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Tchebychef-Like Method for the Simultaneous Finding Zeros of Analytic Functions

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Abstract—Using a suitable approximation in classical Tchebychef's iterative method of the third order, a new method for approximating, simultaneously, all zeros of a class of analytic functions in a given simple smooth closed contour is constructed. It is proved that its order of convergence is three. The analysis of numerical stability and some computational aspects, including a numerical example, are given. Also, the asynchronous implementation of the proposed method on a distributed memory multicomputer is considered from a theoretical point of view. Assuming that the maximum delay r is bounded, a convergence analysis shows that the order of convergence of this version is the unique positive root of the equation $x^{r+1} - 2x^r - 1 = 0$, belonging to the interval $(2, 3]$.

Keywords—Iterative methods, Zeros of analytic functions, Convergence order, Numerical stability, Asynchronous implementation.

1. INTRODUCTION

Let $z \mapsto \Phi(z)$ be an analytic function inside and on the simple smooth closed contour Γ, without zeros on Γ and with the known number n of simple zeros inside Γ. Then Φ is of the form

$$\Phi(z) = X(z) \prod_{j=1}^{n} (z - \zeta_j) \tag{1}$$

inside Γ, where $\zeta_1, \dots, \zeta_n$ are the zeros of Φ (inside Γ) and $X(z)$ is an analytic function without zeros inside Γ (see [1]). In practice, the number of zeros n of Φ inside Γ can be determined by the *computable argument principle* proposed by Gargantini [2]. Following Anastasselou and Ioakimidis [3], $X(z)$ can be represented as $X(z) = \exp(Y(z))$. $Y(z)$ is also an analytic function inside Γ which for an arbitrary complex number $t \in \operatorname{int}\Gamma$ such that $\Phi(t) \neq 0$ is given by

$$Y(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\log[(w-t)^{-n}\Phi(w)]}{w - z}\, dw, \tag{2}$$

whence

$$Y'(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\log[(w-t)^{-n}\Phi(w)]}{(w - z)^2}\, dw. \tag{3}$$

In the recent papers [4–7], some iterative methods for the simultaneous determination of simple zeros of analytic functions of the form (1) have been proposed. The aim of this paper is to present a new iterative method for approximating, simultaneously, zeros of the mentioned class of analytic

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX functions. This method is constructed by a suitable approximation of the term $\Phi''(z)/(2\Phi'(z))$ in the classical Tchebychef third order method (known also as Olver's method, Schröder's method of order 3, etc.)

$$z^{(m+1)} = z^{(m)} - \frac{\Phi\left(z^{(m)}\right)}{\Phi'\left(z^{(m)}\right)} \left(1 + \frac{\Phi\left(z^{(m)}\right)}{\Phi'\left(z^{(m)}\right)} \frac{\Phi''\left(z^{(m)}\right)}{2\Phi'\left(z^{(m)}\right)}\right), \qquad m = 0, 1, \dots. \tag{4}$$

In Section 2, we state a convenient approximation for $\Phi''(z)/(2\Phi'(z))$ which enables us to construct a new method of Tchebychef's type for the simultaneous approximation of all zeros of Φ inside in the given closed contour. The convergence analysis shows that the order of convergence of the proposed method is also *three*. Numerical stability of this method in the presence of the error of numerical integration, necessary for the calculation of $Y'(z)$, is presented in Section 3. Besides, some computational aspects of the considered method, including a numerical example, are given. Finally, in Section 4 we discuss some theoretical aspects of the asynchronous implementation of the proposed method on a distributed memory multicomputer and consider its order of convergence in dependence on the maximum delay.

2. SIMULTANEOUS TCHEBYCHEF-LIKE METHOD

Consider now Tchebychef's method (4) applied to an analytic function Φ belonging to the class of the form (1). Let us introduce the errors

$$\varepsilon_i^{(m+1)} = z_i^{(m+1)} - \zeta_i, \qquad \varepsilon_i^{(m)} = z_i^{(m)} - \zeta_i.$$

For simplicity, we will write in the sequel $\hat{z}_i, z_i, \hat{\varepsilon}_i, \varepsilon_i$ instead of $z_i^{(m+1)}, z_i^{(m)}, \varepsilon_i^{(m+1)}, \varepsilon_i^{(m)}$, respectively. Besides, let $\varepsilon \in \{\varepsilon_1, \dots, \varepsilon_n\}$ denote the error of maximum magnitude. We first state the following necessary approximation.

LEMMA 1. *Let $z_1, \dots, z_n$ be reasonably close approximations to the zeros $\zeta_1, \dots, \zeta_n$. Then*

$$\frac{\Phi''(z_i)}{2\Phi'(z_i)} = Y'(z_i) + \sum_{j \neq i} \frac{1}{z_i - z_j} + O(\varepsilon). \tag{5}$$

PROOF. Applying the logarithmic derivative to (1), we obtain

$$\frac{\Phi'(z)}{\Phi(z)} = Y'(z) + \sum_{j=1}^{n} \frac{1}{z - \zeta_j}; \tag{6}$$

that is,

$$\Phi'(z) = \Phi(z) \left[Y'(z) + \sum_{j=1}^{n} \frac{1}{z - \zeta_j} \right].$$

Hence, we find the second derivative

$$\Phi''(z) = \Phi'(z) \left(Y'(z) + \sum_{j=1}^{n} \frac{1}{z - \zeta_j} \right) + \Phi(z) \left(Y''(z) - \sum_{j=1}^{n} \frac{1}{(z - \zeta_j)^2} \right),$$

whence, using (6),

$$\begin{aligned}
\frac{\Phi''(z)}{\Phi'(z)} &= Y'(z) + \sum_{j=1}^{n} \frac{1}{z - \zeta_j} + \frac{1}{\Phi'(z)/\Phi(z)} \left(Y''(z) - \sum_{j=1}^{n} \frac{1}{(z - \zeta_j)^2} \right) \\
&= \frac{\left(Y'(z) + \sum_{j=1}^{n} (1/(z - \zeta_j))\right)^2 + Y''(z) - \sum_{j=1}^{n} (1/(z - \zeta_j)^2)}{Y'(z) + \sum_{j=1}^{n} (1/(z - \zeta_j))} \\
&= \frac{Y'^2(z) + 2Y'(z) \left(1/(z - \zeta_i) + \sum_{j \neq i} (1/(z - \zeta_j))\right) + Y''(z) + 2\sum_{k=1}^{n-1} \sum_{j=k+1}^{n} (1/(z - \zeta_k))(1/(z - \zeta_j))}{(1/(z - \zeta_i)) \left(1 + (z - \zeta_i)\left(Y'(z) + \sum_{j \neq i} (1/(z - \zeta_j))\right)\right)}.
\end{aligned}$$

By introducing the functions

$$\sigma_i(z) = \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{z-\zeta_j}, \qquad A_i(z) = \sum_{\substack{k=1\\ k\neq i}}^{n-1} \sum_{\substack{j=k+1\\ j\neq i}}^{n} \frac{1}{z-\zeta_k}\frac{1}{z-\zeta_j},$$

after some transformations of the last expression we get

$$\frac{\Phi''(z)}{2\Phi'(z)} = \frac{Y'(z) + \sum_{j\neq i}(1/(z-\zeta_j)) + (z-\zeta_i)\left(\sigma_i(z)Y'(z) + (1/2)Y'^2(z) + (1/2)Y''(z) + A_i(z)\right)}{1 + (z-\zeta_i)\left(Y'(z) + \sigma_i(z)\right)}.$$

Putting $z = z_i$, $\varepsilon_i = z_i - \zeta_i$ and $\sigma_i(z_i) = \sigma_i$, we obtain

$$\frac{\Phi''(z_i)}{2\Phi'(z_i)} = \frac{Y'(z_i) + \sum_{j\neq i}(1/(z_i-\zeta_j)) + \varepsilon_i\left(\sigma_i Y'(z_i) + (1/2)Y'^2(z_i) + (1/2)Y''(z_i) + A_i(z_i)\right)}{1 + \varepsilon_i(Y'(z) + \sigma_i)}.$$

Let us denote the expression in the brackets of the numerator with β_i. Then one yields

$$\begin{aligned}\frac{\Phi''(z_i)}{2\Phi'(z_i)} &= \frac{1}{1+\varepsilon_i(Y'(z)+\sigma_i)}\left(Y'(z_i) + \sum_{j\neq i}\frac{1}{z_i-\zeta_j}\right) + \frac{\varepsilon_i\beta_i}{1+\varepsilon_i(Y'(z)+\sigma_i)}\\ &= \frac{1}{1+\varepsilon_i(Y'(z)+\sigma_i)}\left(Y'(z_i) + \sum_{j\neq i}\frac{1}{(z_i-z_j)\left(1+\varepsilon_j/(z_i-z_j)\right)}\right) + O(\varepsilon_i).\end{aligned}$$

Developing the expressions $(1+\varepsilon_j/(z_i-z_j))^{-1}$ and $(1+\varepsilon_i(Y'(z)+\sigma_i))^{-1}$ in the geometric series (assuming that $|\varepsilon_j|$ and $|\varepsilon_i|$ are sufficiently small) and rearranging the obtained expression, we get (5). ∎

Therefore, $\Phi''(z_i)/(2\Phi'(z_i))$ can be approximated by $Y'(z_i) + \sum_{j\neq i}(z_i - z_j)^{-1}$. Coming back to Tchebychef's formula (4), we obtain a Tchebychef-like iterative method for the simultaneous approximation of all zeros of the analytic function Φ inside Γ

$$z_i^{(m+1)} = z_i^{(m)} - \frac{\Phi\left(z_i^{(m)}\right)}{\Phi'\left(z_i^{(m)}\right)}\left(1 + \frac{\Phi\left(z_i^{(m)}\right)}{\Phi'\left(z_i^{(m)}\right)}\left(Y'\left(z_i^{(m)}\right) + \sum_{j\neq i}\frac{1}{z_i^{(m)} - z_j^{(m)}}\right)\right) \tag{7}$$

$m = 0, 1 \dots.$

The convergence speed of the simultaneous Tchebychef-like method (7) is considered in the following theorem.

THEOREM 1. *Let $\zeta_1, \dots, \zeta_n$ be the zeros of an analytic function of the form (1) and let $z_1^{(0)}, \dots, z_n^{(0)}$ be their sufficiently close approximations. Then the order of convergence of the iterative method (7) is three.*

PROOF. As in the proof of Lemma 1, we will omit the iteration index, for brevity. From (7) we obtain

$$\hat{\varepsilon}_i = \varepsilon_i - \frac{1}{\Phi'(z_i)/\Phi(z_i)}\left(1 + \frac{1}{\Phi'(z_i)/\Phi(z_i)}\left(Y'(z_i) + \sum_{j\neq i}\frac{1}{z_i - z_j}\right)\right),$$

whence, using (6) and the abbreviation $B_i = Y'(z_i) + (1/\varepsilon_i) + \sum_{j\neq i}(1/(z_i - \zeta_j))$,

$$\hat{\varepsilon}_i = \varepsilon_i - \frac{1}{B_i}\left(1 + \frac{1}{B_i}\left(Y'(z_i) + \sum_{j\neq i}\frac{1}{z_i - z_j}\right)\right).$$

Arranging the last expression, we find

$$\begin{aligned}\hat{\varepsilon}_i &= \varepsilon_i - \frac{\varepsilon_i}{1+Y'(z_i)\varepsilon_i+\varepsilon_i\sigma_i}\,\frac{Y'(z_i)\varepsilon_i+\varepsilon_i\sigma_i+1+\varepsilon_i\left(Y'(z_i)+\sum_{j\neq i}(1/(z_i-z_j))\right)}{1+Y'(z_i)\varepsilon_i+\varepsilon_i\sigma_i}\\ &= \frac{\varepsilon_i^2(\sigma_i+Y'(z_i))+\varepsilon_i^3(\sigma_i+Y'(z_i))^2-\varepsilon_i^2Y'(z_i)-\varepsilon_i^2\sum_{j\neq i}(1/(z_i-z_j))}{[1+\varepsilon_i(Y'(z_i)+\sigma_i)]^2}\\ &= \frac{\varepsilon_i^2\sigma_i+\varepsilon_i^3(\sigma_i+Y'(z_i))^2-\varepsilon_i^2\sum_{j\neq i}(1/(z_i-z_j))}{[1+\varepsilon_i(Y'(z_i)+\sigma_i)]^2}\\ &= \frac{\varepsilon_i^2\sigma_i+\varepsilon_i^3(\sigma_i+Y'(z_i))^2-\varepsilon_i^2\sum_{j\neq i}(1/(z_i-\zeta_j))\cdot 1/(1-(\varepsilon_j/(z_i-\zeta_j))}{[1+\varepsilon_i(Y'(z_i)+\sigma_i)]^2}.\end{aligned}$$

Developing the expression $(1-\varepsilon_j/(z_i-\zeta_j))^{-1}$ in the geometric series, we obtain

$$\hat{\varepsilon}_i = \frac{\varepsilon_i^3(\sigma_i+Y'(z_i))^2-\varepsilon_i^2\sum_{j\neq i}(\varepsilon_j/(z_i-\zeta_j)^2+\cdots)}{[1+\varepsilon_i(Y'(z_i)+\sigma_i)]^2}. \tag{8}$$

Hence, taking $|\hat{\varepsilon}| = \max_{1\le i\le n}|\hat{\varepsilon}_i|$ and $|\varepsilon| = \max_{1\le i\le n}|\varepsilon_i|$, we find $|\hat{\varepsilon}| = O(|\hat{\varepsilon}|^3)$, from which we conclude that Tchebychef-like method (7) for calculating the zeros of analytic functions of the form (1) has the cubic convergence. ∎

REMARK 1. Particularly, if $\Phi(z)$ is a monic polynomial with simple zeros $\zeta_1,\dots,\zeta_n$, that is

$$X(z)=1, \qquad Y(z)=0, \qquad \Phi(z)=\prod_{j=1}^{n}(z-\zeta_j),$$

then (7) reduces to the simultaneous method for polynomial zeros.

Algorithm (7) requires the calculation of the derivative $Y'(z)$ at the points $z_1,\dots,z_n$. As it was noted by Ioakimidis and Anastasselou [4], the values $Y'(z_i)$ given by (3) should be computed in practice by applying a sufficiently accurate quadrature rule for contour of the form

$$\frac{1}{2\pi i}\int_\Gamma g(w)\,dw \cong \sum_{k=1}^{\lambda} a_{k\lambda}g(w_{k\lambda}),$$

where $a_{k\lambda}$ are the *weights* and $w_{k\lambda}$ the corresponding *nodes* of the quadrature rule. As recommended in [4], it is convenient to apply trapezoidal quadrature rule along the circumference $\Gamma=\{w : |w|=R\}$ with nodes

$$a_{k\lambda}=R\exp(i\theta_{k\lambda}), \quad \theta_{k\lambda}=\frac{(2k-1)\pi}{\lambda} \qquad k=1,\dots,\lambda.$$

Since an extensive discussion on the computational aspects of numerical integration in the application of iterative formulas for analytic functions has been given in the papers [4–7], we omit details.

3. SOME COMPUTATIONAL ASPECTS

In the previous section, we have noted that the calculation of the values $Y'(z_i)$ appearing in the iterative formula (7) was done by the numerical integration in the complex plane. This approximate calculation involves an error in the determination of the zeros applying method (7). In this section we are going to investigate the influence of the error of numerical integration on the convergence rate of the iterative method (7) and point out some computational features.

Let us assume that δ_i is the absolute value of the upper error bound occurring in the approximate integration applied in order to calculate $Y'(z_i)$. The procedure of the analysis of numerical stability of the method (7) is similar to that presented in [8]. It is based on the substitution of the complex number $Y'(z_i)$ by the disk $\{Y'(z_i);\delta_i\}$ and the use of circular complex arithmetic. For the definition of the basic operations of circular complex arithmetic and their properties, see [9].

For simplicity, we introduce the notations

$$c_i = Y'(z_i) + \sum_{j\neq i} \frac{1}{z_i-\zeta_j}, \qquad d_i = Y'(z_i) + \sum_{j\neq i} \frac{1}{z_i - z_j}, \qquad \gamma_i = \frac{|\varepsilon_i|\delta_i}{|1+\varepsilon_i c_i|}.$$

For the error $\hat{\varepsilon}_i = \hat{z}_i - \zeta_i$ from (7), we obtain

$$\begin{aligned}
\hat{\varepsilon}_i \in \hat{Z}_i &:= z_i - \zeta_i - \frac{\Phi(z_i)}{\Phi'(z_i)} \left(1 + \frac{\Phi(z_i)}{\Phi'(z_i)} \left(\{Y'(z_i);\delta_i\} + \sum_{j\neq i} \frac{1}{z_i - z_j}\right)\right) \\
&= \varepsilon_i - \frac{1}{\left\{Y'(z_i) + \sum_{j=i}^{n} (1/(z_i-\zeta_j))\,;\delta_i\right\}} \left(1 + \frac{\{d_i;\delta_i\}}{\left\{Y'(z_i) + \sum_{j=1}^{n} (1/(z_i-\zeta_j))\,;\delta_i\right\}}\right) \\
&= \varepsilon_i - \frac{1}{\{c_i + 1/\varepsilon_i;\delta_i\}} \left(1 + \frac{\{d_i;\delta_i\}}{\{c_i+1/\varepsilon_i;\delta_i\}}\right);
\end{aligned}$$

that is,

$$\hat{\varepsilon}_i \in \hat{Z}_i = \varepsilon_i - \frac{\varepsilon_i}{\{1+\varepsilon_i c_i; |\varepsilon_i|\delta_i\}} \left(1 + \frac{\{\varepsilon_i d_i; |\varepsilon_i|\delta_i\}}{\{1+\varepsilon_i c_i; |\varepsilon_i|\delta_i\}}\right).$$

Hence, applying circular arithmetic operations, after extensive but elementary calculations, we find the following expressions for the center and radius of the disk $\hat{Z}_i$:

$$\operatorname{mid} \hat{Z}_i = \frac{\varepsilon_i \left[\varepsilon_i (c_i - d_i) + \varepsilon_i^3 c_i^2 - L_i \gamma_i^2 + K_i \gamma_i^4\right]}{M_i^2}, \tag{9}$$

$$\operatorname{rad} \hat{Z}_i = \frac{|\varepsilon_i|^2 \left[\delta_i (2-\gamma_i^2) + \gamma_i (2\delta_i + 2|d_i| + |d_i|\gamma_i)\right]}{|M_i|^2}, \tag{10}$$

where

$$M_i = (1+\varepsilon_i c_i)\left(1-\gamma_i^2\right), \quad L_i = 1 + 2c_i\varepsilon_i + c_i\varepsilon_i^2 + 2c_i^2\varepsilon_i^3, \quad K_i = 1 + c_i\varepsilon_i + c_i\varepsilon_i^2 + c_i^2\varepsilon_i^3$$

are complex numbers which, for sufficiently small $|\varepsilon_i|$, are bounded and tend to 1 when $\varepsilon_i \to 0$.

Let us consider the difference $c_i - d_i$:

$$\begin{aligned}
c_i - d_i &= Y'(z_i) + \sum_{j\neq i} \frac{1}{z_i-\zeta_j} - Y'(z_i) - \sum_{j\neq i} \frac{1}{z_i - z_j} = \sum_{j\neq i} \frac{1}{z_i-\zeta_j} - \sum_{j\neq i} \frac{1}{z_i - z_j} \\
&= \sum_{j\neq i} \frac{1}{z_i-\zeta_j} - \sum_{j\neq i} \frac{1}{z_i-\zeta_j-\varepsilon_j} = \sum_{j\neq i} \frac{1}{z_i-\zeta_j} - \sum_{j\neq i} \frac{1}{z_i-\zeta_j} \cdot \frac{1}{1-\varepsilon_j/(z_i-\zeta_j)}.
\end{aligned}$$

After developing the expression $1/(1-\varepsilon_j/(z_i-\zeta_j))$ into the geometric series and some arrangement, we find

$$c_i - d_i = -\sum_{j\neq i} \left(\frac{\varepsilon_j}{(z_i-\zeta_j)^2} + \cdots\right) + O(\varepsilon^2).$$

Substituting this estimate in (9) we obtain

$$\operatorname{mid} \hat{Z}_i = \frac{\varepsilon_i \left[\varepsilon_i \left(-\sum_{j\neq i} (\varepsilon_j/(z_i-\zeta_j)^2 + \cdots) + O(\varepsilon^2)\right) + \varepsilon_i^3 c_i^2 - L_i\gamma_i^2 + K_i\gamma_i^4\right]}{M_i^2}. \tag{11}$$

Since $\gamma_i = O(|\varepsilon_i|\delta_i)$, from (10) and (11) we estimate

$$\operatorname{rad} \hat{Z}_i = O\left(|\varepsilon_i|^2\delta_i\right) \sim a_i|\varepsilon_i|^2\delta_i, \qquad \operatorname{mid} \hat{Z}_i = O(\varepsilon^3) \sim b_i\varepsilon^3,$$

where $a_i > 0$ and $b_i \in \mathbb{C}$ are bounded numbers. According to this and the obvious implication

$$\hat{\varepsilon}_i \in \hat{Z}_i \Rightarrow |\hat{\varepsilon}_i| \le \left|\operatorname{mid} \hat{Z}_i\right| + \operatorname{rad} \hat{Z}_i,$$

we have the estimate

$$|\hat{\varepsilon}_i| = O\left(|\varepsilon|^2(|b_i|\,|\varepsilon| + a_i\delta_i)\right), \qquad |\varepsilon| = \max_{1\le i\le n} |\varepsilon_i|. \tag{12}$$

From the inequality (12) we conclude that Tchebychef-like method (7) *preserves the cubic convergence* if $\delta_i = O(|\varepsilon|)$. Since $|\Phi(z_i)| = O(|\varepsilon_i|)$, we can say that the order of convergence of (7) is the same as in the absence of the error of numerical integration (calculating $Y'(z_i)$) if this error is of the same order as the absolute value of the function Φ at the point $z = z_i$. If $Y'(z_i)$ was calculated with relatively small accuracy so that the condition $\delta_i = O(|\Phi(z_i)|)$ is not satisfied, then the convergence speed of the method (7) decreases, but not too much. For example, if even $\delta_i = O(1)$ (which often assumes a crude calculation of $Y'(z_i)$), then the order of convergence of the iterative method (7) will be *at least two*. According to this we infer that Tchebychev-like method (7) is rather stable in the presence of quadrature errors, which has been confirmed by numerical experiments.

REMARK 2. If $\Delta_i = \Phi(z_i)/\Phi'(z_i)$ denotes Newton's correction, then from (7) we have

$$\hat{z}_i = z_i - \Delta_i - (\Delta_i)^2 \left[Y'(z_i) + \sum_{j\ne i}(z_i - z_j)^{-1}\right].$$

Hence, in spite of the quadrature errors (if they are reasonably small), the convergence of the method (7) is practically ensured by the main correction term—Newton correction Δ_i. At the same time, $Y'(z_i)$ is multiplied by the quantity $(\Delta_i)^2$, which is very small (in magnitude) if z_i is sufficiently close to the zero ζ_i, so that the influence of the quadrature error is neutralized. This analysis is quite similar to that given in [4].

REMARK 3. In the case when some of the sought zeros are very close to the contour Γ, the presented algorithm (7) (and, also, all other algorithms based on the same principle) can produce poor results if the quadrature formula is not applied with high accuracy. Generally speaking, the price to be paid in order to attain very high convergence and approximations with a great number of accurate digits consists of the requirement for a very high precision arithmetic. Fortunately, at the present time, this is not a problem since multiprecision arithmetic (about 32 significant digits or more) is often built-in on modern computers.

EXAMPLE. In order to demonstrate the convergence rate of Tchebychef-like method (7), we have considered the analytic function

$$\Phi(z) = e^z - 2\cos 3z - 2$$

inside the disk $D = \{z : |z| \le 1.5\}$. Using the computable argument principle [2] it has been found that this analytic function has $n = 3$ zeros inside D. The real numbers $z_1^{(0)} = -1.4$, $z_2^{(0)} = -0.5$, $z_3^{(0)} = 0.9$ (found by a search algorithm including a proximity test for the detection of the presence of a zero) have been taken as starting approximations. Algorithm (7) has been realized in MS-FORTRAN (Microsoft version 5.1) using double-precision arithmetic (about 16 significant digits) on PC 486/66.

Table 1. Approximations obtained by Tchebychef-like method (7). The underlined digit indicates the first incorrect digit. All digits of $z_3^{(3)}$ are correct.

i	$z_i^{(1)}$	$z_i^{(2)}$	$z_i^{(3)}$
1	−1.2485	−1.22974921	−1.2297087181150930
2	−0.8150	−0.82192655	−0.8219322065738026
3	0.5836	0.56406522	0.5640643677390563

The results of the first three iterative steps are displayed in Table 1. The underlined digit indicates the first incorrect digit.

4. ASYNCHRONOUS IMPLEMENTATION OF A TCHEBYCHEF-LIKE METHOD

The parallelization of simultaneous iterative methods for finding zeros on distributed memory multicomputers has been considered in detail in [10–14]. It has been emphasized there that the total cost of such a parallelization per iteration is the sum of a computation time and a communication time needed for a total exchange of data at each iterative step.

Synchronous versions of simultaneous methods have been studied extensively in a number of papers. According to the presented theoretical analysis as well as practical experimentations, it turns out that Weierstrass-Durand-Kerner method is the most efficient regarding the total CPU time. For this reason, we will be concerned in this section with an another implementation of simultaneous method on multicomputers, which can compete with synchronous implementation under suitable conditions. Namely, in order to decrease the communication time the following strategy can be applied (see [12,14,15]). In each iteration, any processor does not have to wait for the end of the total exchange but deals with instantly available data. This type of algorithm is called *asynchronous* by Baudet [16] indicating that the local computation is performed using only a part of the global information. The implementation of an asynchronous method is executed in such a way that, at each iterative step, a processor sends the most recently computed entries to its neighbors only, decreasing the communication time. The decrease of this time is attained on the account of the convergence rate of the asynchronous version, which is the subject of this section.

Since the model of asynchronous implementation has been described extensively in the papers cited above, we will omit details and concentrate to the convergence analysis of asynchronous version of Tchebychef-like method (7). We assume that the number of processors $k(\leq n)$ is given in advance. The starting vector $\mathbf{z}^{(0)} = (z_1^{(0)}, \dots, z_n^{(0)})$ is computed by all the processors $P_1, \dots, P_k$ using some suitable search procedure. Furthermore, each step of the algorithm consists of sharing the computation of n improved approximations $z_1^{(m)}, \dots, z_n^{(m)}$ among the processors and in updating their data $\mathbf{z}^{(m)}$ through a broadcast procedure. If $I(1,m), \dots, I(k,m)$ are disjunctive partitions of the set $\{1, \dots, n\}$ where $\cup I(j,m) = \{1, \dots, n\}$, at the m^{th} iterative step the processor $P_j (j = 1, \dots, k)$ computes $z_i^{(m)}$ for all $i \in I(j,m)$ by the iterative formula (7) and then it transmits these values to the neighbor processors. As explained by Cosnard and Fraigniaud [12], the indices distribution is necessary at each iteration to ensure the safe convergence of the applied method. The program terminates when some stopping criterion (referred to as $STOP(\mathbf{z}^{(m)})$) is fulfilled, for instance, if

$$\max_{1 \leq i \leq n} \left| \Phi\left(z_i^{(m)}\right) \right| < \delta$$

for a given sufficiently small δ.

Let us assume that the new approximation $z_i^{(m+1)}$ is calculated by a processor $P_h, h \in \{1, \dots, k\}$. Evidently, to ensure the convergence, this processor *must know* the value of $z_i^{(m)}$. The improved approximation $z_i^{(m+1)}$ is calculated by the asynchronous Tchebychef-like formula

$$z_i^{(m+1)} = z_i^{(m)} - \frac{\Phi\left(z_i^{(m)}\right)}{\Phi'\left(z_i^{(m)}\right)} \left(1 + \frac{\Phi\left(z_i^{(m)}\right)}{\Phi'\left(z_i^{(m)}\right)} \left(Y'\left(z_i^{(m)}\right) + \sum_{j\neq i} \frac{1}{z_i^{(m)} - z_j^{(m-r(j,m,h))}}\right)\right). \tag{13}$$

In (13), $z_j^{(m-r(j,m,h))}$ is the last approximation to the zero ζ_j known by the processor P_h at step m. Here $r(j,m,h)$ is a **delay** depending on j,m and h and indicating that the processor P_h only knows the value of z_j computed at step $m - r(j,m,h)$. The maximum delay will be denoted by r; that is, $r = \max_{j,m,h} r(j,m,h)$.

According to the previous we give a program in pseudocode for a parallel implementation of the simultaneous method (13) (following [12]):

Program ASYNCHRONOUS TCHEBYCHEF-LIKE METHOD (13)
begin
  for all $j = 1,\dots,k$ **do** determination of the starting approximations $\mathbf{z}^{(0)}$;
  $m := 0$
  do
    for all $j = 1,\dots,k$ **do in parallel**
    begin
(0)   Distribute $I(j,m)$;
(1)   Compute $z_i^{(m+1)}$ by (13), $i \in I(j,m)$
(2)   Send $z_i^{(m+1)}$, $i \in I(j,m)$, to neighbors;
    end
    $m := m + 1$
(3) **until** *STOP* $\left(\mathbf{z}^{(m)}\right)$ holds true;
  OUTPUT $\mathbf{z}^{(m)}$
end

As mentioned above, the asynchronous implementation decreases the convergence speed of the applied method. In the following theorem we give the lower bound of the order of convergence of the asynchronous Tchebychef-like method (13).

THEOREM 2. *Suppose that starting approximations $z_1^{(0)},\dots,z_n^{(0)}$ are reasonably close to the zeros $\zeta_1,\dots,z_n$ of the analytic function Φ of the form (1). Further, assume that $r(j,m,h)$ is bounded for all $j = 1,\dots,n$ and all $h = 1,\dots,k$. Then the asynchronous Tchebychef-like algorithm (13) is locally convergent with the order of convergence at least $\eta > 2$, where η is the unique positive root of the equation*

$$x^{r+1} - 2x^r - 1 = 0, \qquad r = \max_{j,m,h} r(j,m,h). \tag{14}$$

PROOF. For simplicity, the approximations $z_j^{(m-r)}$ to the roots $\zeta_1,\dots,\zeta_n$ at the iterative step m will be shortly denoted with z_j if $r = 0$ and z_j^* if $r > 0$. According to this notation we introduce the errors $\varepsilon_i = z_i - \zeta_i$ and $\varepsilon_j^* = z_j^* - \zeta_j$. The new approximation $z_i^{(m+1)}$ will be denoted with $\hat{z}_i$ and the corresponding error with $\hat{\varepsilon}_i = \hat{z}_i - \zeta_i$.

The proof of Theorem 2 is performed in quite a similar way as the proof of Theorem 1 assuming only that z_j and ε_j are substituted by z_j^* and ε_j^*. According to this, starting from (13) we obtain the relation

$$\hat{\varepsilon}_i = \frac{\varepsilon_i^3\left(\sigma_i + Y'(z_i)\right)^2 - \varepsilon_i^2 \sum_{j\neq i}\left(\varepsilon_j^*/(z_i-\zeta_j)^2 + \cdots\right)}{\left[1 + \varepsilon_i(Y'(z_i) + \sigma_i)\right]^2} \tag{15}$$

which corresponds to (8). Let $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_n)$ and $\mathbf{z}^{(m)} = (z_1^{(m)}, \dots, z_n^{(m)})$ be the vectors of zeros and their approximations. If at the iterative step m we define the absolute error e_m by $e_m := \|\mathbf{z}^{(m)} - \boldsymbol{\zeta}\|_\infty$, then from (15) we obtain

$$e_{m+1} = O\left(\varepsilon_m^2 \varepsilon_{m-r}\right).$$

For large $m \geq m_0$ let us put $e_m = O(E)$, where $0 < E < 1$ according to the assumed closeness of initial approximations. Therefore, the sequence (e_m) tends to 0. Let the order of convergence of the sequence (e_m) be η; that is, $e_{m+1} = O\left(e_m^\eta\right)$. Then

$$e_{m-r} = O\left(E^{1/\eta^r}\right) \qquad r = 0, 1, \dots, m,$$

so that we have

$$e_{m+1} = O\left(e_m^2 \cdot e_{m-r}\right) = O\left(E^{2+1/\eta^r}\right) \quad \text{and} \quad e_{m+1} = O\left(E^\eta\right).$$

By the comparison of the exponents it follows $\eta = 2 + 1/\eta^r$, which reduces to the equation (14). Let $f(\eta) = \eta^{r+1} - 2\eta^r - 1$. The function f is convex and $f(0) = f(2) = -1$, $f(3) = 3^r - 1 \geq 0$ holds. Hence, f has the unique positive zero η belonging to the interval $(2, 3]$. ∎

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