We start with a result that is certainly already known. It generalizes, for metabelian groups, two basic properties of commutators.

Lemma 2.1. *Let G be a metabelian group and x, y, z be elements of G. For all positive integers n, we have*

(i) $[x^{-1},{}_n y]=[x,{}_n y]^{-x^{-1}}$;
(ii) $[xy,{}_n z]=[x,{}_n z][x,{}_n z,y][y,{}_n z]$.

Proof. Since G is metabelian, every g in G induces on G' an endomorphism $-1+g$ that maps u to $u^{-1}u^g$, and any two of these endomorphisms commute. We thus have

$$\left[x^{-1},{}_n y\right]=\left([x,y]^{-x^{-1}}\right)^{(-1+y)^{n-1}}=[x,y]^{-(-1+y)^{n-1}x^{-1}}=[x,{}_n y]^{-x^{-1}}.$$

The proof of (ii) is similar. □

As a consequence of Lemma 2.1, we get:

Lemma 2.2. *If G is a metabelian group generated by an Engel set S, then any $x\in S$ is a left Engel element. In particular, G is locally nilpotent.*

Proof. Take a finite subset of S, say $T=\{x_1,\dots,x_r\}$, and suppose $[x_i,{}_n x_j]=1$ for all $1\leqslant i,j\leqslant r$. By the previous lemma, every x_i is a left n-Engel element in G. Then $(-1+x_i)^n=0$. It follows that any product in the endomorphisms $-1+x_i$ of weight $(n-1)r+1$ is trivial. Hence $\langle T\rangle$ is nilpotent of class at most $(n-1)r+2$. This proves that G is locally nilpotent. □

For a finite Engel set, we then obtain the following:

Theorem 2.3. *Let G be a nilpotent-by-abelian group generated by a finite Engel set. Then G is nilpotent.*

Proof. If N is a normal nilpotent subgroup of G such that G/N is abelian, then G/N' is nilpotent by Lemma 2.2 and so G is nilpotent by a well-known result of P. Hall. □

3. Engel sets of size two

Let $G=\langle x,y\rangle$ be a group and assume that $\{x,y\}$ is an Engel set. Then $[x,{}_n y]=1$ and $[y,{}_m x]=1$ for some positive integers n, m. We also say that the elements x and y are *mutually Engel* and, whenever $n\geqslant m$, that they are *mutually n-Engel*. If $n=m=2$, then G is obviously nilpotent of class at most 2 and the nilpotency still holds for $n=2$ and $m=3$.

Proposition 3.1. *Let $G = \langle x, y\rangle$ be an arbitrary group such that $[x, y, y] = 1$ and $[y, x, x, x] = 1$. Then G is nilpotent of class at most* 3.

Proof. By the Hall–Witt identity we have

$$\big[[y,x], x^{-1}, y\big]^x \big[x, y^{-1}, [y,x]\big]^y \big[y, [y,x]^{-1}, x\big]^{[y,x]} = 1,$$

from which it follows

$$\big[y, x, x^{-1}, y\big] = 1$$

since $[x, y^{-1}] = [x, y]^{-1}$ and $[y, [y,x]^{-1}] = [x, y, y]^{-1} = 1$. Then $[y, x, x, y] = 1$ and hence $[y, x, x] \in Z(G)$. Now $[x, y, y] = [y, x, x] = 1$ modulo $Z(G)$, so $G/Z(G)$ is nilpotent of class $\leqslant 2$ and G is nilpotent of class $\leqslant 3$. □

However, as we will see in the next section, this is not true in general, even in the soluble case. We are therefore led to consider extra conditions for a group generated by an Engel set of size two to be nilpotent. In the sequel, we will turn our attention to groups which are abelian-by-(nilpotent of class 2).

Let G be any abelian-by-(nilpotent of class 2) group generated by two mutually Engel elements x and y. By assumption $[x, {}_n y] = 1$ and $[y, {}_n x] = 1$ for some n. Suppose, by way of contradiction, that G is not nilpotent. Then G has a non-nilpotent finite image by Theorem 10.51 of [7] and so we may assume that G is finite.

Using induction on the order of the group, we may assume that G is a minimal counterexample. It follows that G contains a unique minimal normal subgroup A such that G/A is nilpotent. As G is not nilpotent there is a maximal subgroup H that is not normal. On the other hand G/A is nilpotent, therefore $A \nleqslant H$ (otherwise $H/A \lhd G/A$ implies that $H \lhd G$). Thus $G = AH$. The group $A \cap H$ is normal in G and $A \cap H < A$. The minimality of A then forces $A \cap H = 1$.

Clearly, A is an elementary abelian p-group for some prime p and H is nilpotent. Let P be the Sylow p-subgroup of H. Then $AP/A \lhd G/A$ and so AP is the Sylow p-subgroup of G. Since AP is nilpotent, we have that $[A, AP] < A$ and by the minimality of A, the normal subgroup $[A, AP]$ must be trivial. Thus $[A, P] = 1$ and $P^G = P^{AH} = P^H = P$, that is $P \lhd G$. But $A \nleqslant P$, hence $P = 1$ and H is a Hall p'-subgroup of G.

Lemma 3.2. *Every nontrivial element of $Z(H)$ acts fixed point freely on A by conjugation.*

Proof. For all $z \in Z(H)$ and $h \in H$, $C_A(z)^h = C_A(z)$ and thus $C_A(z) \lhd G$. As $\langle z\rangle$ cannot be normal in G, we get $C_A(z) = 1$ by minimality of A. □

The next lemma shows that H is nilpotent of class 2 and that we can restrict our attention to $n = 3$.

Lemma 3.3. *Let $G = AH = \langle x, y\rangle$ be a minimal counterexample that is abelian-by-(nilpotent of class 2). Then $A = \gamma_3(G)$, $[x, y, y, y] = 1$ and $[y, x, x, x] = 1$.*

Proof. Of course, $A \subseteq \gamma_3(G)$ by minimality of A. Let $q \neq p$ be a prime. Then any q-subgroup of $\gamma_3(G)$ is necessarily trivial. But G/A is a p'-group, therefore $A = \gamma_3(G)$ and H is nilpotent of class 2.

Assuming now $[x, {}_{n-1} y] \neq 1$, we will prove that $n \leqslant 3$. Let $y = ah$ where $a \in A$, $h \in H$, and suppose $n > 3$. We have $[x, y, y] \in A$ and $n - 2 \geqslant 2$, so that $[x, {}_{n-2} y]$ and $[x, {}_{n-2} y, y]$ lie in A. It follows that

$$\big[x, {}_{n-2} y, y^p\big] = [x, {}_{n-2} y, y]^p = 1.$$

Notice that $y^p = a_1 h^p$ with $a_1 \in A$ and $h = h^{\alpha p}$ for some integer α. Thus

$$1 = \left[x,{}_{n-2}y, y^p\right] = \left[x,{}_{n-2}y, a_1 h^p\right] = \left[x,{}_{n-2}y, h^p\right]$$

and

$$1 = \left[x,{}_{n-2}y, h^{\alpha p}\right] = [x,{}_{n-2}y, h].$$

But then

$$1 = [x,{}_{n-2}y, ah] = [x,{}_{n-2}y, y],$$

that is a contradiction. □

We need one more preliminary lemma before proving our main result.

Lemma 3.4. *Let $x = ah$, $y = bk$ where $a, b \in A$ and $h, k \in H$. If $[x, y] = [h, k]$, then*

$$\left[a, k^{-1}\right] = \left[b, h^{-1}\right], \qquad [a, h] = 1 \quad \text{and} \quad [b, k] = 1,$$

with $a \neq 1$ and $b \neq 1$.

Proof. We have

$$[h, k] = [x, y] = [ah, bk] = [a, k]^h [h, k][h, b]^k.$$

This implies $[a, k]^h [h, b]^{h^{-1}kh} = 1$ and then $[a, k]^{k^{-1}} = [b, h]^{h^{-1}}$, or equivalently $[a, k^{-1}] = [b, h^{-1}]$.

As $G \neq H$ we must have that one of a, b is nontrivial. Without loss of generality, we may assume $a \neq 1$. Clearly, $[y, x, x] \in A$ and $1 \neq [y, x] \in Z(H)$. Then $1 = [y, x, x, x] = [y, x, x, h]$ and

$$[x, h]^{[y,x]} = \left[x^{[y,x]}, h\right] = \left[[y, x, x]^{-1} x, h\right] = [x, h].$$

Thus $1 = [x, h, [y, x]] = [[a, h]^h, [y, x]] = [a, h, [y, x]]^h$, so $[a, h]$ is fixed by $[y, x]$. By Lemma 3.2 it follows that $[a, h] = 1$. As a consequence $b \neq 1$, otherwise $[a, k] = 1$ and $[a, [h, k]] = 1$. Arguing as for a, we then conclude that $[b, k] = 1$. □

Theorem 3.5. *Let G be any abelian-by-(nilpotent of class 2) group generated by two mutually Engel elements x and y. Then G is nilpotent.*

Proof. Put $x = ah$, $y = bk$ where $a, b \in A$ and $h, k \in H$. Then $[x, y] = [h, k]c$ with $[h, k] \in Z(H)$ and for some $c \in A$. By Lemma 3.3 we know that

$$[x, y, y], [y, x, x] \in A \quad \text{and} \quad [x, y, y, y] = [y, x, x, x] = 1.$$

This gives

$$\left[x, y, y^p\right] = 1 \quad \text{and} \quad \left[x, y, x^p\right] = 1.$$

If $\langle x^p, y^p\rangle \cap A \neq 1$, the commutator $[x, y]$ commutes with a nontrivial element of A. Thus $[h, k] = 1$ by Lemma 3.2, and $[x, y] \in A$. Indeed $G' \leqslant A$ and G is nilpotent by Lemma 2.2. Therefore $A \cap \langle x^p, y^p\rangle = 1$

and we may assume $H = \langle x^p, y^p\rangle$, since $\langle h,k\rangle \simeq \langle h,k\rangle A/A = \langle x^p, y^p\rangle A/A \simeq \langle x^p, y^p\rangle$. It follows that c must be trivial. Then $1 \neq [x,y] = [h,k]$ and, by Lemma 3.4, we have

$$\left[a,k^{-1}\right] = \left[b,h^{-1}\right] \quad \text{and} \quad [a,h] = 1,$$

with $a \neq 1$.

Now, the Hall–Witt identity

$$\left[a,k^{-1},h\right]^k \left[k,h^{-1},a\right]^h \left[h,a^{-1},k\right]^a = 1$$

implies

$$\left[a,k^{-1},h\right]^k = \left[k,h^{-1},a\right]^{-h}.$$

But $[k,h^{-1},a]$ commutes with h, so $[[a,k^{-1}],h] = [[b,h^{-1}],h]$ commutes with $h^{k^{-1}}$. Then $[b,h,h]^{h^{-1}} = [b,h^{-1},h]^{-1}$ commutes with $h^{k^{-1}}$, in particular $[b,h,h]$ commutes with $h^{k^{-1}h} = h^{k^{-1}}$. Hence $[b,h,h] \in C_A(h^{k^{-1}})$.

Let $B = C_A(h^{k^{-1}})$ and $K = \langle h, h^{k^{-1}}\rangle A$. Then $B \lhd K$ because $[h^{-1},k] \in Z(H)$. If q is the order of h, we also have $B = [b,h^q]B = [b,h]^q B$. However, the order of $[b,h]$ is coprime with q, thus $[b,h] \in B$ and $[a,k^{-1}] = [b,h^{-1}] \in B$. So $[a,k^{-1},h^{k^{-1}}] = 1$ and $[k,a,h] = 1$. Finally, from

$$[a,k,h]^{k^{-1}} \left[k^{-1},h^{-1},a\right]^h \left[h,a^{-1},k^{-1}\right]^a = 1,$$

it follows $[k,h,a] = 1$ which contradicts Lemma 3.2. □

When x and y are mutually 3-Engel elements, we get thanks to GAP that the group G in Theorem 3.5 is nilpotent of class at most 8. In fact, using the ANU Nilpotent Quotient package of W. Nickel (see [6]), we can construct the largest nilpotent quotient of G which is isomorphic to G.

Also notice that the theorem above can be extended to a group generated by more than two mutually Engel elements, provided that none of the generators has order divisible by 2 or 3.

Corollary 3.6. *Let S be a finite Engel set and assume that $G = \langle S\rangle$ is abelian-by-(nilpotent of class 2). If every element in S has order that is not divisible by 2 or 3, then G is nilpotent.*

Proof. For all $x, y \in S$, the subgroup $\langle x,y\rangle$ is nilpotent by Theorem 3.5. Thus the claim follows by Proposition 1 of [3]. □

Using Theorem 3.5, we now present a criterion for nilpotency of a finite soluble group depending on information on its Sylow subgroups.

Corollary 3.7. *Let $G = \langle x,y\rangle$ be a finite soluble group with x and y mutually Engel elements. If all Sylow subgroups of G are nilpotent of class $\leqslant 2$, then G is nilpotent.*

Proof. Let G be a counterexample of least possible order and let N be a minimal normal subgroup of G. Then G/N is nilpotent by minimality. Moreover, all Sylow subgroups of G/N are nilpotent of class $\leqslant 2$, so that G/N is nilpotent of class $\leqslant 2$. On the other hand N is abelian, because G is soluble. Hence G is abelian-by-(nilpotent of class 2) and thus nilpotent by Theorem 3.5: a contradiction. □

4. Examples

Our first example shows that, for any positive integer n, there exists a group generated by two mutually n-Engel elements which are not $(n-1)$-Engel. This is the dihedral group of order 2^{n+1}.

Example 4.1. Let us consider $G = \langle x, y \mid x^2 = y^2 = 1,\ (xy)^{2^n} = 1\rangle$. If $z = xy$, then $[x, y] = z^2$ and $z^x = z^y = z^{-1}$. For any $k \geqslant 1$, we get by induction $[x, {}_k y] = z^{-(-2)^k}$ and $[y, {}_k x] = z^{(-2)^k}$. Therefore $[x, {}_{n-1} y], [y, {}_{n-1} x] \neq 1$ whereas $[x, {}_n y] = [y, {}_n x] = 1$. Thus x and y are mutually n-Engel elements. Furthermore, we have $G = \langle y, z\rangle$ and $[y, {}_2 z] = [z, {}_n y] = 1$, so even y and z are mutually n-Engel elements.

The following is an example obtained by GAP of a non-nilpotent group G generated by two mutually 3-Engel elements, for which $\gamma_4(G)$ is abelian.

Example 4.2. Let $W = S_3\, wr\, \mathbb{Z}_4$ be the wreath product of the symmetric group of degree 3 with the cyclic group of order 4. Thus, $|W| = 2^6 3^4$. We have $W = Q \ltimes N$, where N is an elementary abelian group of order 3^4 and $Q \simeq \mathbb{Z}_2\, wr\, \mathbb{Z}_4$. Moreover, Q is nilpotent of class 4. With the notation of GAP, let ele := Elements(W), x := ele[4] and y := ele[228]. Then $o(x) = o(y) = 4$ and $[x, {}_3 y] = [y, {}_3 x] = 1$. As $o(xy^{-1}) = 6$, the subgroup $G = \langle x, y\rangle$ of W is not nilpotent. Finally, one can check that $G = S \ltimes N$ where S is a group of order 2^5 which is nilpotent of class 3.

For completeness reasons, we point out that $W = \langle x, y'\rangle$ with y' := ele[509] of order 6 and $[x, {}_3 y'] = [y', {}_4 x] = 1$. Hence, W is a generated by two mutually 4-Engel elements and is not nilpotent.

Notice that some more non-nilpotent groups generated by two mutually n-Engel elements can be found in the literature. For instance, Corollary 0.2 of [2] says that, for $n \geqslant 26$, the group $G(n) = \langle x, y \mid [x, {}_n y] = [y, {}_n x] = 1\rangle$ is not nilpotent. We can improve upon this. In fact, we show below that $G(4)$ is not soluble, because it has a quotient isomorphic to the symmetric group S_8.

Example 4.3. Let S_8 be the symmetric group of degree 8, and let $x = (1, 2, 3, 4)(5, 6)(7, 8)$ and $y = (1, 3)(2, 5)(4, 7, 6, 8)$. Put $x_n = [x, {}_n y]$ and $y_n = [y, {}_n x]$, for any $n \geqslant 0$ (so $x_0 = x,\ y_0 = y$). We then have

$$
\begin{aligned}
x_1 &= (1, 6)(2, 7)(3, 8)(4, 5), & y_1 &= (1, 6)(2, 7)(3, 8)(4, 5),\\
x_2 &= (1, 5)(4, 6), & y_2 &= (2, 4)(5, 7),\\
x_3 &= (1, 5)(2, 3)(4, 6)(7, 8), & y_3 &= (1, 3)(2, 4)(5, 7)(6, 8),\\
x_4 &= (1), & y_4 &= (1).
\end{aligned}
$$

In particular, $[x, {}_4 y] = [y, {}_4 x] = 1$. However x and y are of order 4, but $xy = (1, 5, 8, 6, 2)(3, 7, 4)$ is of order 15. The subgroup $G = \langle x, y\rangle$ is thus non-nilpotent. Using GAP, it is easy to see that $|G| = 8!$, so $G = S_8$.

We now discuss the situation of Example 4.3. Clearly, if the pair $(x, y) \in G \times G$ satisfies the condition

$$[x, {}_4 y] = [y, {}_4 x] = 1, \tag{$*$}$$

then all conjugates (x^g, y^g), for all $g \in G$, satisfy the analogous property. Therefore it is sensible to consider classes under conjugation.

It turns out by GAP that the only pairs $(x, y) \in G \times G$ satisfying $(*)$, that generate a non-nilpotent subgroup of G, have both x and y with cycle structure of type (4)(2)(2) and, in addition, x, y necessarily generate the whole group G. Without loss of generality, we may assume $x = (1, 2, 3, 4)(5, 6)(7, 8)$.

For this x, we calculated all solutions $y \in G$ of $(*)$. We ended up with precisely 64 solutions. Of course, the group $C_G(x)$ of order 32 acts on the pairs of solutions. The stabilizer of this action is $C_G(x) \cap C_G(y) = Z(G) = 1$, so that we obtain two essentially distinct solutions.

Other examples? Suppose that in some finite group we can find Sylow p-subgroups P, Q and elements $x \in P$, $y \in Q$ such that $[x, y] \in P \cap Q$. Let c be the nilpotency class of P. Thus, $[x,_{c+1} y] = [y,_{c+1} x] = 1$. If xy is not a p-element, then $\langle x, y\rangle$ is non-nilpotent. The groups in Examples 4.2 and 4.3 are of this form for $p = 2$. It would be very interesting to find analogous examples for all odd primes p.

References

[1] A. Abdollahi, Engel graph associated with a group, J. Algebra 318 (2007) 680–691.
[2] D.J. Collins, A. Juhász, Engel groups III, Israel J. Math. 174 (2009) 73–91.
[3] G. Endimioni, G. Traustason, Groups that are pairwise nilpotent, Comm. Algebra 36 (2008) 4413–4435.
[4] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.4.12, 2008, http://www.gap-system.org.
[5] E.S. Golod, Some problems of Burnside type, in: Proc. Int. Congr. Math., Moscow, 1966, in: Amer. Math. Soc. Transl. Ser. 2, 1968, pp. 284–289.
[6] W. Nickel, Computation of nilpotent Engel groups, J. Aust. Math. Soc. Ser. A 67 (1999) 214–222.
[7] D.J.S. Robinson, Finiteness Conditions and Generalized Soluble Groups, Parts 1 and 2, Springer-Verlag, Berlin, 1972.