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# Schur partial derivative operators

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**Abstract**

A lattice diagram is a finite list $L = ((p_1, q_1), \ldots, (p_n, q_n))$ of lattice cells. The corresponding lattice diagram determinant is $\Delta_L(X; Y) = \det \| x_i^{p_j} y_i^{q_j} \|$. The space $M_L$ is the space spanned by all partial derivatives of $\Delta_L(X; Y)$. We describe here how a Schur function partial derivative operator acts on lattice diagrams with distinct cells in the positive quadrant.


## 1. Introduction

We consider the symmetric group $\mathcal{S}_n$ acting diagonally on $\mathbb{Q}[x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n]$, the polynomial ring in $2n$ variables. More specifically, for $\sigma \in \mathcal{S}_n$, we consider the following (diagonal) action on polynomials:

$$\sigma P(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n) = P(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_n}; y_{\sigma_1}, y_{\sigma_2}, \ldots, y_{\sigma_n}).$$

A polynomial $\Delta = \Delta(x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n)$ is said to be *alternating* if, for all $\sigma \in \mathcal{S}_n$, we have $\sigma\Delta = sign(\sigma)\Delta$. It is well known that the set of all lattice diagram determinants (described in the next section) forms a basis for the space of alternating polynomials.

Given an alternating polynomial $\Delta$ we are interested in the space $\mathcal{L}_\partial[\Delta]$ spanned by all possible partial derivatives of $\Delta$. Since the diagonal action of $\mathcal{S}_n$ commutes with applying

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partial derivatives, the space $\mathcal{L}_\partial[\Delta]$ is an $\mathcal{S}_n$-module. Our goal is to give a complete description of its (graded) character. This is a very hard problem in general and even the simplest cases require elaborate constructions [4,6,7]. The aim of the present work is to develop tools that will allow us to better achieve this goal.

In previous work [1–4] we remark that the first step in describing the structure of $\mathcal{L}_\partial[\Delta]$ is to determine its subspace of alternating polynomials. This subspace corresponds to the space spanned by all *symmetric* partial derivative operators applied to $\Delta$. In view of this, we need to describe explicitly how the different bases of symmetric partial derivative operators act on a given lattice diagram determinant. In the work cited above, we describe the action of power sum symmetric operators and elementary symmetric operators and homogeneous symmetric operators in one set of variables. Yet the action of one of the most important bases of symmetric partial derivative operators, namely the Schur symmetric operators, was still not explicitly given. We give such a description here. Once our formula is established, we encourage the reader to revisit the previous work on the subject. For example, some results of [5] become conceptually simpler using our description and we see exactly why the multiplicity of the sign representation in a row diagram with a hole is as given in Section 4 of [5]. Our result can also be used to give a better description, in terms of partial Schur polynomials, of the vanishing ideal for the diagrams considered in [3]. Our hope is that our contribution will help in describing the generators of the vanishing ideal for the general cases, and this will be the subject of future work.

At first it seems that the description of the Schur symmetric partial derivative operators on $\Delta$ should follow directly from the expansion of Schur symmetric functions in terms of Young tableaux, but this is not quite correct. One has to be careful with the effect of signs when applying partial derivatives to lattice determinants. We thus need to re-derive this expansion from the other basis, carefully keeping track of signs. This can be done in many ways; here we chose to use the method of [10].

## 2. Basic definitions

The lattice cell in the $i+1$-st row and $j+1$-st column of the positive quadrant of the plane is denoted by $(i, j)$. We order the set of all lattice cells using the following *lexicographic* order:

$$(p_1, q_1) < (p_2, q_2) \qquad \Longleftrightarrow \qquad q_1 < q_2 \quad \text{or} \quad [q_1 = q_2 \text{ and } p_1 < p_2]. \tag{2.1}$$

For our purpose, a *lattice diagram* is a finite list $L = ((p_1, q_1), \ldots, (p_n, q_n))$ of lattice cells such that $(p_1, q_1) \le (p_2, q_2) \le \cdots \le (p_n, q_n)$. Following the definitions and conventions of [4], the coordinates $p_i$ and $q_i$ of a cell $(p_i, q_i)$ indicate the row and column positions, respectively, of the cell. For $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_k > 0$, we say that $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ is a *partition* of $n$ if $n = \mu_1 + \cdots + \mu_k$. We associate with a partition $\mu$ the following lattice (Ferrers) diagram $((i, j) : 0 \le i \le k-1, 0 \le j \le \mu_{i+1} - 1)$, distinct cells ordered with (2.1), and we use the symbol $\mu$ for both the partition and its associated Ferrers diagram. For example, given the partition (4, 2, 1), its Ferrers diagram is

| | | | |
|---|---|---|---|
| 2, 0 | | | |
| 1, 0 | 1, 1 | | |
| 0, 0 | 0, 1 | 0, 2 | 0, 3 |

This consists of the lattice cells ((0, 0), (1, 0), (2, 0), (0, 1), (1, 1), (0, 2), (0, 3)).

Given a lattice diagram $L = ((p_1, q_1), (p_2, q_2), \ldots, (p_n, q_n))$ we define the *lattice diagram determinant*

$$\Delta_L(X; Y) = \det \left\| \frac{x_i^{p_j} y_i^{q_j}}{p_j! q_j!} \right\|_{i,j=1}^{n},$$

where $X = x_1, x_2, \ldots, x_n$ and $Y = y_1, y_2, \ldots, y_n$. This determinant clearly vanishes if any cell has multiplicity greater than one, and we set $\Delta_L(X; Y) = 0$ if a coordinate of any cell is negative. The determinant $\Delta_L(X; Y)$ is bihomogeneous of degree $|p| = p_1 + \cdots + p_n$ in $X$ and degree $|q| = q_1 + \cdots + q_n$ in $Y$. The factorials will ensure that the lattice diagram determinants behave nicely under partial derivatives.

For a polynomial $P(X; Y)$ we denote by $P(\partial X; \partial Y)$ the differential operator obtained from $P$, substituting for every variable $x_i$ the operator $\frac{\partial}{\partial x_i}$ and for every variable $y_j$ the operator $\frac{\partial}{\partial y_j}$. Under the diagonal action, $\Delta_L(X; Y)$ is clearly an alternant.

These lattice diagram determinants are crucial in the study of the so-called "$n$! conjecture" of Garsia and Haiman [6], recently proven by Haiman [7], and in generalizations of this question (see [2,4] for example). To be more precise the key object is the vector space spanned by all partial derivatives of a given lattice diagram determinant $\Delta_L$, which we denote by

$$\mathbf{M}_L = \mathcal{L}_\partial[\Delta_L].$$

Very useful in the comprehension of the structure of the $\mathbf{M}_L$ spaces are the "shift operators". These operators are special symmetric derivative operators, whose action on the lattice diagram determinants could be easily described in terms of movements of cells.

Another area of interest related to the shift operators is the hope of obtaining a description of the vanishing ideal of $\mathbf{M}_L$, which is defined as

$$\mathcal{I}_L = \{ f \in \mathbb{Q}[X; Y];\ f(\partial X; \partial Y)\Delta_L(X; Y) = 0 \}.$$

The structures of $\mathbf{M}_L$ and of $\mathcal{I}_L$ are closely related and the shift operators are crucial tools for studying $\mathcal{I}_L$ (see [1–3] for some applications).

Let us recall results of [2] that describe the effects of power sums and elementary and homogeneous symmetric differential operators on lattice diagram determinants.

For the sake of simplicity, we limit our descriptions to $X$-operators; the $Y$-operators are similar. Recall that

$$P_k(X) = \sum_{i=1}^{n} x_i^k$$

$$e_k(X) = \sum_{1\le i_1<i_2<\cdots<i_k\le n} x_{i_1}x_{i_2}\cdots x_{i_k}$$
$$h_k(X) = \sum_{1\le i_1\le i_2\le\cdots\le i_k\le n} x_{i_1}x_{i_2}\cdots x_{i_k}$$

are respectively the $k$-th power sum and the elementary and homogeneous symmetric polynomials.

Now, to state the next proposition, we need to introduce some notation. For a lattice diagram $L$, we denote by $\overline{L}$ its complement in the positive quadrant (it is an infinite subset). Again we order $\overline{L} = \{(\overline{p}_1, \overline{q}_1), (\overline{p}_2, \overline{q}_2), \ldots\}$ using the lexicographic order (2.1). Let $L$ be a lattice diagram with $n$ distinct cells in the positive quadrant. For any integer $k \ge 1$ we have:

**Proposition 2.1** (*Proposition I.1 [4], Propositions 2.4, 2.6 [2]*).

$$P_k(\partial X)\Delta_L(X, Y) = \sum_{i=1}^{n} \pm\Delta_{P_k(i;L)}(X, Y), \tag{2.2}$$

*where $P_k(i; L)$ is the diagram obtained by replacing the $i$-th biexponent $(p_i, q_i)$ by $(p_i - k, q_i)$. The sign in (2.2) is the sign of the permutation that reorders the biexponents obtained with respect to the lexicographic order (2.1).*

$$e_k(\partial X)\Delta_L(X; Y) = \sum_{1\le i_1<i_2<\cdots<i_k\le n} \Delta_{e_k(i_1,\ldots,i_k;L)}(X; Y) \tag{2.3}$$

*where $e_k(i_1, \ldots, i_k; L)$ is the lattice diagram obtained from $L$ by replacing the biexponents $(p_{i_1}, q_{i_1}), \ldots, (p_{i_k}, q_{i_k})$ with $(p_{i_1} - 1, q_{i_1}), \ldots, (p_{i_k} - 1, q_{i_k})$:*

$$h_k(\partial X)\Delta_L(X, Y) = \sum_{1\le i_1<i_2<\cdots<i_k} \Delta_{h_k(i_1,\ldots,i_k;L)}(X, Y) \tag{2.4}$$

*where $h_k(i_1, \ldots, i_k; L)$ is the lattice diagram with the following complement diagram. Replace the biexponents $(\overline{p}_{i_1}, \overline{q}_{i_1}), \ldots, (\overline{p}_{i_k}, \overline{q}_{i_k})$ of the complement $\overline{L}$ with $(\overline{p}_{i_1} + 1, \overline{q}_{i_1}), \ldots, (\overline{p}_{i_k} + 1, \overline{q}_{i_k})$ and keep the others unchanged.*

The aim of this work is to obtain a description similar to the previous proposition of the effect of a partial Schur differential symmetric operator on a lattice diagram determinant. We obtain such a result in the next section and prove it.

## 3. Schur operators

Following [9], recall that for a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ the conjugate (transpose) partition is denoted by $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_\ell)$. With this in mind, the Schur polynomial indexed by $\lambda$ is

$$S_\lambda(X) = \det \|e_{\lambda'_j+i-j}(X)\|$$

with the understanding that $e_0(X) = 1$ and $e_k(X) = 0$ if $k < 0$. The Schur polynomials also have a description in terms of column-strict Young tableaux. Given $\lambda$ a partition of $n$, a tableau of shape $\lambda$ is a map $T : \lambda \to \{1, 2, \ldots, n\}$. We say that $T$ is a column-strict Young tableau if it is weakly increasing along the rows and strictly increasing along the columns of $\lambda$. That is, $T(i, j) \leq T(i, j+1)$ and $T(i, j) < T(i+1, j)$ wherever it is defined. We denote by $\mathcal{T}_\lambda$ the set of all column-strict Young tableaux of shape $\lambda$. For any tableau $T$, we define $X^T = \prod_{i=1}^{n} x_i^{|T^{-1}(i)|}$. As seen in [9], we have

$$S_\lambda(X) = \sum_{T \in \mathcal{T}_\lambda} X^T.$$

It is convenient to define the following function on lattice diagrams:

$$\epsilon(L) = \begin{cases} 1 & \text{if } L \text{ has } n \text{ distinct cells in the positive quadrant,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

Let $L$ be a lattice diagram with $n$ distinct cells in the positive quadrant. For any partition $\lambda$ of an integer $k \geq 1$ we have

**Theorem 3.1.**

$$S_\lambda(\partial X)\Delta_L(X; Y) = \sum_{T \in \mathcal{T}_\lambda} \epsilon'(T, L)\Delta_{\partial T(L)}(X; Y)$$

*where $\partial T(L)$ is the lattice diagram obtained from $L$ by replacing the biexponents $(p_i, q_i)$ with $(p_i - |T^{-1}(i)|, q_i)$ for $1 \leq i \leq n$. The coefficient $\epsilon'(T, L)$ is described as follows. Let $T_1, T_2, \ldots, T_\ell$ be the $\ell$ columns of $T$; then $\partial T(L) = \partial T_1 \partial T_2 \cdots \partial T_\ell(L)$ and*

$$\epsilon'(T, L) = \epsilon(\partial T(L)) \cdots \epsilon(\partial T_{\ell-1}\partial T_\ell(L))\, \epsilon(\partial T_\ell(L)) \tag{3.2}$$

*where $\epsilon$ is defined in (3.1). Hence $\epsilon'(T, L)$ is 0 or 1.*

We shall prove this result using Proposition 2.1 and an adaptation of the involution defined in [10]. We will see in the proof at the end of this section that the order in which we apply the operators $\partial T_j$ to the lattice diagram $L$ in Eq. (3.2) is not arbitrary. The result and the proof depend on that precise order and no known results covered that aspect before.

To start, we remark that the Theorem 3.1 and Proposition 2.1 agree on their domain of definition. This is because $e_k = S_{1^k}$ and the tableau of shape $1^k$ corresponds to sequences $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Now let $\ell$ be the number of components of $\lambda'$ and expand the determinant:

$$S_\lambda(X) = \det \| e_{\lambda'_j + i - j}(X) \| = \sum_{\sigma \in \mathcal{S}_\ell} sgn(\sigma) e_{\sigma(\lambda' + \delta_\ell) - \delta_\ell}. \tag{3.3}$$

Here $\delta_\ell = (\ell - 1, \ell - 2, \ldots, 1, 0)$ and $e_{\alpha_j} = 0$ if $\alpha_j < 0$. If we have $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_\ell$ a sequence of integers, we let $e_\alpha = e_{\alpha_1} e_{\alpha_2} \cdots e_{\alpha_\ell}$. Here the order in which we write this product matters. For $\ell = 1$, as noted before, Proposition 2.1 can be rewritten as

$$e_{\alpha_1}(\partial X)\Delta_L(X; Y) = \sum_{T_1 \in \mathcal{T}_{1^{\alpha_1}}} \epsilon(\partial T_1(L))\Delta_{\partial T_1(L)}(X; Y) \tag{3.4}$$

where $\mathcal{T}_{1^{\alpha_1}}$ is the set of $\alpha_1$-column tableaux with content in $\{1, 2, \ldots, n\}$, strictly increasing in the column. Here $\epsilon'(T_1, L) = \epsilon(\partial T_1(L))$. Suppose now that $\ell = 2$. We use (3.4) with

$e_{\alpha_2}(\partial X)$ and apply $e_{\alpha_1}(\partial X)$ on both side. That gives

$$\begin{aligned} e_\alpha(\partial X)\Delta_L(X;Y) &= e_{\alpha_1}(\partial X)e_{\alpha_2}(\partial X)\Delta_L(X;Y) \\ &= \sum_{T_2\in\mathcal{T}_{1^{\alpha_2}}} \epsilon(\partial T_2(L))e_{\alpha_1}(\partial X)\Delta_{\partial T_2(L)}(X;Y) \\ &= \sum_{T_1\in\mathcal{T}_{1^{\alpha_1}}}\ \sum_{T_2\in\mathcal{T}_{1^{\alpha_2}}} \epsilon(\partial T_2(L))\epsilon(\partial T_1\partial T_2(L))\Delta_{\partial T_1\partial T_2(L)}(X;Y). \end{aligned}$$

Now let $\mathcal{CT}_\alpha = \mathcal{CT}_{\alpha_1,\alpha_2,\ldots,\alpha_\ell}$ be the set of $\ell$ columns $\mathbf{T} = (T_1, T_2, \ldots, T_\ell)$ where $T_j \in \mathcal{T}_{1^{\alpha_j}}$. We can represent $\mathbf{T}$ as a tableau $\alpha \to \{1, 2, \ldots, n\}$ where as before we identify the composition $\alpha$ with the lattice diagram $((i, j) \mid 0 \le i \le \alpha_{j+1} - 1, 0 \le j \le \ell - 1)$, with distinct cells ordered by (2.1). The tableau $\mathbf{T}$ is strictly increasing along every column and has no restriction along rows. Note that the shape $\alpha$ is not necessarily a partition. We can now simplify our computation above and write, for $\ell = 2$,

$$e_\alpha(\partial X)\Delta_L(X;Y) = \sum_{T\in\mathcal{CT}_\alpha} \epsilon'(\mathbf{T}, L)\Delta_{\partial\mathbf{T}(L)}\,(X;Y), \tag{3.5}$$

where $\partial\mathbf{T}(L) = \partial T_1\partial T_2\cdots\partial T_\ell(L)$ is the lattice diagram obtained from $L$ by replacing the biexponents $(p_i, q_i)$ with $(p_i - |\mathbf{T}^{-1}(i)|, q_i)$ for $1 \le i \le n$ and

$$\epsilon'(\mathbf{T}, L) = \epsilon(\partial\mathbf{T}(L))\ \cdots\ \epsilon(\partial T_{\ell-1}\partial T_\ell(L))\ \epsilon(\partial T_\ell(L)). \tag{3.6}$$

It is clear, by induction, that this is true for all $\ell \ge 2$ as well. We must also remark here that if one of the $\alpha_j < 0$, the sum (3.5) must be set to zero.

We can now start the computation of the operator (3.3) using (3.5):

$$\begin{aligned} S_\lambda(\partial X)\Delta_L(X;Y) &= \sum_{\sigma\in\mathcal{S}_\ell} sgn(\sigma)e_{\sigma(\lambda'+\delta_\ell)-\delta_\ell}(\partial X)\Delta_L(X;Y) \\ &= \sum_{\sigma\in\mathcal{S}_\ell}\ \sum_{\mathbf{T}\in\mathcal{CT}_{\sigma(\lambda'+\delta_\ell)-\delta_\ell}} sgn(\sigma)\epsilon'(\mathbf{T}, L)\Delta_{\partial\mathbf{T}(L)}(X;Y). \end{aligned} \tag{3.7}$$

Now we need to construct an involution on the set indexing the double sum such that all terms cancel, unless $\sigma$ is the identity permutation and $\mathbf{T} \in \mathcal{T}_\lambda$. Here is an example of a $\mathbf{T} \in \mathcal{CT}_{(1,0,3,2,4,1)}$:

$$\mathbf{T} \;=\; \begin{array}{cccccc} & & & & \boxed{8} & \\ & & \boxed{10} & & \boxed{6} & \\ & & \boxed{8} & \boxed{9} & \boxed{5} & \\ \boxed{2} & & \boxed{7} & \boxed{3} & \boxed{4} & \boxed{4} \end{array}$$

The only requirement is that $\mathbf{T}$ is strictly increasing in columns.

Let us first concentrate on $\ell = 2$ and let $\lambda' = (\lambda'_1, \lambda'_2)$. We have two possible shapes $\alpha = (\alpha_1, \alpha_2)$, either $\lambda' = Id(\lambda' + \delta_2) - \delta_2$ or $(\lambda'_2 - 1, \lambda'_1 + 1) = (1, 2)(\lambda' + \delta_2) - \delta_2$, where $(i, j)$ is the usual notation for transpositions. These two cases are completely characterized by $\alpha_1 < \alpha_2$ or $\alpha_1 \ge \alpha_2$. We now define an involution similar to [10].

With each $\mathbf{T} \in \mathcal{CT}_\alpha$ we associate two words $w_{\mathbf{T}}$ and $\widehat{w}_{\mathbf{T}}$. This method is originally due to Lascoux and Schützenberger (cf. [8]). The first $w_{\mathbf{T}}$ consists of all the entries $\mathbf{T}(i, j)$ of $\mathbf{T}$ sorted in increasing order. For example if

$$\mathbf{T} \quad = \quad \begin{array}{cc} & \boxed{9} \\ & \boxed{6} \\ \boxed{9} & \boxed{5} \\ \boxed{3} & \boxed{9} \end{array}$$

then $w_{\mathbf{T}} = 3\ 4\ 5\ 6\ 9\ 9$. Now we associate with $w_{\mathbf{T}}$ its parentheses structure $\widehat{w}_{\mathbf{T}}$. For this, we list the entries in $w_{\mathbf{T}}$ and associate with an entry from the first column of $\mathbf{T}$ a left parenthesis, and with an entry of the second column a right parenthesis. For two columns, the same entry appears at most twice, in which case the first one that we read in $w_{\mathbf{T}}$ is assumed to be from the first column of $\mathbf{T}$. In the example above, $w_{\mathbf{T}} = 3\ 4\ 5\ 6\ 9\ 9$ and $\widehat{w}_{\mathbf{T}} = (\,)))(\,)$.

There is a natural way to pair parentheses under the usual rule of parenthesization. In any word $\widehat{w}_{\mathbf{T}}$ some parentheses will be paired and others will be unpaired. In our example, $\widehat{w}_{\mathbf{T}} = (\,)\big)\big)(\,)$, the first two parentheses and the last two are paired and the two parentheses in the middle are unpaired. The subword of any $\widehat{w}_{\mathbf{T}}$ consisting of unpaired parentheses must be of the form $))\cdots)\,(\cdots(($.

We have the following useful result.

**Proposition 3.2** (*[10], Proposition 5*). *A tableau $\mathbf{T} = (T_1, T_2, \ldots, T_\ell) \in \mathcal{CT}_\alpha$ is a column-strict Young tableau $\mathbf{T} \in \mathcal{T}_\lambda$ if and only if there are no unpaired right parentheses in $\widehat{w}_{T_j,T_{j+1}}$ for all $1 \le j \le \ell - 1$ and two columns $T_j, T_{j+1}$ of $\mathbf{T}$.*

Remark here that if $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ is not a partition, that is $\alpha_j < \alpha_{j+1}$ for some $1 \le j \le \ell - 1$, then necessarily $\widehat{w}_{T_j,T_{j+1}}$ will contain more right parentheses than left parentheses and some will be left unpaired and no $\mathbf{T} \in \mathcal{CT}_\alpha$ could be a column-strict Young tableau.

We return to the construction of the involution from [10] for $\lambda' = (\lambda'_1, \lambda'_2)$. Let

$$A = \mathcal{CT}_{(\lambda'_1,\lambda'_2)} \cup \mathcal{CT}_{(\lambda'_2-1,\lambda'_1+1)}.$$

The involution is a map $\Psi : A \to A$ defined as follows. Let $\mathbf{T} \in \mathcal{CT}_{(\alpha_1,\alpha_2)} \subset A$ and consider $\widehat{w}_{\mathbf{T}}$. The subword of unpaired parentheses contains $r \ge 0$ unpaired right parentheses followed by $l \ge 0$ unpaired left parentheses. We have that $l - r = \alpha_1 - \alpha_2$.

- If $r = 0$, then $\mathbf{T} \in \mathcal{T}_\lambda \subset \mathcal{CT}_{\lambda'}$ and we define $\Psi(\mathbf{T}) = \mathbf{T}$.
- If $l \ge r > 0$, then $\mathbf{T} \in \mathcal{CT}_{\lambda'} \backslash \mathcal{T}_\lambda$ and we define $\Psi(\mathbf{T}) = \mathbf{T}' \in \mathcal{CT}_{(\lambda'_2-1,\lambda'_1+1)}$, the unique tableau such that $w_{\mathbf{T}'} = w_{\mathbf{T}}$ and $\widehat{w}_{\mathbf{T}'}$ is obtained from $\widehat{w}_{\mathbf{T}}$ replacing the $l-r+1$ leftmost unpaired left parentheses by right parentheses.
- If $r > l$, then $\mathbf{T} \in \mathcal{CT}_{(\lambda'_2-1,\lambda'_1+1)}$ and we define $\Psi(\mathbf{T}) = \mathbf{T}' \in \mathcal{CT}_{\lambda'} \backslash \mathcal{T}_\lambda$, the unique tableau such that $w_{\mathbf{T}'} = w_{\mathbf{T}}$ and $\widehat{w}_{\mathbf{T}'}$ is obtained from $\widehat{w}_{\mathbf{T}}$ replacing the $r - l - 1$ rightmost unpaired right parentheses by left parentheses.

Now in the general case, that is if $\ell \geq 2$, let

$$A = \bigcup_{\sigma \in \mathcal{S}_\ell} \mathcal{CT}_{\sigma(\lambda'+\delta_\ell)-\delta_\ell}.$$

For $\mathbf{T} \in \mathcal{CT}_\alpha \subset A$, the composition $\alpha$ completely characterizes the permutation $\sigma \in \mathcal{S}_\ell$ such that $\alpha = \sigma(\lambda' + \delta_\ell) - \delta_\ell$. In particular, $\alpha$ is a partition if and only if $\sigma = Id$. We read the rows of $\mathbf{T}$ from right to left, bottom to top. We find in this way the first pair $(i, j)$ and $(i, j+1)$ such that

$$T(i, j) > T(i, j+1) \qquad \text{or} \qquad (i, j) \notin \alpha \text{ and } (i, j+1) \in \alpha.$$

- If there is no such pair, then we have $\mathbf{T} \in \mathcal{T}_\lambda \subset \mathcal{CT}_{\lambda'}$ and we define $\Psi(\mathbf{T}) = \mathbf{T}$.
- If we find such a pair, then we have $\mathbf{T} \in \mathcal{CT}_\alpha \subset A \backslash \mathcal{T}_\lambda$. We define $\Psi(\mathbf{T}) = \mathbf{T}' \in \mathcal{CT}_\beta \subset A \backslash \mathcal{T}_\lambda$ where $\mathbf{T}'$ is obtained from $\mathbf{T}$ using the procedure above applied to the two columns $T_{j+1}, T_{j+2}$. By construction, if $\alpha = \sigma(\lambda'+\delta_\ell)-\delta_\ell$, then $\beta = \sigma(j, j+1)(\lambda'+\delta_\ell)-\delta_\ell$.

The fact that $\Psi$ is a well defined involution is shown in several papers, for example, in [10], Section 3. Let us give one example:

$$\mathbf{T} \quad = \quad \begin{array}{|c|c|c|} \cline{3-3} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 8 \\ \cline{1-1}\cline{3-3} 10 & \multicolumn{1}{c|}{} & 6 \\ \hline 8 & 9 & 5 \\ \hline 7 & 3 & 4 \\ \hline \end{array} \quad \text{we have} \quad \Psi(\mathbf{T}) \quad = \quad \begin{array}{|c|c|c|} \hline 10 & 9 & 6 \\ \hline 8 & 8 & 5 \\ \hline 7 & 3 & 4 \\ \hline \end{array}$$

The pair (1, 2) and (1, 3) is the first one where $T(1, 2) > T(1, 3)$. We thus apply the involution to the second and third column. We have here $w_{T_2,T_3} = 3\ 4\ 5\ 6\ 8\ 9$ and $\widehat{w}_{T_2,T_3} = ())))\,($. There are $r = 3$ unpaired right parentheses followed by $l = 1$ unpaired left parenthesis. We must change $r - l - 1 = 1$ unpaired left parenthesis for a right one. That is, $\widehat{w}_{T'_2,T'_3} = ()))((.$ That moves the entry 8 from the third column to the second column.

**Proof of Theorem 3.1.** We return to the computation (3.7) using the notation that we have developed:

$$S_\lambda(\partial X)\Delta_L(X; Y) = \sum_{\mathbf{T} \in \mathcal{CT}_{\sigma(\lambda'+\delta_\ell)-\delta_\ell} \subset A} sgn(\sigma)\epsilon'(\mathbf{T}, L)\Delta_{\partial \mathbf{T}(L)}(X; Y).$$

The involution constructed above matches the term in the sum corresponding to $\mathbf{T} \in \mathcal{CT}_{\sigma(\lambda'+\delta_\ell)-\delta_\ell} \subset A \backslash \mathcal{T}_\lambda$ with $\mathbf{T}' \in \mathcal{CT}_{\sigma(j,j+1)(\lambda'+\delta_\ell)-\delta_\ell} \subset A \backslash \mathcal{T}_\lambda$. Clearly, we have that $sgn(\sigma) = -sgn(\sigma(j, j+1))$ and $\partial\mathbf{T}(L) = \partial\mathbf{T}'(L)$. Once we show that

$$\epsilon'(\mathbf{T}, L) = \epsilon'(\mathbf{T}', L) \tag{3.8}$$

the Theorem 3.1 will follow from the fact that all the terms in $A \backslash \mathcal{T}_\lambda$ will cancel out and the remaining terms are in $\mathcal{T}_\lambda$ with the desired coefficient.

To establish (3.8) we need to show that if $\epsilon'(\mathbf{T}, L) \neq 0$ then $\epsilon'(\mathbf{T}', L) \neq 0$, for they will then both be equal to 1. From (3.6)

$$\epsilon'(\mathbf{T}, L) = \epsilon'((T_1, T_2, \dots, T_\ell), L) = \epsilon(\partial \mathbf{T}(L)) \cdots \epsilon(\partial T_{\ell-1}\partial T_\ell(L))\, \epsilon(\partial T_\ell(L)).$$

Similarly, $\epsilon'(\mathbf{T}', L) = \epsilon'((T_1, T_2, \dots, T'_{j+1}, T'_{j+2}, \dots, T_\ell), L)$ for some $0 \le i \le \ell - 1$. If $\epsilon'(\mathbf{T}, L) \neq 0$, then $\epsilon(\partial T_k \cdots \partial T_\ell(L)) = 1$ for $1 \le k \le \ell$. For $1 \le k \le j+1$ we clearly have

$$\epsilon(\partial T_k \cdots \partial T_{j+1}\partial T_{j+2} \cdots \partial T_\ell(L)) = \epsilon(\partial T_k \cdots \partial T'_{j+1}\partial T'_{j+2} \cdots \partial T_\ell(L)).$$

For $j+3 \le k \le \ell$, the corresponding terms of $\epsilon'(\mathbf{T}, L)$ and $\epsilon'(\mathbf{T}', L)$ are the same. Let $\tilde{L} = \partial T_{j+3} \cdots \partial T_{\ell-1}\partial T_\ell L$; the equality (3.8) will follow as soon as we show that

$$\epsilon(\partial T_{j+2}(\tilde{L})) = 1 \text{ and } \epsilon(\partial T_{j+1}\partial T_{j+2}(\tilde{L})) = 1 \quad \Longrightarrow \quad \epsilon(\partial T'_{j+2}(\tilde{L})) = 1 \tag{3.9}$$

for all $\tilde{L}$ such that $\epsilon(\tilde{L}) = 1$.

Let $\beta = \sigma(j, j+1)(\lambda' + \delta_\ell) - \delta_\ell$, the shape of $\mathbf{T}$. Suppose that $\epsilon(\partial T'_{j+2}(\tilde{L})) = 0$. This implies that there is an entry $1 \le k = \mathbf{T}'(i, j+2) \le n$ such that the cells $(p_k, q_k) \in \tilde{L}$ and $(p_{k-1}, q_{k-1}) = (p_k - 1, q_k) \in \tilde{L}$, and $k - 1 \neq \mathbf{T}'(i-1, j+2)$ is not an entry of $T'_{j+2}$. Now since $\epsilon(\partial T_{j+1}\partial T_{j+2}(\tilde{L})) = 1$ we must have that both $k$ and $k-1$ are entries of $T_{j+1}, T_{j+2}$. This implies that $k-1$ is an entry of $T'_{j+1}$ and $k$ is not. This analysis shows that $k-1$ and $k$ are entries of $w_{T'_{j+1}T'_{j+2}}$ with multiplicity one, $k-1$ is in the column $T'_{j+1}$ and $k$ is in the column $T'_{j+2}$. They will be consecutive entries in $w_{T'_{j+1}T'_{j+2}}$ and will be paired in $\widehat{w}_{T'_{j+1}T'_{j+2}}$. This would imply that $T_{j+2}$ in $\Psi(\mathbf{T}') = \mathbf{T}$ contains the entry $k$ but not $k-1$ and $\epsilon(\partial T_{j+2}(\tilde{L})) = 0$, contrary to our hypothesis. This completes our proof. $\square$

**Remark 3.3.** Given a lattice diagram $L$ and a column-strict tableau $T \in \mathcal{T}_\lambda$, we have that $\epsilon'(T, L) = 1$ exactly when we can *slide down* the cells of $L$ by one, reading $T$ column by column, from right to left, without having any cells colliding.

**Corollary 3.4.** *For $h_k(X) = s_{(k)}(X)$ we have*

$$h_k(\partial X)\Delta_L(X; Y) = \sum_{1 \le j_1 \le j_2 \le \cdots \le j_k \le n} \epsilon'((j_1, \dots, j_k), L)\Delta_{\partial_{j_1}\dots\partial_{j_k}(L)}(X; Y).$$

This is equivalent to the description in [2]. The only way to have $\epsilon'((j_1, \dots, j_k), L) \neq 0$ is if the cells $j_1, \dots, j_k$ that move down are moved into holes. This can be described as holes moving up.

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