Leibniz Representations of Lie Algebras

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A Leibniz representation of the Lie algebra \( g \) is a vector space \( M \) equipped with two actions (left and right) \([\cdot, \cdot] \colon g \otimes M \to M \) and \([\cdot, \cdot] \colon M \otimes g \to M \) which satisfy the relations

\[
[x, [y, z]] - [[x, y], z] = [[x, z], y],
\]

when one of the variables is in \( M \) and the two others are in \( g \). In this paper we show that the category \( L(g) \) of finite dimensional Leibniz representations of a finite dimensional semi-simple Lie algebra is not semi-simple, but that \( L(g) \) has global dimension 2. We give an explicit description of the extensions of simple objects and we obtain the description of the quiver of \( L(g) \) (in the sense of Gabriel). It turns out that \( L(g) \) is tame only for \( g = \mathfrak{sl}_2 \). We give the complete list of indecomposable objects in that case.

1. INTRODUCTION

A Leibniz algebra over a field \( k \) is a vector space \( g \) equipped with a linear map \([\cdot, \cdot] \colon g \otimes g \to g \) which satisfies the Leibniz relation

\[
[x, [y, z]] = [[x, y], z] - [[x, z], y],
\]

for any \( x, y, z \) in \( g \). Obviously a Lie algebra is an example of a Leibniz algebra. A Leibniz representation of \( g \) (cf. [6]) is a vector space \( M \)

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equipped with two actions (left and right) of $\mathfrak{g}$,

$$[-, -]: \mathfrak{g} \otimes M \to M \quad \text{and} \quad [-, -]: M \otimes \mathfrak{g} \to M$$

which satisfy the following three axioms:

$$[m, [x, y]] = [[m, x], y] - [[m, y], x],$$
$$[x, [m, y]] = [[x, m], y] - [[x, y], m],$$
$$[x, [y, m]] = [[x, y], m] - [[x, m], y],$$

for any $m \in M$ and $x, y \in \mathfrak{g}$. Such representations arise naturally when considering extensions in the category of Leibniz algebras. We refer the reader to [4–6, 9] for more information about Leibniz algebras, Leibniz representations, and Leibniz (co)homologies. In this paper we always assume that $\mathfrak{g}$ is a Lie algebra. Observe that $M$ has natural right $\mathfrak{g}$-module structure (in the Lie sense). Conversely starting with a right $\mathfrak{g}$-module $M$, one can define Leibniz representations $M'$ and $M''$, whose underlying right module structures are $M$, but the left actions are given by $[x, m] = -[m, x]$ and $[x, m] = 0$ respectively. The representations obtained under the first process are called symmetric representations, and representations obtained under the second process are called antisymmetric representations.

For any right $\mathfrak{g}$-module $A$, we denote by $\overline{A}$ the Leibniz representation whose underlying right $\mathfrak{g}$-module is $A \oplus \text{Hom}(\mathfrak{g}, A)$, and the left action is given by

$$[x, (a, f)] = (f(x), -[f, x]), \quad a \in A, x \in \mathfrak{g}, f \in \text{Hom}(\mathfrak{g}, A).$$

Then one has a short exact sequence of Leibniz representations:

$$0 \to A^* \to \overline{A} \to \text{Hom}(\mathfrak{g}, A)^* \to 0,$$  \hspace{1cm} (1.1)

which generally does not split. So the category $\mathcal{L}(\mathfrak{g})$ of finite dimensional Leibniz representations of the finite dimensional semi-simple Lie algebra $\mathfrak{g}$ is not semi-simple, even when the characteristic of $k$ is 0. In this paper we prove that this category has global dimension 2.

The classification of simple objects is an easy task: the simple objects are the symmetric and antisymmetric Leibniz representations associated with usual simple right $\mathfrak{g}$-modules.

Next, we calculate the Ext group of two simple objects. We show that all nontrivial extensions between simple objects are related by the above short exact sequence. In this way we obtain the description of the quiver of the category $\mathcal{L}(\mathfrak{g})$ in the sense of Gabriel [3]. It turns out that this category is
tame only for \( g = \mathfrak{sl}_2 \). We give the complete list of indecomposable objects in that case.

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2. ON Ext IN THE CATEGORY OF LEIBNIZ REPRESENTATIONS

It is well known that for any Lie algebra \( \mathfrak{g} \) one has a natural isomorphism

\[
\text{Ext}_{\mathcal{U}(\mathfrak{g})}^n(M, N) \cong H^n(\mathfrak{g}, \text{Hom}(M, N)),
\]

where \( M, N \) are \( \mathfrak{g} \)-modules and \( \mathcal{U}(\mathfrak{g}) \) is the universal enveloping algebra of \( \mathfrak{g} \). We shall consider the relation between \( \text{Ext} \) in the category of Leibniz representations and Leibniz cohomologies. We recall that the category of Leibniz representations is isomorphic to the category of right \( \mathcal{U}\mathfrak{L}(\mathfrak{g}) \)-modules (cf. [6]), where \( \mathcal{U}\mathfrak{L}(\mathfrak{g}) \) is defined as a factor algebra of the tensor algebra:

\[
\mathcal{U}\mathfrak{L}(\mathfrak{g}) := T(\mathfrak{g}^l \oplus \mathfrak{g}^r) / \mathfrak{I}.
\]

Here \( \mathfrak{g}^l \) and \( \mathfrak{g}^r \) are two copies of \( \mathfrak{g} \), and \( \mathfrak{I} \) is a two-sided ideal corresponding to the relations

\[
\begin{align*}
    r_{[x, y]} &= r_x r_y - r_y r_x, \\
    l_{[x, y]} &= l_x r_y - r_y l_x, \\
    (r_x + l_x)l_x &= 0,
\end{align*}
\]

where \( l_x \) and \( r_x \) denote the elements of \( \mathfrak{g}^l \) and \( \mathfrak{g}^r \) corresponding to \( x \in \mathfrak{g} \). Let

\[
d_0, d_1: \mathcal{U}\mathfrak{L}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})
\]

be the homomorphisms defined in [6] by \( d_0(l_x) = 0, d_0(r_x) = x, d_1(l_x) = -x, d_1(r_x) = x \). There are two ways to look at a \( \mathcal{U}(\mathfrak{g}) \)-module \( M \) as a module over \( \mathcal{U}\mathfrak{L}(\mathfrak{g}) \) either under \( d_0 \) or under \( d_1 \). The first one gives an antisymmetric representation \( M^a \) and the second one gives a symmetric representation \( M^s \). Moreover, by 1.10 of [6], for each Leibniz representation \( X \) there exists a short exact sequence

\[
0 \to X_a \to X \to X_s \to 0
\]
with symmetric representation $X$, and antisymmetric representation $X_a$. Therefore, up to exact sequences, the problem of studying $\text{Ext}_{UL(\mathfrak{g})}^n(X, -)$ for general $X$ reduces to considering the symmetric and antisymmetric cases separately. In [6] we proved that

$$HL^*(\mathfrak{g}, X) \cong \text{Ext}^*_x_{UL(\mathfrak{g})}(U(\mathfrak{g})^x, X).$$

Therefore the next proposition is an immediate consequence of the change of ring spectral sequence for $d_0$ and $d_1$ respectively.

2.2. **Proposition.** Let $X$ be a Leibniz representation of the Lie algebra $\mathfrak{g}$ and $Y$ and $Z$ be right $\mathfrak{g}$-modules. Then one has two spectral sequences:

$$E_2^{pq} = H^p(\mathfrak{g}, \text{Hom}(Y, HL^q(\mathfrak{g}, X))) \Rightarrow \text{Ext}^*_{UL(\mathfrak{g})}(Y^a, X),$$

$$E_2^{pq} = H^p(\mathfrak{g}, \text{Hom}(Z, \text{Ext}^*_{UL(\mathfrak{g})}(U(\mathfrak{g})^x, X))) \Rightarrow \text{Ext}^*_{UL(\mathfrak{g})}(Z^a, X).$$

The next proposition relates $\text{Ext}^*(U(\mathfrak{g})^x, -)$ with $HL^*$.

2.3. **Proposition.** Let $X$ be a Leibniz representation of $\mathfrak{g}$. One has natural isomorphisms:

$$\text{Ext}^*_{UL(\mathfrak{g})}(U(\mathfrak{g})^x, X) \cong \text{Hom}(\mathfrak{g}, HL^*(\mathfrak{g}, X)), \quad \text{for } i > 0,$$

$$\cong \text{Coker } f, \quad \text{for } i = 0,$$

$$\cong \text{Ker } f, \quad \text{for } i = -1,$$

where $f: X \rightarrow \text{Hom}(\mathfrak{g}, HL^0(\mathfrak{g}, X))$ is given by

$$f(x)(g) = [x, g] + [g, x]; \quad x \in X, g \in \mathfrak{g}.$$

**Proof.** We consider $\mathfrak{g} \otimes UL(\mathfrak{g})$ as a right $UL(\mathfrak{g})$-module with the following action: $(x \otimes r)s = x \otimes rs$, where $x \in \mathfrak{g}$, and $r, s \in UL(\mathfrak{g})$. Let

$$f_1: \mathfrak{g} \otimes UL(\mathfrak{g}) \rightarrow UL(\mathfrak{g})$$

be a homomorphism of right $UL(\mathfrak{g})$-modules given by $f_1(x \otimes 1) = l_x + r_x$. It follows from the relation $(r_y + l_y)l_x = 0$ that $f_1$ factors through

$$f_2: \mathfrak{g} \otimes U(\mathfrak{g})^a \rightarrow UL(\mathfrak{g}).$$

Based on Proposition 2.4 of [6], one shows that $f_2$ is a monomorphism with

$$\text{Coker } f_2 \cong U(\mathfrak{g})^x.$$

From this follows the proposition.
3. THE CASE OF SIMPLE LIE ALGEBRAS

In this section we assume that $\text{char } k = 0$ and $\mathfrak{g}$ is a finite dimensional simple Lie algebra. We recall that $\mathbf{L}(\mathfrak{g})$ denotes the category of finite dimensional Leibniz representations of $\mathfrak{g}$.

3.1. Theorem. The simple objects in $\mathbf{L}(\mathfrak{g})$ are exactly the representations of the form $M^a$ and $N^s$, where $M$, $N$ are simple right $\mathfrak{g}$-modules. All groups $\text{Ext}^2_{\mathbf{U}(\mathfrak{g})}(M, N)$ between simple finite dimensional representations $M$, $N$ are zero except

$$\text{Ext}^2_{\mathbf{U}(\mathfrak{g})}(\mathfrak{g}^a, \mathfrak{g}^a)$$

which is one-dimensional. Moreover,

$$\text{Ext}^1_{\mathbf{U}(\mathfrak{g})}(M^s, N^s) \cong \text{Hom}_{\mathbf{U}(\mathfrak{g})}(M, \hat{N}),$$

where

$$\hat{N} = \text{Coker}(h: N \to \text{Hom}(\mathfrak{g}, N)), h(n)(x) = [n, x]$$

and all other groups $\text{Ext}^1_{\mathbf{U}(\mathfrak{g})}(M, N)$ between simple finite dimensional representations $M$, $N$ are zero.

In fact, the proof gives also the complete computation of higher Ext-groups in terms of usual cohomology groups of $\mathfrak{g}$.

3.2. Remark. Observe that $h$ is a homomorphism of $\mathfrak{g}$-modules. The extension of $N^s$ by $M^s$, corresponding to the $\mathfrak{g}$-module homomorphism $\alpha: M \to \hat{N}$ can be obtained as a pull-back along $\alpha_1: M \to \text{Hom}(\mathfrak{g}, N)$ of the extension (1.1). Here $\alpha_1$ is a lifting of $\alpha$. We consider two particular cases of this construction. First we take $N = k$, $M = \mathfrak{g}$, and $\alpha = 1$.

In this case $h = 0$ and $\hat{N} \equiv \mathfrak{g}$, thus we get an extension:

$$0 \to k \to X_0 \to \mathfrak{g}^s \to 0.$$ 

Then we take $M = k$, $N = \mathfrak{g}$, $\alpha(1) = 1$, and we obtain an extension

$$0 \to \mathfrak{g}^a \to X_1 \to k \to 0.$$ 

By gluing these extensions together we obtain a generator of $\text{Ext}^2_{\mathbf{U}(\mathfrak{g})}(\mathfrak{g}^a, \mathfrak{g}^a)$. Based on Theorem 3.1, we easily check that $X_1$ is a projective cover of $k$ and $X_0$ is an injective hull of $k$. Thus the simple objects $M^a$ and $M^s$ are respectively projective and injective objects in $\mathbf{L}(\mathfrak{g})$ when $M$ is a nontrivial simple right $\mathfrak{g}$-module. In the terminology of [7], $M^s$ is a source, $M^a$ is a sink, and $k^a = k^s$ is a node.
Proof of Theorem 3.1. The statement about the structure of simple objects follows immediately from the existence of the exact sequence (2.1). We calculate \( \text{Ext}^\ast_{UL(\mathfrak{g})}(M, N) \), where \( M, N \) are simple objects. There are several possibilities.

Case 1. Suppose that \( M = k = N \) is the trivial representation. We apply Proposition 2.2 to \( Y = k \) and \( X = k \). Since \( HL^q(\mathfrak{g}, k) = 0 \), for \( q > 0 \) (cf. [8, 9]), we obtain:

\[
\text{Ext}^\ast_{UL(\mathfrak{g})}(k, k) \cong H^\ast(\mathfrak{g}, k).
\]

Case 2. Suppose that \( M = k \) is the trivial representation and that \( N \) is a nontrivial simple symmetric representation. We apply Proposition 2.2 to \( Y = k \) and \( X = N \). Since \( HL^q(\mathfrak{g}, N) = 0 \) [9], we obtain:

\[
\text{Ext}^\ast_{UL(\mathfrak{g})}(k, N) = 0.
\]

Case 3. Suppose that \( M = k \) is the trivial representation and that \( N \) is a nontrivial simple antisymmetric representation. We apply Proposition 2.2 to \( M = k \) and \( X = N^a \). By [9] we know that

\[
HL^q(\mathfrak{g}, N^a) = 0,
\]

for \( q > 1 \),

\[
\cong \text{Hom}_{UL(\mathfrak{g})}(\mathfrak{g}, N), \quad \text{for} \ q = 1
\]

\[
\cong N, \quad \text{for} \ q = 0.
\]

Since \( H^\ast(\mathfrak{g}, N) = 0 \), we obtain:

\[
\text{Ext}^\ast_{UL(\mathfrak{g})}(k, N^a) \cong H^{\ast-1}(\mathfrak{g}, \text{Hom}_{UL(\mathfrak{g})}(\mathfrak{g}, N)).
\]

Observe that this last vector space is zero when \( N \) and \( \mathfrak{g} \) are nonisomorphic and is \( H^{\ast-1}(\mathfrak{g}, k) \) when \( N \cong \mathfrak{g}^a \).

Case 4. Suppose that \( M \) is a nontrivial simple antisymmetric representation and \( N \) is symmetric. In this case \( HL^q(\mathfrak{g}, N^a) = 0 \) for \( q > 0 \) and the action of \( \mathfrak{g} \) on \( HL^q(\mathfrak{g}, N^a) = N^a \) is trivial. Hence \( \text{Hom}(M, HL^q(\mathfrak{g}, N^a)) \) is isomorphic to the direct sum of \( \text{dim} N^a \) copies of \( M \), and we get \( E^q = 0 \). Thus one obtains

\[
\text{Ext}^\ast_{UL(\mathfrak{g})}(M^a, N^a) = 0.
\]

Case 5. Suppose that \( M \) is a nontrivial simple antisymmetric representation and \( N \) is antisymmetric. As in Case 3, \( HL^q(\mathfrak{g}, N^a) \) is nonzero only for \( q = 1 \) and \( q = 0 \). Moreover, \( HL^1(\mathfrak{g}, N^a) \) is a trivial \( \mathfrak{g} \)-module and we
obtain, as in Case 4, that \( E_2^{pq} = 0 \) for \( q > 0 \). Thus one gets

\[
\Ext_{UL(q)}^p(M^a, N^a) \cong H^*(\mathcal{g}, \text{Hom}(M, N)),
\]

which is zero when \( M \) and \( N \) are non-isomorphic and is isomorphic to \( H^*(\mathcal{g}, k) \) when \( N \cong M \).

Case 6. Suppose that \( M \) is a simple nontrivial symmetric representation and that \( N = k \). We put \( X = k \) in Proposition 2.3 and obtain

\[
\Ext_{UL(q)}^i((U\mathcal{g})^\mathcal{r}, k) \cong 0, \quad \text{if } i > 1,
\]

\[
\cong q \quad \text{if } i = 1,
\]

\[
\cong k \quad \text{if } i = 0,
\]

because \( f = 0 \). Now we use Proposition 2.2 with \( X = k, N = M \). Since \( \text{Hom}(M, k) \) does not have invariant part we get:

\[
\Ext_{UL(q)}^t(M^a, k) \cong H^{-1}(\mathcal{g}, \text{Hom}(M, \mathcal{g})),
\]

which is non-zero only for \( M \cong \mathcal{g}^a \). In this case it is isomorphic to \( H^{-1}(\mathcal{g}, k) \).

Case 7. Suppose that \( M \) and \( N \) are simple nontrivial symmetric representations. It follows from Proposition 2.3 that \( \Ext_{UL(q)}^i((U\mathcal{g})^\mathcal{r}, N^a) \) is 0 for \( i > 0 \) and is isomorphic to \( N \) for \( i = 0 \). By Proposition 2.2 we get

\[
\Ext_{UL(q)}^t(M^a, N^a) \cong H^*(\mathcal{g}, \text{Hom}(M, N)),
\]

which is nontrivial only when \( M \cong N \). In this case it is isomorphic to \( H^*(\mathcal{g}, k) \).

Case 8. Suppose that \( M \) is a simple nontrivial symmetric representation and \( N \) is a simple nontrivial antisymmetric representation. We put \( X = N \) in Proposition 2.3 and we obtain: \( \Ext_{UL(q)}^i((U\mathcal{g})^\mathcal{r}, N^a) \) is 0 for \( i > 2 \), is isomorphic to \( \text{Hom}(\mathcal{g}, \text{Hom}_{UL(q)}(\mathcal{g}, N)) \) for \( i = 2 \), is isomorphic to \( \text{Coker } h \) for \( i = 1 \), and is isomorphic to \( \text{Ker } h \) for \( i = 0 \). Since \( N \) is nontrivial and \( h \) is a \( \mathcal{g} \)-homomorphism, we get \( \text{Ker } h = 0 \). Thus \( E_2^{pq} = 0 \) for \( q > 2 \) and \( q = 0 \). Moreover, one has isomorphisms

\[
E_2^{pq} \cong H^p(\mathcal{g}, \text{Hom}(M, N)),
\]

and

\[
E_2^{pq} \cong H^p(\mathcal{g}, \text{Hom}(M, \text{Hom}(\mathcal{g}, \text{Hom}_{UL(q)}(\mathcal{g}, N))))
\]

\[
\cong H^p(\mathcal{g}, \text{Hom}(M, \text{Hom}(\mathcal{g}, \text{Hom}_{UL(q)}(\mathcal{g}, N)))).
\]
Observe that $E_2^{p,2} \equiv H^{p-2}(g, k)$ if $M \cong g^i$ and $N \cong g^j$, and $E_2^{p,2} = 0$ if $(M, N)$ and $(g^i, g^j)$ are not isomorphic.

Since $H^3(g, k) = 0 = H^2(g, k)$, it follows from the above calculation that $\text{Ext}^2_{L(g)}(M, N) = 0$ except in the case when $(M, N) \cong (g^i, g^j)$, which is one-dimensional. Similarly, $\text{Ext}^2_{L(g)}(M, N)$ is nontrivial if and only if $M$ is symmetric, $N$ is antisymmetric, and $\text{Hom}_{L(g)}(M, N)$ is nontrivial. So the theorem is completed.

The category $L(g)$ of finite-dimensional Leibniz representations of $g$ is abelian. Moreover, based on Theorem 3.1 and Remark 3.2, one can easily prove that this category has sufficiently enough projective and injective objects. Hence the Ext-groups in this category are well-defined. We denote them by $\text{Ext}^i_{L(g)}$.

3.3. Corollary. The binatural transformation $\text{Ext}^i_{L(g)} \rightarrow \text{Ext}^i_{L(g)}$ is an isomorphism for $i = 0, 1, 2$, and $\text{Ext}^i_{L(g)} = 0$ when $i > 2$. So the global dimension of the category $L(g)$ is 2.

Proof. The statement is obvious for $i = 0, 1$ and it is enough to consider values on simple objects for $i \geq 2$. Let $L(g)^s$ (resp. $L(g)^a$) be the full subcategory of $L(g)$ whose objects are symmetric (resp. antisymmetric) representations. Obviously both of them are isomorphic to the category of right $g$-modules and $L(g)^s \cap L(g)^a$ is isomorphic to the category of vector spaces, considered as trivial representations. It follows from the theorem that if $\text{Ext}^i_{L(g)}(M, N) \neq 0$, with simple objects $M, N$, then $M \in L(g)^s$ and $N \in L(g)^a$. If moreover $M$ (resp. $N$) is in $L(g)^s \cap L(g)^a$, then $N \cong g^a$ (resp. $M \cong g^s$). Thus all possible two-fold extensions of a simple representation by another simple one are split, except for the extension constructed in Remark 3.2. From this follows the statement.

4. The Gabriel Quiver of $L(g)$

We start by recalling the definition of the Gabriel quiver of an abelian category (see [3]). A quiver is a directed graph. Let $A$ be a $k$-linear abelian category, whose objects have finite length. Vertices of the Gabriel quiver $Q(A)$ are isomorphism classes of simple objects of $A$. If $S_1$ and $S_2$ are simple objects, then $Q(A)$ has exactly $\text{dim}_k \text{Ext}_A^1(S_1, S_2)$ arrows from $S_1$ to $S_2$. An abelian category $A$ has wild representation type if there exists an exact functor $T: k\langle x, y \rangle\text{-mod} \rightarrow A$ which preserves indecomposability and reflects isomorphisms. Here $k\langle x, y \rangle\text{-mod}$ is the category of finite dimensional modules over the free associative $k$-algebra with two-generators. So $A$ is as “bad” as the category of representations of $k\langle x, y \rangle$, which are, in some sense, unclassifiable.
We describe the Gabriel quiver $Q(L_{(g)})$ of $L_{(g)}$. First let us consider the simplest example $g = sl_2$. In this case, the simple objects are classified by natural numbers: for any $n \geq 0$, there exists a unique simple $g$-module $W_n$, whose dimension is $n + 1$. Therefore the vertices of $Q(L_{(g)})$ are $W_n$ and $W_n^\vee$, $n \geq 0$. Moreover, $W_n = W_n^\vee$. It follows from Theorem 3.1 that there is exactly $\dim \text{Hom}_{L_{(g)}}(W_m, W_n)$ arrows from $W_m$ to $W_n$ and only such vertices are connected. Here

$$\hat{W} = \text{Coker}(W \to \text{Hom}(g, W)).$$

Thanks to the Clebsch–Gordon formula one has $\hat{W}_n = W_{n+2} \oplus W_{n-2}$, for $n \geq 2$; $\hat{W}_1 = W_3$; and $\hat{W}_0 = W_2$. Thus $Q(L(sl_2))$ looks as follows:

![Diagram](image)

Here $n$ corresponds to $W_n$, if $n \geq 0$, and to $W_n^\vee$, if $n \leq 0$. Since the quiver $Q(L(sl_2))$ has three components, the category $L(sl_2)$ is isomorphic to the product of three subcategories, two of them have global dimension 1, corresponding to the components containing 1 and $-1$, respectively, and the third one has global dimension 2. In order to describe the indecomposable objects consider the following linear ordered sets:

$$-1 < 3 < -5 < 7 < -9 < 11 < \cdots,$$
$$1 < -3 < 5 < -7 < 9 < -11 < \cdots,$$

and

$$\cdots -6 < 4 < -2 < 0 < 2 < -4 < 6 < \cdots.$$
As was remarked in 3.2, in the last case 0 is a node. The classification of indecomposable objects of the corresponding subcategories can be deduced from the results of [3] and [7]. For any nonempty interval \([a, b]\) of the above ordered sets (in the third case, the interval must satisfy the following condition: if \(0 \in [a, b]\), then \(a = 0\) or \(b = 0\)) there exists a unique indecomposable object \(W_{[a, b]}\) in \(\text{L(l)}\), for which the sequence

\[
0 \to W_{[a, b]}^a \to W_{[a, b]} \to W_{[a, b]}^s \to 0
\]

is exact, where \(W_{[a, b]}^s\) (resp. \(W_{[a, b]}^a\)) is the sum of all \(W_{[x, y]}^s\) (resp. \(W_{[x, y]}^a\)) where \(x \in [a, b]\) and \(x \geq 0\) (resp. \(x \leq 0\)). Moreover, each indecomposable object is isomorphic to exactly one \(W_{[a, b]}\).

In order to describe \(Q(L(g))\) when the rank of \(g\) is \(> 1\), we fix some notation. Let \(g\) be a simple complex Lie algebra having a Cartan subalgebra \(g\) and corresponding set of roots \(R\). Choose a set of simple roots \(B\), or, equivalently, a set of positive roots \(R\). Let \(g_a\) be the root space corresponding to \(\alpha \in R\). Choose \(H_\alpha \in \{g_\alpha, g_\alpha^{-1}\}\) and \(X_\alpha \in g_\alpha\) in such a way that \(\alpha(H_\alpha) = 2\) and \([X_\alpha, X_{-\alpha}] = -H_\alpha\). Let \(P\) be a set of highest weights and let \(E(\lambda)\) denotes the irreducible finite dimensional \(g\)-module, whose highest weight is \(\lambda\). We will consider \(0 \in P\) as base point. Let \(LP\) be a wedge of two copies of \(P\), say \(P^s\) and \(P^a\). For any element \(\lambda \in P\) we write \(\lambda^s\) and \(\lambda^a\) for the corresponding element of \(P^s\) and \(P^a\) and \(d(\lambda)\) denotes the number of nonzero integers of the form \(\lambda(H_\alpha), \alpha \in B\).

4.1. Proposition. The vertices of \(Q(L(g))\) can be identified with \(LP\) and the arrows are only those which are described as follows. For any \(\lambda \in P\) there is exactly \(\max(0, d(\lambda) - 1)\) arrows from \(\lambda^s\) to \(\lambda^a\). For \(\lambda \in P\), \(\omega \in R\) with \(\lambda + \omega \in P\) there is at most one arrow from \((\lambda + \omega)^s\) to \(\lambda^a\). Such an arrow exists if and only if

\[
(ad \ X_\alpha)^{\lambda(H_\alpha)} X_{\alpha + \omega} = 0, \quad \alpha \in B.
\]

In the case when all roots have the same length (i.e., \(g\) is of type \(A_n, D_n, E_6, E_7, E_8\)) this condition is always fulfilled.

Proof. We use the following formula (see [1, Exercise 14, Section 9, Chap. VIII]):

\[
\text{Hom}(g, E(\lambda)) \cong d(\lambda) E(\lambda) \oplus \bigoplus d(\omega, \lambda) E(\lambda + \omega),
\]

where the sum is taken over all \(\omega \in R\). Here

\[
d(\omega, \lambda) = \dim \left( g^\omega \cap \bigcap \ker(\text{ad} \ X_\alpha)^{\lambda(H_\alpha) + 1} \right),
\]
where intersection is taken over all $\alpha \in B$. Thus $d(\omega, \lambda) = 1$ if and only if

$$(\text{ad} \ X_\alpha)^\lambda\Lambda H_{\alpha}\lambda \ X_\alpha, X_\omega = 0; \quad \alpha \in B.$$ 

But one has $[X_\alpha, X_\omega] = H_\alpha$ if $\alpha + \omega = 0$, $[X_\alpha, X_\omega] = X_{\alpha + \omega}$ if $\alpha + \omega \in R$, and $[X_\alpha, X_\omega] = 0$ in the other cases. Hence $d(\omega, \lambda) = 1$ if and only if the condition (4.2) is fulfilled for all $\alpha \in B$ and $(\text{ad} X_\alpha)^\lambda\Lambda H_{\alpha} = 0$ if $\alpha = -\omega \in B$. Observe that this last condition always holds, because $\lambda + \omega = \lambda - \alpha \in P$ yields $\lambda(H_\alpha) \geq 2$ and $(\text{ad} X_\alpha)^2 H_\alpha = 0$. If all roots have the same length, then Condition 4.2 is always fulfilled, because in this case $2\alpha + \omega$ is not a root vector and $\lambda(H_\alpha) > 0$. The last inequality follows from $\lambda + \omega \in P$ and $\omega(\alpha) < 0$. Now the statement is a consequence of Theorem 3.1.

4.3. Example. When $\mathfrak{g} = \mathfrak{sl}_3$, i.e., $A_2$, the vertices of the quiver can be identified with the wedge of two lattices $P^+$ and $P^-$. There are arrows only from $P^+$ to $P^-$. From a generic point of $P^+$ there are 7 arrows, one to its symmetric counterpart and the 6 others to the adjacent vertices (see Fig. 1).

Based on the classification of simple Lie algebras, it can be proved that the number of connected components of the Gabriel quiver of $L(\mathfrak{g})$, when rank of $\mathfrak{g}$ is $\geq 2$ and all roots have same length, is equal to the determinant of the corresponding Cartan matrix.

![Fig. 1. Fireworks.](image-url)
4.4. **Corollary.** If the rank of $\mathfrak{g}$ is $\geq 2$, then the category $\mathbf{L}(\mathfrak{g})$ has wild type.

**Proof.** Thanks to Proposition 4.1 this follows from the I.10.8 of [2].

**References**