A finite difference scheme for solving the Timoshenko beam equations with boundary feedback

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Abstract

In this study, we derive a finite difference for a Timoshenko beam with boundary feedback by the method of reduction of order on uniform meshes. It is proved by the discrete energy method that the scheme is uniquely solvable, unconditionally stable and second order convergent in $L_\infty$ norm. Numerical results demonstrate the theoretical results.

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1. Introduction

As already pointed out by Xu and Feng [15], the boundary control problem of flexible structure has recently attracted much attention with the rapid development of high technology such as space science and flexible robots. If the cross-section dimension of the beam is negligible in comparison with its length, the transversal vibration of an elastic beam is described by the Euler–Bernoulli beam equation. If the cross-section dimension is not negligible, then it is necessary to consider the effect of the rotatory inertia. If the deflection due to shear is not negligible either, then the transversal vibration is described by the so-called Timoshenko beam equation. Up to now, a large number of interesting results on the boundary feedback of Timoshenko model have been obtained by many researchers (e.g. see [7,14,15,17] and references there in). Recently, much numerical work has also been done to the Timoshenko beam model, which (assuming dimensionless variables) is given by the system

\[-\frac{\partial^2 \theta(x)}{\partial x^2} + \frac{1}{d^2} \left[ \theta(x) - \frac{\partial w(x)}{\partial x} \right] = 0, \quad 0 \leq x \leq 1, \quad (1.1)\]

\[\frac{1}{d^2} \left[ \frac{\partial^2 \theta(x)}{\partial x^2} - \frac{\partial^2 w(x)}{\partial x^2} \right] = g(x), \quad 0 \leq x \leq 1, \quad (1.2)\]

\[\theta(0) = \theta(1) = w(0) = w(1) = 0, \quad (1.3)\]

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where \( d \) represents the thickness of the beam, \( \bar{\theta}(x) \) is the rotation of vertical fibers in the beam and \( w(x) \) is the vertical displacement of the beam’s centerline (under a vertical load given by \( g(x) \)). The model was studied by Timoshenko [12] and is considered an improvement over the standard Euler–Bernoulli beam model since shear deformation is taken into account. Li [8] considered discretization of problem (1.1)–(1.3) by the \( p \) and the \( h – p \) versions of the finite element method and obtained the optimal error estimates which is independent of the thickness of the beam. Brandts [1] studied the discretization by the mixed finite element and obtained the superconvergence of the mixed finite element solutions to projections of the real solutions on the approximating spaces in the global \( H^1 \)-norm uniform in \( d \).

Semper [9], Feng et al. [3] considered the following time dependent vibrating Timoshenko beam equations

\[
\rho \frac{\partial^2 w(x,t)}{\partial t^2} - K \left[ \frac{\partial^2 w(x,t)}{\partial x^2} - \frac{\partial \varphi(x,t)}{\partial x} \right] = f_1(x,t), \quad 0 < x < l, \quad t > 0, \tag{1.9}
\]

\[
I \rho \frac{\partial^2 \varphi(x,t)}{\partial t^2} - EI \frac{\partial^2 \varphi(x,t)}{\partial x^2} - K \left[ \frac{\partial w(x,t)}{\partial x} - \varphi(x,t) \right] = f_2(x,t), \quad 0 < x < l, \quad t > 0, \tag{1.10}
\]

\[
w(0, t) = w(l, t) = \varphi(0, t) = \varphi(l, t) = 0, \quad 0 \leq t \leq T, \tag{1.11}
\]

\[
w(x, 0) = w_0(x), \quad \frac{\partial w(x, 0)}{\partial t} = w_1(x), \quad 0 \leq x \leq 1, \tag{1.12}
\]

\[
\varphi(x, 0) = \varphi_0(x), \quad \frac{\partial \varphi(x, 0)}{\partial t} = \varphi_1(x), \quad 0 \leq x \leq 1, \tag{1.13}
\]

where \( \delta \) represents a damping constant. Semper discussed some semi-discrete and fully discrete Galerkin method for this model. He obtained optimal-order error estimates with constants independent of the beam thickness under the assumption of the regularity of the solution of (1.4)–(1.8). Feng et al. studied the semi-discrete and fully discrete schemes for the vibrating beam model (1.4)–(1.8) using the partial projection finite element method, and also obtained optimal convergence rates with constants independent of the beam thickness when assuming that a smooth solution exists. Cheng and Xue [2] considered the linear finite element approximations for the Timoshenko beam. Franca and Loula [4] studied the mixed finite element method for the Timoshenko beam. Jou [6] investigated the least-squares finite element approximations to the Timoshenko beam. Feng et al. [3] considered the following time dependent vibrating Timoshenko beam equations

\[
\frac{\partial^2 w(x,t)}{\partial t^2} + \delta \frac{\partial w(x,t)}{\partial t} - \frac{1}{d^2} \left[ \frac{\partial^2 w(x,t)}{\partial x^2} - \frac{\partial \varphi(x,t)}{\partial x} \right] = g(x, t), \quad 0 \leq x \leq 1, \quad 0 < t \leq T, \tag{1.4}
\]

\[
\frac{\partial^2 \varphi(x,t)}{\partial t^2} + \delta \frac{\partial \varphi(x,t)}{\partial t} - \frac{1}{d^2} \left[ \frac{\partial w(x,t)}{\partial x} - \varphi(x,t) \right] = 0, \quad 0 \leq x \leq 1, \quad 0 < t \leq T, \tag{1.5}
\]

\[
w(0, t) = w(1, t) = \varphi(0, t) = \varphi(1, t) = 0, \quad 0 \leq t \leq T, \tag{1.6}
\]

\[
w(x, 0) = w_0(x), \quad \frac{\partial w(x, 0)}{\partial t} = w_1(x), \quad 0 \leq x \leq 1, \tag{1.7}
\]

\[
\varphi(x, 0) = \varphi_0(x), \quad \frac{\partial \varphi(x, 0)}{\partial t} = \varphi_1(x), \quad 0 \leq x \leq 1, \tag{1.8}
\]

where \( \rho \) represents the mass density, \( I \) is the moment of inertia, \( E \) is the Young’s modulus, \( l \) is the length of the beam, \( \alpha, \beta \) are given positive gain feedback constants, \( w(x, t) \) is the transversal displacement and \( \varphi(x, t) \) is the rotational angle of the beam. The boundary conditions in (1.11) and (1.12) mean that the beam is clamped at \( x = 0 \) and controlled at \( x = l \) by the force and moment feedback. Xu and Feng [15] studied the Riesz basis property of the generalized eigenvector system of (1.9)–(1.13).
Throughout this article, for simplicity, we assume that \( f_1(x, t), f_2(x, t) \in C([0, l] \times [0, T]), g_1(t), g_2(t) \in C[0, T], g_3(x), g_4(x), g_5(x), g_6(x) \in C([0, l]) \) and \( g_3(0) = g_5(0) = 0 \) such that problem (1.9)–(1.13) has a smooth solution \( w(x, t) \in C^{3,5}([0, l] \times [0, T]), \phi(x, t) \in C^{3,3}([0, l] \times [0, T]). \)

Take two positive integers \( M \) and \( N \) and denote \( h = l / M, \tau = T / N. \) Let \( \Omega_h = \{ x_i : x_i = i h, 0 \leq i \leq M \}, \Omega = \{ t_k : t_k = k \tau, 0 \leq k \leq N \} \) and \( \Omega_{h\tau} = \Omega_h \times \Omega. \) Suppose \( u = \{ u_i^k \mid 0 \leq i \leq M, 0 \leq k \leq N \} \) be a discrete function on \( \Omega_{h\tau}. \) Introduce the following notations:

\[
\begin{align*}
&u_{i+1/2}^k = \frac{u_i^k + u_{i+1}^k}{2}, \quad \delta_x u_{i+1/2}^k = \frac{u_{i+1}^k - u_i^k}{h}, \quad \delta_t u_{i+1/2}^k = \frac{u_{i+1}^k - u_i^k}{\tau}, \\
&u_i^k = \frac{u_i^{k+1} + u_i^{k-1}}{2}, \quad D_t u_i^k = \frac{u_i^{k+1} - u_i^{k-1}}{2\tau}, \quad \delta_t^2 u_i^k = \frac{\delta_t u_i^{k+1/2} - \delta_t u_i^{k-1/2}}{\tau}, \\
&\|u^k\| = \left[ h \sum_{i=1}^{M} (u_{i-1/2}^k)^2 \right]^{1/2}, \quad \|\delta_x u^k\| = \left[ h \sum_{i=1}^{M} (\delta_x u_{i-1/2}^k)^2 \right]^{1/2}, \quad x_{i-1/2} = \frac{1}{2}(x_i + x_{i-1}),
\end{align*}
\]

where \( u_{i+1/2}^k \) is an average of \( u \) at the points \((x_i, t_k)\) and \((x_{i+1}, t_k)\), and \( \delta_x u_{i+1/2}^k \) is the difference quotient of \( u \) based on the two points; \( \delta_t u_{i+1/2}^k \) is the difference quotient of \( u \) based on the two points \((x_i, t_k)\) and \((x_i, t_{k+1})\); \( u_i^k \) is an average of \( u \) at the points \((x_i, t_{k+1})\) and \((x_i, t_k)\) and \( D_t u_i^k \) is the difference quotient of \( u \) based on these points; \( \delta_t^2 u_i^k \) is the second order difference quotient of \( u \) based on the points \((x_i, t_{k+1}), (x_i, t_k)\) and \((x_i, t_{k+1})\); \( \|u^k\| \) and \( \|\delta_x u^k\| \) are some norms if \( u_0^k = 0 \). In addition, if \( g = \{ g_i \mid 0 \leq i \leq M \} \) is a grid function on \( \Omega_h \) with \( g_0 = 0 \), we have

\[
\|g\|_\infty \leq \sqrt{h}\|\delta_x g\|, \quad \|g\| \leq \frac{l}{\sqrt{2}}\|\delta_x g\|. \tag{1.14}
\]

The difference scheme we will consider for (1.9)–(1.13) is as follows:

\[
\begin{align*}
\frac{\rho}{2} (\delta_t^2 w_{i-1/2}^k + \delta_x^2 w_{i+1/2}^k) - K \left[ \delta_x w_i^k - \frac{1}{2} (\delta_x \phi_{i-1/2}^k + \delta_x \phi_{i+1/2}^k) \right] &= \frac{1}{2} [(f_1)^k_{i-1/2} + (f_1)^k_{i+1/2}], \\
&1 \leq i \leq M - 1, \quad 1 \leq k \leq N - 1, \tag{1.15}
\end{align*}
\]

\[
\begin{align*}
\frac{I_0}{2} (\delta_t^2 \phi_{i-1/2}^k + \delta_x^2 \phi_{i+1/2}^k) - EI \delta_x^2 \phi_i^k - \frac{K}{2} [(\delta_x w_{i+1/2}^k + \delta_x w_{i-1/2}^k) - (\phi_{i-1/2}^k + \phi_{i+1/2}^k)] &= \frac{1}{2} [(f_2)^k_{i-1/2} + (f_2)^k_{i+1/2}], \\
&1 \leq i \leq M - 1, \quad 1 \leq k \leq N - 1, \tag{1.16}
\end{align*}
\]

\[
\begin{align*}
w_0^k &= 0, \quad \phi_0^k = 0, \quad 1 \leq k \leq N, \tag{1.17}
\end{align*}
\]

\[
\begin{align*}
\delta_x w_{M-1/2}^k - \phi_M^k &= -\alpha D_t w_M^k - \frac{h}{2K} [K \delta_x \phi_{M-1/2}^k + \rho \delta_x^2 w_{M-1/2}^k - (f_1)^k_{M-1/2}] + (g_1)^k, \\
&1 \leq k \leq N - 1, \tag{1.18}
\end{align*}
\]

\[
\begin{align*}
\delta_x \phi_{M-1/2}^k &= -\beta D_t \phi_M^k - \frac{h}{2EI} [I_0 \delta_x^2 \phi_M^k - K (\delta_x w_{M-1/2}^k - \phi_{M-1/2}^k) - (f_2)^k_{M-1/2}] + (g_2)^k, \\
&1 \leq k \leq N - 1, \tag{1.19}
\end{align*}
\]

\[
\begin{align*}
w_i^0 &= (g_3)_i, \quad w_i^1 = (g_3)_i + \tau (g_4)_i + \frac{\tau^2}{2\rho} [K ((g_3')_i - (g_5')_i) + (f_1)^0_i], \quad 0 \leq i \leq M, \tag{1.20}
\end{align*}
\]

\[
\begin{align*}
\phi_i^0 &= (g_5)_i, \quad \phi_i^1 = (g_5)_i + \tau (g_6)_i + \frac{\tau^2}{2I_0} [EI (g_5'')_i + K ((g_5')_i - (g_5')_i) + (f_2)^0_i], \quad 0 \leq i \leq M. \tag{1.21}
\end{align*}
\]
Here the primes denote derivatives with respect to space variable \( x \), \((f_1)_i^{k-1/2} = f_1(x_{i-1/2}, t_k)\), \((g_1)^k = g_1(t_k)\), \((g_2)^k, (g_4)^k\), \((g_5)^k\), \((g_6)^k\), etc., are similar.

The remainder of the rest of the article is arranged as follows. In Section 2, the difference scheme (1.15)–(1.21) is derived by the method of reduction order \([10,11,13]\). In Section 3, the unique solvability, unconditional convergence and stability of the difference scheme are proved by the energy method. The convergence order is of \( O(h^2 + \tau^2) \). In Section 4, some numerical results are provided to demonstrate the theoretical results.

2. The derivation of the difference scheme

Let \( v = \partial w / \partial x \) and \( \psi = \partial \phi / \partial x \), then (1.9)–(1.13) is equivalent to the following system of equations:

\[
\rho \frac{\partial^2 w(x, t)}{\partial t^2} - K \left[ \frac{\partial v(x, t)}{\partial x} - \psi(x, t) \right] = f_1(x, t), \quad 0 < x < l, \quad t > 0, \tag{2.1}
\]

\[
v(x, t) - \frac{\partial w(x, t)}{\partial x} = 0, \quad 0 < x < l, \quad t > 0, \tag{2.2}
\]

\[
I_\rho \frac{\partial^2 \phi(x, t)}{\partial t^2} - EI \frac{\partial \psi(x, t)}{\partial x} - K [v(x, t) - \varphi(x, t)] = f_2(x, t), \quad 0 < x < l, \quad t > 0, \tag{2.3}
\]

\[
\psi(x, t) - \frac{\partial \phi(x, t)}{\partial x} = 0, \quad 0 < x < l, \quad t > 0, \tag{2.4}
\]

\[
w(0, t) = 0, \quad \varphi(0, t) = 0, \quad t > 0, \tag{2.5}
\]

\[
w(l, t) - \varphi(l, t) = -2 \frac{\partial w(l, t)}{\partial t} + g_1(t), \quad \psi(l, t) = -\beta \frac{\partial \phi(l, t)}{\partial t} + g_2(t), \quad t > 0, \tag{2.6}
\]

\[
w(x, 0) = g_3(x), \quad \frac{\partial w(x, 0)}{\partial t} = g_4(x), \quad \varphi(x, 0) = g_5(x), \quad \frac{\partial \phi(x, 0)}{\partial t} = g_6(x), \quad 0 \leq x \leq l. \tag{2.7}
\]

In (2.1)–(2.7), the maximal order of the derivatives with respect to \( x \) is only one, which is less than that of the original problem (1.9)–(1.13). We will derive the difference (1.15)–(1.21) for (1.9)–(1.13) by considering (2.1)–(2.7). This indirect constructing-difference-scheme method is called the method of reduction of order.

Define the grid functions:

\[
W_i^k = w(x_i, t_k), \quad V_i^k = v(x_i, t_k), \quad \Phi_i^k = \varphi(x_i, t_k), \quad \Psi_i^k = \psi(x_i, t_k), \quad 0 \leq i \leq M, \quad 0 \leq k \leq N.
\]

Using the Taylor expansion, we have

\[
\rho \delta_t^2 W_{i-1/2}^k - K (\delta_x V_{i-1/2}^k - \Phi_{i-1/2}^k) = (f_1)_{i-1/2}^k + (e_1)_{i-1/2}^k, \quad 1 \leq i \leq M, \quad 1 \leq k \leq N - 1, \tag{2.8}
\]

\[
V_{i-1/2}^k - \delta_x W_{i-1/2}^k = (e_2)_{i-1/2}^k, \quad 1 \leq i \leq M, \quad 0 \leq k \leq N, \tag{2.9}
\]

\[
I_\rho \delta_t^2 \Phi_{i-1/2}^k - EI \delta_x \Psi_{i-1/2}^k - K (V_{i-1/2}^k - \Phi_{i-1/2}^k) = (f_2)_{i-1/2}^k + (e_3)_{i-1/2}^k, \quad 1 \leq i \leq M, \quad 1 \leq k \leq N - 1, \tag{2.10}
\]

\[
\Psi_{i-1/2}^k - \delta_x \Phi_{i-1/2}^k = (e_4)_{i-1/2}^k, \quad 1 \leq i \leq M, \quad 0 \leq k \leq N, \tag{2.11}
\]

\[
W_0^k = 0, \quad \Phi_0^k = 0, \quad 1 \leq k \leq N, \tag{2.12}
\]

\[
V_M^k - \Phi_M^k = -\alpha D_t W_M^k + (g_1)^k + (e_5)^k, \quad 1 \leq k \leq N - 1, \tag{2.13}
\]

\[
\Psi_M^k + \beta D_t \Phi_M^k = (g_2)^k + (e_6)^k, \quad 1 \leq k \leq N - 1, \tag{2.14}
\]
Thanks to the assumption of the existence of smooth solution, there exists a positive constant $c_1$ such that

$$| (e_1)_i | \leq c_1 (\tau^2 + h^2), \quad 1 \leq i \leq M, \quad 1 \leq k \leq N - 1, \quad (2.17)$$

$$| (e_2)_i | \leq c_1 h^2, \quad 1 \leq i \leq M, \quad 0 \leq k \leq N, \quad (2.18)$$

$$| (e_3)_i | \leq c_1 (\tau^2 + h^2), \quad 1 \leq i \leq M, \quad 1 \leq k \leq N - 1, \quad (2.19)$$

$$| (e_4)_i | \leq c_1 h^2, \quad 1 \leq i \leq M, \quad 0 \leq k \leq N, \quad (2.20)$$

$$| (e_5)_i | \leq c_1 \tau^2, \quad 1 \leq k \leq N - 1, \quad (2.21)$$

$$| (e_6)_i | \leq c_1 \tau^2, \quad 1 \leq k \leq N - 1, \quad (2.22)$$

$$| (e_7)_i | \leq c_1 \tau^3, \quad 0 \leq i \leq M, \quad (2.23)$$

$$| (e_8)_i | \leq c_1 \tau^3, \quad 0 \leq i \leq M, \quad (2.24)$$

$$| \delta_x (e_7)_i | \leq c_1 \tau^3, \quad 1 \leq i \leq M, \quad (2.25)$$

$$| \delta_x (e_8)_i | \leq c_1 \tau^3, \quad 1 \leq i \leq M. \quad (2.26)$$

Here, we should note that, by Taylor expansion with the integration remainder term and by means of differential (1.1)–(1.2), we can easily obtain

$$(e_7)_i = \frac{1}{2} \int_0^\tau (\tau - s)^2 \frac{\partial^3 w(x_i, s)}{\partial t^3} \, ds, \quad (e_8)_i = \frac{1}{2} \int_0^\tau (\tau - s)^2 \frac{\partial^3 \phi(x_i, s)}{\partial t^3} \, ds.$$

Consequently,

$$\delta_x (e_7)_i = \frac{(e_7)_i - (e_7)_{i-1}}{h} = \frac{1}{2h} \int_0^\tau (\tau - s)^2 \left( \frac{\partial^3 w(x_i, s)}{\partial t^3} - \frac{\partial^3 w(x_{i-1}, s)}{\partial t^3} \right) \, ds$$

$$= \frac{1}{2} \int_0^\tau (\tau - s)^2 \frac{\partial^4 w(\zeta_i, s)}{\partial x \partial t^3} \, ds,$$

$$\delta_x (e_8)_i = \frac{(e_8)_i - (e_8)_{i-1}}{h} = \frac{1}{2h} \int_0^\tau (\tau - s)^2 \left( \frac{\partial^3 \phi(x_i, s)}{\partial t^3} - \frac{\partial^3 \phi(x_{i-1}, s)}{\partial t^3} \right) \, ds$$

$$= \frac{1}{2} \int_0^\tau (\tau - s)^2 \frac{\partial^4 \phi(\eta_i, s)}{\partial x \partial t^3} \, ds,$$

where $\zeta_i, \eta_i \in (x_{i-1}, x_i)$, with which (2.25) and (2.26) follows.

Based on equalities (2.8)–(2.16), neglecting the small terms, we construct the difference scheme for (2.1)–(2.7) as follows:

$$\rho^2_t w_{i-1/2} - K (\delta_x v_{i-1/2} - \psi_{i-1/2}) = (f_1)^k_{i-1/2}, \quad 1 \leq i \leq M, \quad 1 \leq k \leq N - 1, \quad (2.27)$$

$$v_{i-1/2} - \delta_x w_{i-1/2} = 0, \quad 1 \leq i \leq M, \quad 0 \leq k \leq N, \quad (2.28)$$

$$I_x^2 \psi_{i-1/2} = E I \delta_x \psi_{i-1/2} - K (v_{i-1/2} - \phi_{i-1/2}) = (f_2)^k_{i-1/2}, \quad 1 \leq i \leq M, \quad 1 \leq k \leq N - 1, \quad (2.29)$$
\[ \psi^k_{i-1/2} - \delta_x \psi^k_i = 0, \quad 1 \leq i \leq M, \quad 0 \leq k \leq N, \]  
(2.30)

\[ w^k_i = 0, \quad \varphi^k_0 = 0, \quad 1 \leq k \leq N, \]  
(2.31)

\[ v^k_M - \varphi^k_M = -\alpha D_l w^k_M + (g_1)^k, \quad 1 \leq k \leq N - 1, \]  
(2.32)

\[ \psi^k_M + \beta D_l \varphi^k_M = (g_2)^k, \quad 1 \leq k \leq N - 1, \]  
(2.33)

\[ w^i_0 = (g_3)_i, \quad w^i_1 = (g_3)_i + \tau(g_4)_i + \frac{\tau^2}{2\rho} [K((g_3')_i - (g_5)_i) + (f_1)_i^0], \quad 0 \leq i \leq M, \]  
(2.34)

\[ \varphi^i_0 = (g_5)_i, \quad \varphi^i_1 = (g_5)_i + \tau(g_6)_i + \frac{\tau^2}{2\rho} [EI((g_5')_i + K((g_3')_i - (g_5)_i) + (f_2)_i^0), \quad 0 \leq i \leq M. \]  
(2.35)

At the \((k + 1)\)th time level, we regard (2.27)–(2.35) as a system of linear algebraic equations with respect to unknowns \( \{w^{k+1}_i, \varphi^{k+1}_i, v^{k+1}_i, \psi^{k+1}_i, 0 \leq i \leq M\}. \)

We can prove the following theorem.

**Theorem 1.** The difference scheme (2.27)–(2.35) is equivalent to (1.15)–(1.21) and

\[ v^0_{i-1/2} = \delta_x w^0_{i-1/2}, \quad v^1_{i-1/2} = \delta_x w^1_{i-1/2}, \quad 1 \leq i \leq M, \]  
(2.36)

\[ \psi^0_{i-1/2} = \delta_x \varphi^0_{i-1/2}, \quad \psi^1_{i-1/2} = \delta_x \varphi^1_{i-1/2}, \quad 1 \leq i \leq M, \]  
(2.37)

\[ v^k_i = \delta_x w^k_{i+1/2} - \frac{h}{2} \left[ \delta_x \varphi^k_{i+1/2} + \frac{\rho}{K} \delta^2_x w^k_{i+1/2} - \frac{1}{K} (f_1)_i^k \right], \quad 0 \leq i \leq M - 1, \quad 1 \leq k \leq N - 1, \]  
(2.38)

\[ v^k_M = \delta_x w^k_{M-1/2} + \frac{h}{2} \left[ \delta_x \varphi^k_{M-1/2} + \frac{\rho}{K} \delta^2_x w^k_{M-1/2} - \frac{1}{K} (f_1)_i^k \right], \quad 1 \leq k \leq N - 1, \]  
(2.39)

\[ \psi^k_i = \delta_x \varphi^k_{i+1/2} + \frac{h}{2EI} \left[ K(\delta_x w^k_{i+1/2} - \varphi^k_{i+1/2}) - I_\rho \delta^2_x \varphi^k_{i+1/2} + (f_2)_i^k \right], \quad 0 \leq i \leq M - 1, \quad 1 \leq k \leq N - 1, \]  
(2.40)

\[ \psi^k_M = \delta_x \varphi^k_{M-1/2} - \frac{h}{2EI} \left[ K(\delta_x w^k_{M-1/2} - \varphi^k_{M-1/2}) - I_\rho \delta^2_x \varphi^k_{M-1/2} + (f_2)_i^k \right], \quad 1 \leq k \leq N - 1. \]  
(2.41)

**Proof.** (2.28) is equivalent to (2.36) and

\[ v^k_{i-1/2} = \delta_x w^k_{i-1/2}, \quad 1 \leq i \leq M, \quad 1 \leq k \leq N - 1. \]  
(2.42)

(2.30) is equivalent to (2.37) and

\[ \psi^k_{i-1/2} = \delta_x \varphi^k_{i-1/2}, \quad 1 \leq i \leq M, \quad 1 \leq k \leq N - 1. \]  
(2.43)

Rewrite (2.27) as follows:

\[ \delta_x v^k_{i-1/2} = \delta_x \varphi^k_{i-1/2} + \frac{1}{K} (\rho \delta^2_x w^k_{i-1/2} - (f_1)_i^k), \quad 1 \leq i \leq M, \quad 1 \leq k \leq N - 1. \]  
(2.44)
Multiplying (2.44) by \( \frac{1}{h} \) and adding the result with (2.42), we have
\[
v_i^k = \delta_x v_i^{k-1} + \frac{h}{2} \left[ \delta_x \varphi_i^{k-1} + \frac{1}{K} (\rho \delta_t^2 w_i^{k-1} - (f_1)^i_{k-1} / 2) \right], \quad 1 \leq i \leq M, \quad 1 \leq k \leq N - 1. \tag{2.45}
\]

Multiplying (2.44) by \( \frac{1}{h} \) and subtracting the result from (2.42), we have
\[
v_{i-1}^k = \delta_x v_{i-1}^{k-1} - \frac{h}{2} \left[ \delta_x \varphi_{i-1}^{k-1} + \frac{1}{K} (\rho \delta_t^2 w_{i-1}^{k-1} - (f_1)^i_{k-1} / 2) \right], \quad 1 \leq i \leq M, \quad 1 \leq k \leq N - 1,
\]
that is
\[
v_i^k = \delta_x v_i^{k-1} - \frac{h}{2} \left[ \delta_x \varphi_i^{k-1} + \frac{1}{K} (\rho \delta_t^2 w_i^{k-1} - (f_1)^i_{k-1} / 2) \right], \quad 0 \leq i \leq M - 1, \quad 1 \leq k \leq N - 1. \tag{2.46}
\]

Using the equations in (2.45) and (2.46) for \( 1 \leq i \leq M - 1, 1 \leq k \leq N - 1 \), we have
\[
\delta_x w_{i-1/2}^k + \frac{h}{2} \left[ \delta_x \varphi_{i-1/2}^k + \frac{1}{K} (\rho \delta_t^2 w_{i-1/2}^k - (f_1)^i_{i-1/2}^k) \right] \\
= \delta_x w_{i+1/2}^k - \frac{h}{2} \left[ \delta_x \varphi_{i+1/2}^k + \frac{1}{K} (\rho \delta_t^2 w_{i+1/2}^k - (f_1)^i_{i+1/2}^k) \right],
\]
which is (1.15). It is not difficult to check that (2.27) and (2.28) are equivalent to (2.36) and (2.38) and (1.15). In addition, (2.32) is equivalent to (2.39).

Similarly, we have (2.29) and (2.30) are equivalent to (2.37) and (2.40) and (1.16). In addition, (2.33) is equivalent to (2.41), (2.34) and (2.35) are just (1.20) and (1.21). (2.31) is just (1.17). This completes the proof. \( \square \)

3. Analysis of the difference scheme

In this section we will discuss the solvability, convergence and stability of the difference (1.15)–(1.21).

**Lemma 1.** Suppose \( \{w_i^k, \varphi_i^k, v_i^k, \psi_i^k\} \) be the solution of
\[
\rho \delta_t^2 w_i^{k-1} - K (\delta_x v_i^{k-1} - \psi_i^{k-1}) = (P_1)^i_{k-1}, \quad 1 \leq i \leq M, \quad 1 \leq k \leq N - 1, \tag{3.1}
\]
\[
v_i^{k-1/2} - \delta_x w_i^{k-1/2} = (P_2)^i_{k-1/2}, \quad 1 \leq i \leq M, \quad 0 \leq k \leq N, \tag{3.2}
\]
\[
I \rho \delta_t^2 \varphi_i^{k-1} - EI \delta_x \psi_i^{k-1} - K (v_i^{k-1} - \varphi_i^{k-1}) = (P_3)^i_{k-1}, \quad 1 \leq i \leq M, \quad 1 \leq k \leq N - 1, \tag{3.3}
\]
\[
\psi_i^{k-1/2} - \delta_x \varphi_i^{k-1/2} = (P_4)^i_{k-1/2}, \quad 1 \leq i \leq M, \quad 0 \leq k \leq N, \tag{3.4}
\]
\[
w_0^k = 0, \quad \varphi_0^k = 0, \quad 1 \leq k \leq N, \tag{3.5}
\]
\[
v_M^k - \varphi_M^k = -\alpha D_i w_M^k + (P_5)^k, \quad 1 \leq k \leq N - 1, \tag{3.6}
\]
\[
\psi_M^k + \beta D_i \varphi_M^k = (P_6)^k, \quad 1 \leq k \leq N - 1, \tag{3.7}
\]
\[
w_0^1 = 0, \quad w_1^1 = (P_7)i, \quad 0 \leq i \leq M, \tag{3.8}
\]
\[
\varphi_0^1 = 0, \quad \varphi_1^1 = (P_8)i, \quad 0 \leq i \leq M. \tag{3.9}
\]
Let
\[ F(k) = \rho \| \delta_i w_i^{k+1/2} \|^2 + I_\rho \| \delta_i \phi_i^{k+1/2} \|^2 + \frac{EI}{2} (\| \psi^{k+1} \|^2 + \| \psi^k \|^2) + \frac{K}{2} (\| v_i^{k+1} - \phi_i^{k+1} \|^2 + \| v_i^k - \phi_i^k \|^2), \quad 0 \leq k \leq N - 1. \]

Then, we have
\[ F(k) \leq \left( 1 + \frac{3}{2} \tau \right)^k \left[ F(0) + \frac{3}{2} \tau \sum_{n=1}^{k} G(n) \right], \quad 1 \leq k \leq N - 1, \]

where
\[ G(n) = \frac{2}{\rho} \| (P_i)^n \|^2 + \frac{1}{I_\rho} \| (P_i)^n \|^2 + \frac{2K^2}{\rho} \| (P_i)^n \|^2 + \frac{K}{2\rho} \| (P_i)^n \|^2 + \frac{EI}{2\rho} \| (P_i)^n \|^2 + 2K \| D_i (P_i)^n \|^2 + 2EI \| D_i (P_i)^n \|^2. \]

**Proof.** From (3.2) and (3.4), we have
\[ D_i \psi_i^{k+1/2} - D_i \psi_{i-1/2} = D_i \delta_i \psi_i^{k+1/2} + D_i (P_i)^{k+1/2}, \quad 1 \leq i \leq M, \quad 1 \leq k \leq N - 1, \tag{3.10} \]
\[ D_i \psi_{i-1/2} = D_i \delta_i \psi_{i-1/2} + D_i (P_i)^{k-1/2}, \quad 1 \leq i \leq M, \quad 1 \leq k \leq N - 1. \tag{3.11} \]

Multiplying (3.1) by \( 2h D_i w_i^{k+1/2} \) and then summing up for \( i \) from 1 to \( M \), we have
\[ 2h \rho \sum_{i=1}^{M} D_i w_i^{k+1/2} \cdot \delta_i^2 w_i^{k+1/2} - 2Kh \sum_{i=1}^{M} D_i w_i^{k+1/2} \cdot (\delta_i w_i^{k+1/2} - \phi_i^{k+1/2}) \]
\[ = 2h \sum_{i=1}^{M} D_i w_i^{k+1/2} \cdot (P_i)^{k+1/2}, \quad 1 \leq k \leq N - 1. \tag{3.12} \]

Multiplying (3.10) by \( 2Kh (w_i^{k+1/2} - \phi_i^{k+1/2}) \) and then summing up for \( i \) from 1 to \( M \), we obtain
\[ 2Kh \sum_{i=1}^{M} (w_i^{k+1/2} - \phi_i^{k+1/2}) \cdot D_i w_i^{k+1/2} - 2Kh \sum_{i=1}^{M} (w_i^{k+1/2} - \phi_i^{k+1/2}) \cdot D_i \delta_i w_i^{k+1/2} \]
\[ = 2Kh \sum_{i=1}^{M} (w_i^{k+1/2} - \phi_i^{k+1/2}) \cdot D_i (P_i)^{k+1/2}, \quad 1 \leq k \leq N - 1. \tag{3.13} \]

Similarly, multiplying (3.3) and (3.11) by \( 2h D_i \psi_i^{k+1/2} \) and \( 2EIh (\psi_i^{k+1/2}) \), respectively, then summing up for \( i \) from 1 to \( M \), we get
\[ 2I \rho h \sum_{i=1}^{M} D_i \psi_i^{k+1/2} \cdot \delta_i^2 \psi_i^{k+1/2} - 2EIh \sum_{i=1}^{M} D_i \psi_i^{k+1/2} \cdot \delta_i \psi_i^{k+1/2} \]
\[ - 2Kh \sum_{i=1}^{M} D_i \psi_i^{k+1/2} \cdot (\psi_i^{k+1/2} - \phi_i^{k+1/2}) = 2h \sum_{i=1}^{M} D_i \psi_i^{k+1/2} \cdot (P_i)^{k+1/2}, \quad 1 \leq k \leq N - 1, \tag{3.14} \]
\[ 2EIh \sum_{i=1}^{M} (\psi_i^{k+1/2} \cdot D_i \psi_i^{k+1/2} - 2EIh \sum_{i=1}^{M} \psi_i^{k+1/2} \cdot D_i \delta_i \psi_i^{k+1/2} \]
\[ = 2EIh \sum_{i=1}^{M} (\psi_i^{k+1/2} \cdot D_i (P_i)^{k+1/2}, \quad 1 \leq k \leq N - 1. \tag{3.15} \]
Summing up (3.12)–(3.15) and moving some terms to the right, we have

\[
2\rho h \sum_{i=1}^{M} D_t w_i^{k-1/2} \cdot \delta^2 w_i^{k-1/2} + 2I_\rho h \sum_{i=1}^{M} D_t \phi_i^{k} \cdot \delta^2 \phi_i^{k-1/2} \\
+ 2Kh \sum_{i=1}^{M} (v_i^{k-1/2} - \phi_i^{k-1/2}) \cdot D_t (v_i^{k-1/2} - \phi_i^{k-1/2}) + 2EIh \sum_{i=1}^{M} \psi_i^{k} \cdot D_t \psi_i^{k-1/2} \\
= 2EI \left( h \sum_{i=1}^{M} \psi_i^{k} \cdot D_t \delta \phi_i^{k-1/2} + h \sum_{i=1}^{M} D_t \phi_i^{k} \cdot \delta \psi_i^{k-1/2} \right) \\
+ 2K \left[ h \sum_{i=1}^{M} D_t w_i^{k-1/2} \cdot (\delta \phi_i^{k-1/2} - \psi_i^{k-1/2}) + h \sum_{i=1}^{M} (v_i^{k-1/2} - \phi_i^{k-1/2}) \cdot D_t \delta \phi_i^{k-1/2} \right] \\
+ 2h \sum_{i=1}^{M} D_t w_i^{k-1/2} \cdot (P_1)_{i-1/2} + 2h \sum_{i=1}^{M} D_t \phi_i^{k-1/2} \cdot (P_3)_{i-1/2} + 2EIh \sum_{i=1}^{M} \psi_i^{k} \cdot D_t (P_4)_{i-1/2} \\
+ 2Kh \sum_{i=1}^{M} (v_i^{k-1/2} - \phi_i^{k-1/2}) \cdot D_t (P_2)_{i-1/2}, \quad 1 \leq k \leq N - 1.
\]  

(3.16)

The each term on the left of (3.16) can be transformed into

\[
2\rho h \sum_{i=1}^{M} D_t w_i^{k-1/2} \cdot \delta^2 w_i^{k-1/2} = \frac{\rho}{\tau} (\|\delta_t w^{k+1/2}\|^2 - \|\delta_t w^{k-1/2}\|^2), \quad 1 \leq k \leq N - 1,
\]  

(3.17)

\[
2I_\rho h \sum_{i=1}^{M} D_t \phi_i^{k} \cdot \delta^2 \phi_i^{k-1/2} = \frac{I_\rho}{\tau} (\|\delta_t \phi^{k+1/2}\|^2 - \|\delta_t \phi^{k-1/2}\|^2), \quad 1 \leq k \leq N - 1,
\]  

(3.18)

\[
2Kh \sum_{i=1}^{M} (v_i^{k-1/2} - \phi_i^{k-1/2}) \cdot D_t (v_i^{k-1/2} - \phi_i^{k-1/2}) = \frac{K}{2\tau} (\|v^{k+1} - \phi^{k+1}\|^2 - \|v^{k-1} - \phi^{k-1}\|^2), \quad 1 \leq k \leq N - 1,
\]  

(3.19)

\[
2EIh \sum_{i=1}^{M} \psi_i^{k} \cdot D_t \psi_i^{k-1/2} = \frac{EI}{2\tau} (\|\psi^{k+1}\|^2 - \|\psi^{k-1}\|^2), \quad 1 \leq k \leq N - 1.
\]  

(3.20)

For the first term on the right-hand side of (3.16), using (3.5), (3.7) and (3.9) and the inequality \(2(b - a)a \leq \frac{1}{2} b^2\), we have

\[
2EI \left( h \sum_{i=1}^{M} \psi_i^{k} \cdot D_t \delta \phi_i^{k-1/2} + h \sum_{i=1}^{M} D_t \phi_i^{k} \cdot \delta \psi_i^{k-1/2} \right) \\
= 2EI \left( \psi_M^k \cdot D_t \phi_M^k - \psi_0^k \cdot D_t \phi_0^k \right) \\
= 2EI [(P_0)^k - \beta D_t \phi_M^k] \cdot D_t \phi_M^k \\
\leq \frac{EI}{2\beta} [(P_0)^k]^2.
\]  

(3.21)
For the second term on the right-hand side of (3.16), noting (3.4)–(3.6) and (3.8), we obtain

\[
2K \left[ h \sum_{i=1}^{M} D_i u_{i-1/2}^k \cdot (\delta_x v_{i-1/2}^k - \psi_{i-1/2}^k) + h \sum_{i=1}^{M} (v_{i-1/2}^k - \phi_{i-1/2}^k) \cdot D_i \delta_x u_{i-1/2}^k \right]
\]

\[
= 2Kh \sum_{i=1}^{M} (\delta_x (v_{i-1/2}^k - \phi_{i-1/2}^k)) \cdot D_i u_{i-1/2}^k + (v_{i-1/2}^k - \phi_{i-1/2}^k) \cdot D_i \delta_x u_{i-1/2}^k \]

\[
+ 2Kh \sum_{i=1}^{M} (\delta_x \phi_{i-1/2}^k - \psi_{i-1/2}^k) \cdot D_i u_{i-1/2}^k
\]

\[
= 2K [(v_M^k - \phi_M^k) \cdot D_i u_M^k - (v_0^k - \phi_0^k) \cdot D_i u_0^k] - 2Kh \sum_{i=1}^{M} (D_i u_{i-1/2}^k) \cdot (P_i \hat{k})_{i-1/2}
\]

\[
= 2K (-\alpha D_i u_M^k + (P_5)^k) \cdot D_i u_M^k - 2Kh \sum_{i=1}^{M} (D_i u_{i-1/2}^k) \cdot (P_i \hat{k})_{i-1/2}
\]

\[
\leq K \frac{2}{\rho} (P_5)^k + \frac{\rho}{4} \| \delta_t u^{k+1/2} \|^2 + \| \delta_t u^{k-1/2} \|^2 + \frac{2K^2}{\rho} \| (P_4)^k \|^2.
\]  

(3.22)

For the last four terms on the right-hand side of (3.16), we have

\[
| 2h \sum_{i=1}^{M} D_i w_{i-1/2}^k \cdot (P_1)_{i-1/2} \| \leq h \sum_{i=1}^{M} \left[ \frac{\rho}{2} (D_i w_{i-1/2}^k)^2 + \frac{2}{\rho} ((P_1)_{i-1/2})^2 \right]
\]

\[
\leq \frac{\rho}{4} (\| \delta_t u^{k+1/2} \|^2 + \| \delta_t u^{k-1/2} \|^2) + \frac{2}{\rho} \| (P_1)^k \|^2,
\]  

(3.23)

\[
| 2h \sum_{i=1}^{M} D_i \phi_{i-1/2}^k \cdot (P_3)_{i-1/2} \| \leq h \sum_{i=1}^{M} \left[ I_{\rho} (D_i \phi_{i-1/2}^k)^2 + \frac{1}{I_{\rho}} ((P_3)_{i-1/2})^2 \right]
\]

\[
\leq \frac{I_{\rho}}{2} (\| \delta_t \phi^{k+1/2} \|^2 + \| \delta_t \phi^{k-1/2} \|^2) + \frac{1}{I_{\rho}} \| (P_3)^k \|^2,
\]  

(3.24)

\[
| 2EIh \sum_{i=1}^{M} \psi_{i-1/2}^k \cdot D_i (P_4)_{i-1/2}^k \| \leq EIh \sum_{i=1}^{M} \left[ \frac{1}{4} (\psi_{i-1/2}^k)^2 + 2(D_i (P_4)_{i-1/2}^k)^2 \right]
\]

\[
\leq \frac{EI}{4} (\| \psi^{k+1} \|^2 + \| \psi^{k-1} \|^2) + 2EI \| D_i (P_4)^k \|^2,
\]  

(3.25)

\[
| 2Kh \sum_{i=1}^{M} (v_{i-1/2}^k - \phi_{i-1/2}^k) \cdot D_i (P_2)_{i-1/2}^k \| \leq Kh \sum_{i=1}^{M} \left[ \frac{1}{4} (v_{i-1/2}^k - \phi_{i-1/2}^k)^2 + 2(D_i (P_2)_{i-1/2}^k)^2 \right]
\]

\[
\leq \frac{K}{4} (\| v^{k+1} - \phi^{k+1} \|^2 + \| v^{k-1} - \phi^{k-1} \|^2) + 2K \| D_i (P_2)^k \|^2.
\]  

(3.26)
Substituting (3.17)–(3.26) into (3.16), we can obtain
\[
\frac{\rho}{\tau} \left( \| \delta_t w^{k+1/2} \|^2 - \| \delta_t w^{k-1/2} \|^2 \right) + \frac{L_p}{\tau} \left( \| \delta_t \varphi^{k+1/2} \|^2 - \| \delta_t \varphi^{k-1/2} \|^2 \right) \\
+ \frac{EI}{2\tau} \left( \| \psi^{k+1} \|^2 - \| \psi^{k-1} \|^2 \right) + \frac{K}{2\tau} \left( \| v^{k+1} - \varphi^{k+1} \|^2 - \| v^{k-1} - \varphi^{k-1} \|^2 \right)
\leq \frac{\rho}{2} \left( \| \delta_t w^{k+1/2} \|^2 + \| \delta_t w^{k-1/2} \|^2 \right) + \frac{L_p}{2} \left( \| \delta_t \varphi^{k+1/2} \|^2 + \| \delta_t \varphi^{k-1/2} \|^2 \right) \\
+ \frac{EI}{4} \left( \| \psi^{k+1} \|^2 + \| \psi^{k-1} \|^2 \right) + \frac{K}{4} \left( \| v^{k+1} - \varphi^{k+1} \|^2 + \| v^{k-1} - \varphi^{k-1} \|^2 \right)
\leq \frac{2}{\rho} \| (P_1)^k \|^2 + \frac{1}{L_p} \| (P_3)^k \|^2 + \frac{2K^2}{\rho} \| (P_4)^k \|^2 + \frac{K}{2\rho} \| (P_5)^k \|^2 + \frac{EI}{2\beta} \| (P_6)^k \|^2 \\
+ 2K \| D_I (P_2)^k \|^2 + 2EI \| D_I (P_4)^k \|^2, \quad 1 \leq k \leq N - 1.
\]

Denoting
\[
F(k) = \rho \| \delta_t w^{k+1/2} \|^2 + L_p \| \delta_t \varphi^{k+1/2} \|^2 + \frac{EI}{2} \left( \| \psi^{k+1} \|^2 + \| \psi^{k-1} \|^2 \right) \\
+ \frac{K}{2} \left( \| v^{k+1} - \varphi^{k+1} \|^2 + \| v^{k-1} - \varphi^{k-1} \|^2 \right), \quad 0 \leq k \leq N - 1,
\]
we have
\[
\frac{1}{\tau} [F(k) - F(k - 1)] \leq \frac{1}{2} [F(k) + F(k - 1)] + G(k), \quad (3.27)
\]
where
\[
G(k) = \frac{2}{\rho} \| (P_1)^k \|^2 + \frac{1}{L_p} \| (P_3)^k \|^2 + \frac{2K^2}{\rho} \| (P_4)^k \|^2 + \frac{K}{2\rho} \| (P_5)^k \|^2 + \frac{EI}{2\beta} \| (P_6)^k \|^2 \\
+ 2K \| D_I (P_2)^k \|^2 + 2EI \| D_I (P_4)^k \|^2, \quad 1 \leq k \leq N - 1.
\]

It follows from (3.27), when \( \tau \leq \frac{2}{\tau^2} \), that
\[
F(k) \leq (1 + \frac{3}{2} \tau) F(k - 1) + \frac{3}{2} \tau G(k), \quad 1 \leq k \leq N - 1.
\]

Gronwall inequality gives
\[
F(k) \leq \left( 1 + \frac{3}{2} \tau \right)^k \left[ F(0) + \frac{3}{2} \tau \sum_{n=1}^{k} G(n) \right], \quad 1 \leq k \leq N - 1.
\]

This completes the proof. \( \square \)

**Theorem 2.** The difference scheme of (1.15)–(1.21) is uniquely solvable.

**Proof.** According to Theorem 1, it suffices to prove that the difference (2.27)–(2.35) has a unique solution. Suppose \( \{(w_i^k, \varphi_i^k, v_i^k, \psi_i^k) | 0 \leq i \leq M \} \) and \( \{(w_i^{-1/2}, \varphi_i^{-1/2}, v_i^{-1/2}, \psi_i^{-1/2}) | 0 \leq i \leq M \} \) have been determined, we are to determine \( \{(w_i^k, \varphi_i^k, v_i^k, \psi_i^k) | 0 \leq i \leq M \} \) from (2.27)–(2.35). Since (2.27)–(2.35) is a system of linear algebraic equations with respect to \( \{(w_i^k, \varphi_i^k, v_i^k, \psi_i^k) | 0 \leq i \leq M \} \), we only need to prove that its homogeneous system

\[
\frac{\rho}{\tau^2} (w_i^{k-1/2} - \frac{K}{2} (\delta_x v_i^{k+1/2} - \psi_i^{k+1/2}) = 0, \quad 1 \leq i \leq M, \quad (3.28) \\
v_i^{k+1/2} - \delta_x w_i^{k-1/2} = 0, \quad 1 \leq i \leq M, \quad (3.29) \\
\frac{L_p}{\tau^2} (\varphi_i^{k+1/2} - \frac{EI}{2} \delta_x \psi_i^{k+1/2} - \frac{K}{2} (v_i^{k+1/2} - \varphi_i^{k+1/2}) = 0, \quad 1 \leq i \leq M, \quad (3.30)
\]
Theorem 3. The solution \( \{ w_i^k, \phi_i^k \} \) of the difference scheme (1.15)–(1.21) is convergent to the solution \( \{ W_i^k, \Phi_i^k \} \) of problem (1.9)–(1.13) with the convergence order of \( O(\tau^2 + h^2) \) in the \( L_\infty \) norm.

Proof. According to Theorem 1, it suffices to prove that the solution of the difference (2.27)–(2.35) is convergent to the solution of problem (2.1)–(2.7) and the convergence order is \( O(\tau^2 + h^2) \) in the \( L_\infty \) norm. Denote

\[
\tilde{w}_i^k = W_i^k - w_i^k, \quad \tilde{v}_i^k = V_i^k - v_i^k, \quad \tilde{\phi}_i^k = \Phi_i^k - \phi_i^k, \quad \tilde{\psi}_i^k = \Psi_i^k - \psi_i^k, \quad 0 \leq i \leq M, \quad 0 \leq k \leq N.
\]
Subtracting (2.27)–2.35 from (2.8)–(2.16), respectively, we obtain the error equations:

\[ \rho \partial^2_t \tilde{w}^k_i - \mathcal{K} (\partial_x \tilde{v}^k_i - \tilde{\psi}^k_i) = (e_1)^k_i, \quad 1 \leq i \leq M, \quad 1 \leq k \leq N - 1, \]  

\[ \tilde{v}^k_i - \partial_x \tilde{w}^k_i = (e_2)^k_i, \quad 1 \leq i \leq M, \quad 0 \leq k \leq N, \]  

\[ I_\rho \partial^2_t \tilde{\phi}^k_i - \mathcal{E} I \partial_x \tilde{\psi}^k_i - \mathcal{K} (\tilde{v}^k_i - \tilde{\phi}^k_i) = (e_3)^k_i, \quad 1 \leq i \leq M, \quad 1 \leq k \leq N - 1, \]  

\[ \tilde{\psi}^k_i - \partial_x \tilde{\phi}^k_i = (e_4)^k_i, \quad 1 \leq i \leq M, \quad 0 \leq k \leq N, \]  

\[ \tilde{w}^k_0 = 0, \quad \tilde{\phi}^k_0 = 0, \quad 1 \leq k \leq N, \]  

\[ \tilde{v}^k_M - \tilde{\phi}^k_M = -\alpha D_t \tilde{w}^k_M + (e_5)^k, \quad 1 \leq k \leq N - 1, \]  

\[ \tilde{\psi}^k_M + \beta D_t \tilde{\phi}^k_M = (e_6)^k, \quad 1 \leq k \leq N - 1, \]  

\[ \tilde{w}^0_i = 0, \quad \tilde{w}^1_i = (e_7)_i, \quad 0 \leq i \leq M, \]  

\[ \tilde{\phi}^0_i = 0, \quad \tilde{\phi}^1_i = (e_8)_i, \quad 0 \leq i \leq M. \]  

Using Lemma 1, we have

\[ F(k) \leq \left( 1 + \frac{3}{2} \tau \right)^k \left[ F(0) + \frac{3}{2} \tau \sum_{n=1}^k G(n) \right], \quad 1 \leq k \leq N - 1, \]  

where

\[ F(k) = \rho \| \partial_t \tilde{w}^{k+1/2} \|^2 + I_\rho \| \partial_t \tilde{\phi}^{k+1/2} \|^2 + \frac{EI}{2} (\| \tilde{\psi}^{k+1} \|^2 + \| \tilde{\psi}^k \|^2) \]  

\[ + \frac{K}{2} (\| \tilde{w}^{k+1} - \tilde{\phi}^{k+1} \|^2 + \| \tilde{w}^k - \tilde{\phi}^k \|^2), \]  

\[ G(n) = \frac{2}{\rho} \| (e_1)^n \|^2 + \frac{1}{I_\rho} \| (e_3)^n \|^2 + \frac{2K^2}{\rho} \| (e_4)^n \|^2 + \frac{K}{2\tau} \| (e_5)^n \|^2 + \frac{EI}{2\beta} \| (e_6)^n \|^2 \]  

\[ + 2K \| D_t(e_7)^n \|^2 + 2EI \| D_t(e_8)^n \|^2. \]

Next we will estimate \( G(n) \) and \( F(0) \).

It follows from (2.28) and (2.30) that

\[ D_t v^k_i - D_t \delta_i w^k_i = 0, \quad 1 \leq i \leq M, \quad 1 \leq k \leq N - 1, \]  

\[ D_t \psi^k_i - D_t \delta_i \phi^k_i = 0, \quad 1 \leq i \leq M, \quad 1 \leq k \leq N - 1 \]  

which can be regarded as the discretization of the two equations

\[ \frac{\partial \tilde{v}}{\partial t} - \frac{\partial^2 w}{\partial x \partial t} = 0, \quad \frac{\partial \tilde{\psi}}{\partial t} - \frac{\partial^2 \phi}{\partial x \partial t} = 0. \]

It is obvious that the equations above can be obtained by differentiating the both sides of (2.2) and (2.4) with respect to \( t \). Using the Taylor expansion again, we have

\[ D_t v^k_i - D_t \delta_i W^k_i = (e_9)^k_i, \quad 1 \leq i \leq M, \quad 1 \leq k \leq N - 1, \]  

\[ D_t \psi^k_i - D_t \delta_i \Phi^k_i = (e_{10})^k_i, \quad 1 \leq i \leq M, \quad 1 \leq k \leq N - 1, \]
and there exists a positive constant $c_2$ such that
\[
|\langle e_9 \rangle_{i-1/2}^k| \leq c_2 (\tau^2 + h^2), \quad |\langle e_{10} \rangle_{i-1/2}^k| \leq c_2 (\tau^2 + h^2), \quad 1 \leq i \leq M, \quad 1 \leq k \leq N - 1.
\] (3.52)

Subtracting (3.48) and (3.49) from (3.50) and (3.51) respectively, we obtain
\[
D_t \hat{v}^k_{i-1/2} - D_t \delta_x \hat{w}^k_{i-1/2} = \langle e_9 \rangle_{i-1/2}^k, \quad 1 \leq i \leq M, \quad 1 \leq k \leq N - 1,
\] (3.53)
\[
D_t \hat{v}^k_{i-1/2} - D_t \delta_x \hat{p}^k_{i-1/2} = \langle e_{10} \rangle_{i-1/2}^k, \quad 1 \leq i \leq M, \quad 1 \leq k \leq N - 1.
\] (3.54)

Comparing (3.53) with (3.39) and (3.54) with (3.41), respectively, we have
\[
D_t \langle e_2 \rangle_{i-1/2}^n = \langle e_9 \rangle_{i-1/2}^n, \quad D_t \langle e_4 \rangle_{i-1/2}^n = \langle e_{10} \rangle_{i-1/2}^n, \quad 1 \leq i \leq M, \quad 1 \leq k \leq N - 1.
\]

Using (3.52), we obtain
\[
\|D_t \langle e_2 \rangle^n\| \leq \sqrt{T} c_2 (\tau^2 + h^2), \quad \|D_t \langle e_4 \rangle^n\| \leq \sqrt{T} c_2 (\tau^2 + h^2).
\] (3.55)

By means of (2.17), (2.19), (2.21), (2.22) and (3.55), we have
\[
G(n) \leq \left( \frac{2}{\rho} c_1^2 + \frac{1}{\rho} c_2^2 + \frac{2K^2}{\rho} c_1^2 + \frac{K}{2aI} c_1^2 + \frac{E l}{2bI} c_1^2 + 2Kc_2^2 + 2EIc_2^2 \right) l (\tau^2 + h^2)^2
\equiv c_3 (\tau^2 + h^2)^2,
\] (3.56)

where
\[
c_3 = \left( \frac{2}{\rho} c_1^2 + \frac{1}{\rho} c_2^2 + \frac{2K^2}{\rho} c_1^2 + \frac{K}{2aI} c_1^2 + \frac{E l}{2bI} c_1^2 + 2Kc_2^2 + 2EIc_2^2 \right) l.
\]

Using (3.45) and (3.46), we have
\[
|\delta_x \hat{w}_i^{1/2}| = \left| \frac{1}{\tau} \hat{w}_i \right| \leq \left| \frac{\langle e_7 \rangle_i}{\tau} \right|, \quad |\delta_x \hat{p}_i^{1/2}| = \left| \frac{1}{\tau} \hat{p}_i \right| \leq \left| \frac{\langle e_8 \rangle_i}{\tau} \right|,
\]
\[
\delta_x \hat{w}_{i-1/2} = \delta_x \langle e_7 \rangle_{i-1/2}, \quad \delta_x \hat{p}_{i-1/2} = \delta_x \langle e_8 \rangle_{i-1/2}.
\]

Noting (2.23)–(2.26), we obtain
\[
\|\delta_x \hat{w}^{1/2}\| \leq \sqrt{T} c_1 \tau^2, \quad \|\delta_x \hat{p}^{1/2}\| \leq \sqrt{T} c_1 \tau^2,
\] (3.57)
\[
\|\delta_x \hat{w}^1\| \leq \sqrt{T} c_1 \tau^2, \quad \|\delta_x \hat{p}^1\| \leq \sqrt{T} c_1 \tau^2.
\] (3.58)

Using (3.39), (3.41) and (3.45)–(3.46), we have
\[
\|\tilde{v}^0\| \leq \|\tilde{e}_2\| \leq c_2^2 l h^4,
\] (3.59)
\[
\|\tilde{v}^1\| \leq 2(\|\delta_x \tilde{w}^1\|^2 + \|e_2^1\|^2) \leq 4c_1^2 l (\tau^2 + h^2)^2,
\] (3.60)
\[
\|\tilde{p}^0\| \leq \|e_4^0\|^2 \leq c_2^2 l h^4,
\] (3.61)
\[
\|\tilde{p}^1\| \leq 2(\|\delta_x \tilde{p}^1\|^2 + \|e_4^1\|^2) \leq 4c_1^2 l (\tau^2 + h^2)^2,
\] (3.62)
\[
\|\tilde{w}^0\| = 0, \quad \|\tilde{p}^0\| = 0,
\] (3.63)
\[
\|\tilde{w}^1\| = \|e_7\| \leq \sqrt{T} c_1 \tau^2, \quad \|\tilde{p}^1\| = \|e_8\| \leq \sqrt{T} c_1 \tau^2.
\] (3.64)

Using the triangle inequality, we can obtain
\[
\|\tilde{v}^1 - \tilde{p}^1\| \leq \|\tilde{v}^1 - \tilde{p}^1\|^2 \leq 2(\|\tilde{v}^1\|^2 + \|\tilde{p}^1\|^2) \leq 10c_1^2 l (\tau^2 + h^2)^2.
\] (3.65)
By means of (3.57), (3.61)–(3.62) and (3.65), we get
\[
 F(0) \leq \rho c_1^2 l x^4 + I_\rho c_1^2 l x^4 + \frac{5EI}{2} c_1^2 l (\tau^2 + h^2)^2 + \frac{11K}{2} c_1^2 l (\tau^2 + h^2)^2
 \leq c_4 (\tau^2 + h^2)^2,
\]  
(3.66)

where
\[
c_4 = \left( \rho + I_\rho + \frac{5EI}{2} + \frac{11K}{2} \right) c_1^2 l (\tau^2 + h^2)^2.
\]

Substituting (3.56) and (3.66) into the right-hand side of (3.47), we obtain
\[
 F(k) \leq e^{(3/2)T} \left( c_4 + \frac{3}{2} c_3 T \right) (\tau^2 + h^2)^2, \quad 1 \leq k \leq N - 1.
\]

Consequently, we have
\[
\| \tilde{\psi}^k \|^2 \leq \frac{2}{EI} c_5^2 (\tau^2 + h^2)^2, \quad 0 \leq k \leq N,
\]  
(3.67)

\[
\| \tilde{v}^k - \tilde{\phi}^k \|^2 \leq \frac{2}{K} c_5^2 (\tau^2 + h^2)^2, \quad 0 \leq k \leq N,
\]  
(3.68)

where
\[
c_5 = [e^{(3/2)T} (c_4 + \frac{3}{2} c_3 T)]^{1/2}.
\]

It follows from (3.39), that
\[
\delta_x \tilde{w}_{i-1/2}^k = \tilde{v}_{i-1/2}^k - (e2)_{i-1/2}^k, \quad 1 \leq i \leq M, \quad 0 \leq k \leq N,
\]

which gives
\[
\| \delta_x \tilde{w}^k \|^2 \leq 2(\| \tilde{v}^k \|^2 + \| e_2^k \|^2), \quad 0 \leq k \leq N,
\]  
(3.69)

Similarly, it follows from (3.41) that
\[
\delta_x \phi_i^k = \tilde{\psi}_{i-1/2}^k - (e4)_{i-1/2}^k, \quad 1 \leq i \leq M, \quad 0 \leq k \leq N
\]

which gives
\[
\| \delta_x \phi^k \|^2 \leq 2(\| \tilde{\psi}^k \|^2 + \| e_4^k \|^2), \quad 0 \leq k \leq N.
\]  
(3.70)

By (1.14), (3.67) and (3.70), we have
\[
\| \tilde{\phi}^k \|_\infty \leq \sqrt{7} \| \delta_x \phi^k \| \leq c_6 (\tau^2 + h^2)
\]

and
\[
\| \tilde{\phi}^k \| \leq \frac{1}{\sqrt{2}} \| \delta_x \phi^k \| \leq c_7 (\tau^2 + h^2),
\]  
(3.71)

where
\[
c_6 = \left( \frac{4l}{EI} c_3^2 + 2c_1^2 l^2 \right)^{1/2}, \quad c_7 = \left( \frac{2l^2}{EI} c_3^2 + c_1^2 l^2 \right)^{1/2}.
\]

Using the triangle inequality, (3.68) and (3.71), we have
\[
\| \tilde{v}^k \| \leq \| \tilde{v}^k - \tilde{\phi}^k \| + \| \tilde{\phi}^k \| \leq c_8 (\tau^2 + h^2),
\]  
(3.72)
Theorem 4. The difference scheme of (1.15)–(1.21) is stable to the initial values and the inhomogeneous terms. More precisely, if \( \{w^k_i, \varphi^k_i\} \) is the solution of the difference scheme

\[
\frac{\rho}{2} (\delta^2 w^k_{i-1/2} + \delta^2 w^k_{i+1/2}) - K \left[ \delta^2 w^k_i - \frac{1}{2} (\delta \varphi^k_{i-1/2} + \delta \varphi^k_{i+1/2}) \right] = \frac{1}{2} [(f_1)^k_{i-1/2} + (f_1)^k_{i+1/2}],
\]

\[
1 \leq i \leq M - 1, \quad 1 \leq k \leq N - 1,
\]

\[
\frac{I_0}{2} (\delta^2 \varphi^k_{i-1/2} + \delta^2 \varphi^k_{i+1/2}) - EI \delta^2 \varphi^k_i - \frac{K}{2} [(\delta \varphi^k_{i+1/2} + \delta \varphi^k_{i-1/2}) - (\varphi^k_{i+1/2} + \varphi^k_{i-1/2})]
\]

\[
= \frac{1}{2} [(f_2)^k_{i-1/2} + (f_2)^k_{i+1/2}], \quad 1 \leq i \leq M - 1, \quad 1 \leq k \leq N - 1,
\]

\[
w^k_0 = 0, \quad \varphi^k_0 = 0, \quad 1 \leq k \leq N,
\]

\[
\delta \varphi^k_M = -\varepsilon D \varphi^k_M - \frac{h}{2K} [(K \delta \varphi^k_{M-1/2} + \rho \delta^2 w^k_{M-1/2} - (f_1)^k_{M-1/2}) + (g_1)^k], \quad 1 \leq k \leq N - 1,
\]

\[
\delta \varphi^k_{M-1/2} = -\beta D \varphi^k_{M-1/2} - \frac{h}{2EI} [I_0 \delta^2 \varphi^k_{M-1/2} - K (\delta \varphi^k_{M-1/2} - \varphi^k_{M-1/2}) - (f_2)^k_{M-1/2}] + (g_2)^k,
\]

\[
1 \leq k \leq N - 1,
\]

\[
w^0_i = (g_3)_i, \quad w^1_i = (g_3)_i + \tau A_i, \quad 0 \leq i \leq M,
\]

\[
\varphi^0_i = (g_5)_i, \quad \varphi^1_i = (g_5)_i + \tau B_i, \quad 0 \leq i \leq M,
\]

we have

\[
\rho \|\delta \varphi^{k+1/2}\|^2 + I_0 \|\delta \varphi^{k+1/2}\|^2 + \frac{E I}{2} (\|\delta \varphi^{k+1}\|^2 + \|\delta \varphi^k\|^2)
\]

\[
+ \frac{K}{2} (\|\delta \varphi^{k+1} - \varphi^{k+1}\|^2 + \|\delta \varphi^k - \varphi^k\|^2)
\]

\[
\leq e^{(3/2)T} \left[ \rho \|A\|^2 + I_0 \|B\|^2 + \frac{EI}{2} (\|\delta (g_5 + \tau B)\|^2 + \|\delta g_3\|^2)
\]

\[
+ \frac{K}{2} (\|\delta (g_3 + \tau A) - (g_5 + \tau B)\|^2 + \|\delta g_3 - g_5\|^2)
\]

\[
+ \frac{3}{2} \tau \sum_{n=1}^{k} \left( \|f_1^n\|^2 + \|f_2^n\|^2 + \frac{K}{2\tau} (\|g_1^n\|^2 + \frac{EI}{2\beta} (g_2)^n) \right), \quad 0 \leq k \leq N - 1.
\]
Furthermore, if we denote
\[
F^k \equiv e^{(3/2)T} \left[ \rho \|A\|^2 + I_\rho \|B\|^2 + \frac{EI}{2}(\|\delta_x(g_5 + \tau B)\|^2 + \|\delta_xg_3\|^2) \\
+ \frac{K}{2}(\|\delta_x(g_3 + \tau A) - (g_5 + \tau B)\|^2 + \|\delta_xg_3 - g_5\|^2) \\
+ \frac{3}{2} \tau \sum_{n=1}^{k} (\|(f_1)^n\|^2 + \|(f_2)^n\|^2 + \frac{K}{2\tau} |(g_1)^n|^2 + \frac{EI}{2\beta |(g_2)^n|^2}) \right], \quad 0 \leq k \leq N - 1,
\]
then we have
\[
\|\varphi^k\|_{\infty} \leq \left( \frac{2l}{EI} F^k \right)^{1/2}, \quad 1 \leq k \leq N, \quad (3.73)
\]
\[
\|w^k\|_{\infty} \leq \left( \frac{2l}{K} F^k \right)^{1/2} + l \left( \frac{2l}{EI} F^k \right)^{1/2}, \quad 1 \leq k \leq N. \quad (3.74)
\]

**Proof.** Let
\[
v_i^k = \delta_xw_{i+1/2}^k - \frac{h}{2} \left[ \delta_x\varphi_{i+1/2}^k + \frac{\rho}{K} \delta_t^2 w_{i+1/2}^k - \frac{1}{K} (f_1)_{i+1/2}^k \right], \quad 0 \leq i \leq M - 1, \quad 1 \leq k \leq N - 1,
\]
\[
v_M^k = \delta_xw_{M-1/2}^k + \frac{h}{2} \left[ \delta_x\varphi_{M-1/2}^k + \frac{\rho}{K} \delta_t^2 w_{M-1/2}^k - \frac{1}{K} (f_1)_{M-1/2}^k \right], \quad 1 \leq k \leq N - 1,
\]
\[
\psi_i^k = \delta_x\varphi_{i+1/2}^k + \frac{h}{2EI} \left[ K(\delta_xw_{i+1/2}^k - \varphi_{i+1/2}^k) - I_\rho \delta_t^2 \varphi_{i+1/2}^k + (f_2)_{i+1/2}^k \right], \quad 0 \leq i \leq M - 1, \quad 1 \leq k \leq N - 1,
\]
\[
\psi_M^k = \delta_x\varphi_{M-1/2}^k - \frac{h}{2EI} \left[ K(\delta_xw_{M-1/2}^k - \varphi_{M-1/2}^k) - I_\rho \delta_t^2 \varphi_{M-1/2}^k + (f_2)_{M-1/2}^k \right], \quad 1 \leq k \leq N - 1,
\]
and
\[
v_i^0 = \delta_xw_{i-1/2}^0, \quad v_i^1 = \delta_xw_{i-1/2}^1, \quad 1 \leq i \leq M,
\]
\[
\psi_i^0 = \delta_x\varphi_{i-1/2}^0, \quad \psi_i^1 = \delta_x\varphi_{i-1/2}^1, \quad 1 \leq i \leq M.
\]
According to Theorem 1, \{w, v, \varphi, \psi\} is the solution of the following difference scheme
\[
\rho \delta_t^2 w_{i-1/2}^k - K(\delta_xw_{i-1/2}^k - \varphi_{i-1/2}^k) = (f_1)_{i-1/2}^k, \quad 1 \leq i \leq M, \quad 1 \leq k \leq N - 1, \quad (3.75)
\]
\[
v_{i-1/2}^k - \delta_xw_{i-1/2}^k = 0, \quad 1 \leq i \leq M, \quad 0 \leq k \leq N, \quad (3.76)
\]
\[
I_\rho \delta_t^2 \varphi_{i-1/2}^k - EI \delta_x\psi_{i-1/2}^k - K(v_{i-1/2}^k - \varphi_{i-1/2}^k) = (f_2)_{i-1/2}^k, \quad 1 \leq i \leq M, \quad 1 \leq k \leq N - 1, \quad (3.77)
\]
\[
\psi_{i-1/2}^k - \delta_x\varphi_{i-1/2}^k = 0, \quad 1 \leq i \leq M, \quad 0 \leq k \leq N, \quad (3.78)
\]
\[
w_0^k = 0, \quad \varphi_0^k = 0, \quad 1 \leq k \leq N, \quad (3.79)
\]
\[
v_M^k - \varphi_M^k = -\alpha D_tw_M^k + (g_1)^k, \quad 1 \leq k \leq N - 1, \quad (3.80)
\]
\[
\psi_M^k + \beta D_t\varphi_M^k = (g_2)^k, \quad 1 \leq k \leq N - 1, \quad (3.81)
\]
\[
w_i^0 = (g_3)_i, \quad w_i^1 = (g_3)_i + \tau A_i, \quad 0 \leq i \leq M, \quad (3.82)
\]
\[
\varphi_i^0 = (g_5)_i, \quad \psi_i^1 = (g_5)_i + \tau B_i, \quad 0 \leq i \leq M. \quad (3.83)
\]
Using Lemma 1, noting (3.76) and (3.78), we have

\[
\rho \| \delta_t w^{k+1/2} \|^2 + I_\rho \| \delta_t \varphi^{k+1/2} \|^2 + \frac{EI}{2} (\| \delta_x \varphi^{k+1} \|^2 + \| \delta_x w^k \|^2) \\
+ \frac{K}{2} (\| \delta_x w^{k+1} - \varphi^{k+1} \|^2 + \| \delta_x w^k - \varphi^k \|^2) \\
\leq e^{(3/2)T} \left[ \rho \| A \|^2 + I_\rho \| B \|^2 + \frac{EI}{2} (\| \delta_x (g_5 + \tau B) \|^2 + \| \delta_x g_3 \|^2) \\
+ \frac{K}{2} (\| \delta_x (g_3 + \tau A) - (g_5 + \tau B) \|^2 + \| \delta_x g_3 - g_5 \|^2) \\
+ \frac{3}{2} \tau \sum_{n=1}^{k} \left( \| (f_1)^n \|^2 + \| (f_2)^n \|^2 + \frac{K}{2\beta} (g_1)^n \|^2 + \frac{EI}{2\beta} (g_2)^n \|^2 \right) \right], \quad 0 \leq k \leq N - 1.
\]

By means of (1.14) and (3.84), we have

\[
\| \varphi^k \|_\infty \leq \sqrt{T} \| \delta_x \varphi^k \| \leq \left( \frac{2l}{EI} F^k \right)^{1/2}
\]

and

\[
\| \varphi^k \| \leq \frac{l}{\sqrt{2}} \| \delta_x \varphi^k \| \leq l \left( \frac{1}{EI} F^k \right)^{1/2}.
\]

Using the triangle inequality and (3.84)–(3.85), we can obtain

\[
\| \delta_x w^k \| \leq \| \delta_x w^k - \varphi^k \| + \| \varphi^k \| \leq \left( \frac{2K}{l} F^k \right)^{1/2} + l \left( \frac{1}{EI} F^k \right)^{1/2}.
\]

Using (1.14), we have

\[
\| w^k \|_\infty \leq \sqrt{T} \| \delta_x w^k \| \leq \left( \frac{2K}{l} F^k \right)^{1/2} + l \left( \frac{1}{EI} F^k \right)^{1/2}.
\]

This completes the proof. □

4. Numerical example

To test our difference scheme (1.15)–(1.21), we consider a simple initial and boundary value problem. We choose the coefficients to be \( l = 1, \ T = 1, \ \rho = 1, \ K = 1.5, \ I_\rho = 2, \ EI = 7.5, \ \alpha = 3.5, \ \beta = 4.1 \). The right hand functions \( f_1(x, t), f_2(x, t) \) and the functions in the initial and boundary conditions are determined by the exact solution \( w(x, t) = e^{-t} \sin(\pi/2)x \) and \( \varphi(x, t) = e^{-1.5t} \sin \pi x \). In order to show the convergence order, let \( M = 40, 80, 160, 320, 640 \), respectively, and \( N = M \). Tables 1 and 2 give the numerical and the exact solutions. Tables 3 and 4 give the absolute error of the numerical solutions of \( w \) and \( \varphi \) at some points at \( t = 1 \), respectively. The curve of absolute errors are plotted in Figs. 1 and 2. Tables 5 and 6 give the maximal norm of numerical solutions of \( w \) and their respective bounds in inequalities (3.73) and (3.74) when \( N = M = 320 \). The stability of the difference scheme of (1.15)–(1.21) to the initial values and the inhomogeneous terms is apparent by the data in Tables 5 and 6. Our estimates (3.73) and (3.74) are not very sharp because of many times amplifying in the process of deriving them.

Tables 7 and 8 give the errors of the numerical solutions, in which the maximal errors are defined as follows:

\[
\| w - w_{ht} \|_\infty = \max_{0 \leq k \leq N} \left\{ \max_{0 \leq i \leq M} \| w(x_i, t_k) - w_i^k \| \right\},
\]

\[
\| \varphi - \varphi_{ht} \|_\infty = \max_{0 \leq k \leq N} \left\{ \max_{0 \leq i \leq M} \| \varphi(x_i, t_k) - \varphi_i^k \| \right\}.
\]
Table 1
The numerical solutions of \( w(x, t) \) at \( t = 1 \)

<table>
<thead>
<tr>
<th>( M \times (x, t) )</th>
<th>( \frac{1}{5}, 1 )</th>
<th>( \frac{2}{5}, 1 )</th>
<th>( \frac{3}{5}, 1 )</th>
<th>( \frac{4}{5}, 1 )</th>
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<tbody>
<tr>
<td>40</td>
<td>0.113438</td>
<td>0.216027</td>
<td>0.297600</td>
<td>0.349912</td>
<td>0.367730</td>
</tr>
<tr>
<td>80</td>
<td>0.113616</td>
<td>0.216180</td>
<td>0.297617</td>
<td>0.349889</td>
<td>0.367843</td>
</tr>
<tr>
<td>160</td>
<td>0.113664</td>
<td>0.216220</td>
<td>0.297619</td>
<td>0.349878</td>
<td>0.367870</td>
</tr>
<tr>
<td>320</td>
<td>0.113677</td>
<td>0.216230</td>
<td>0.297621</td>
<td>0.349875</td>
<td>0.367877</td>
</tr>
<tr>
<td>640</td>
<td>0.113680</td>
<td>0.216233</td>
<td>0.297621</td>
<td>0.349874</td>
<td>0.367879</td>
</tr>
</tbody>
</table>

The exact solution 0.113681 0.216234 0.297621 0.349874 0.367879

Table 2
The numerical solutions of \( \varphi(x, t) \) at \( t = 1 \)

<table>
<thead>
<tr>
<th>( M \times (x, t) )</th>
<th>( \frac{1}{5}, 1 )</th>
<th>( \frac{2}{5}, 1 )</th>
<th>( \frac{3}{5}, 1 )</th>
<th>( \frac{4}{5}, 1 )</th>
<th>(1, 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>0.131645</td>
<td>0.213001</td>
<td>0.212982</td>
<td>0.131578</td>
<td>−1.784e−04</td>
</tr>
<tr>
<td>80</td>
<td>0.131277</td>
<td>0.212409</td>
<td>0.212403</td>
<td>0.131260</td>
<td>−4.241e−05</td>
</tr>
<tr>
<td>160</td>
<td>0.131183</td>
<td>0.212260</td>
<td>0.212258</td>
<td>0.131179</td>
<td>−1.037e−05</td>
</tr>
<tr>
<td>320</td>
<td>0.131160</td>
<td>0.212222</td>
<td>0.212221</td>
<td>0.131159</td>
<td>−2.568e−06</td>
</tr>
<tr>
<td>640</td>
<td>0.131154</td>
<td>0.212216</td>
<td>0.212212</td>
<td>0.131154</td>
<td>−6.392e−07</td>
</tr>
</tbody>
</table>

The exact solution 0.131153 0.212209 0.212209 0.131153 0

Table 3
The errors of the numerical solution of \( w \) at \( t = 1 \)

<table>
<thead>
<tr>
<th>( M \times x )</th>
<th>( \frac{1}{5} )</th>
<th>( \frac{2}{5} )</th>
<th>( \frac{3}{5} )</th>
<th>( \frac{4}{5} )</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>2.4327e−04</td>
<td>2.0667e−04</td>
<td>2.0666e−05</td>
<td>3.8776e−05</td>
<td>1.4933e−04</td>
</tr>
<tr>
<td>80</td>
<td>6.5204e−05</td>
<td>5.4043e−05</td>
<td>3.6447e−06</td>
<td>1.5036e−05</td>
<td>3.6503e−05</td>
</tr>
<tr>
<td>160</td>
<td>1.6738e−05</td>
<td>1.3788e−05</td>
<td>7.4034e−07</td>
<td>4.2811e−06</td>
<td>9.0722e−06</td>
</tr>
<tr>
<td>320</td>
<td>4.2339e−06</td>
<td>3.4770e−06</td>
<td>1.6679e−07</td>
<td>1.1267e−06</td>
<td>2.2622e−06</td>
</tr>
<tr>
<td>640</td>
<td>1.0641e−06</td>
<td>8.7290e−07</td>
<td>3.9503e−08</td>
<td>2.8830e−07</td>
<td>5.6489e−07</td>
</tr>
</tbody>
</table>

Table 4
The errors of the numerical solution of \( \varphi \) at \( t = 1 \)

<table>
<thead>
<tr>
<th>( M \times x )</th>
<th>( \frac{1}{5} )</th>
<th>( \frac{2}{5} )</th>
<th>( \frac{3}{5} )</th>
<th>( \frac{4}{5} )</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td>4.9245e−04</td>
<td>7.9138e−04</td>
<td>7.7236e−04</td>
<td>4.2504e−04</td>
<td>1.7847e−04</td>
</tr>
<tr>
<td>80</td>
<td>1.2461e−04</td>
<td>2.0017e−04</td>
<td>1.9367e−04</td>
<td>1.0721e−04</td>
<td>4.2408e−05</td>
</tr>
<tr>
<td>160</td>
<td>3.1296e−05</td>
<td>5.0338e−05</td>
<td>4.8670e−05</td>
<td>2.6418e−05</td>
<td>1.0371e−05</td>
</tr>
<tr>
<td>320</td>
<td>7.8379e−06</td>
<td>1.2621e−05</td>
<td>1.2213e−05</td>
<td>6.6005e−06</td>
<td>2.5677e−06</td>
</tr>
<tr>
<td>640</td>
<td>1.9613e−06</td>
<td>3.1575e−06</td>
<td>3.0569e−06</td>
<td>1.6526e−06</td>
<td>6.3915e−07</td>
</tr>
</tbody>
</table>

Suppose \( \tau = h \) and \( \| w - w_{h\tau} \| \approx c_1 h^{p_1} \), \( \| \varphi - \varphi_{h\tau} \| \approx c_2 h^{p_2} \), then taking log on both sides of the approximate equalities above, we have

\[- \log \| w - w_{h\tau} \| \approx - \log c_1 + p_1 (−\log h), \quad - \log \| \varphi - \varphi_{h\tau} \| \approx - \log c_2 + p_2 (−\log h).\]
The error of $\omega$ at $t=1$

Fig. 1. Curves of the errors of $w$ between the exact solution and numerical solution.

Fig. 2. Curves of the errors of $\varphi$ between the exact solution and numerical solution.

Table 5
The numerical solutions of $w$ in maximal norm and their respective bounds in inequalities (3.73) when $N = M = 320$

<table>
<thead>
<tr>
<th>$k$</th>
<th>40</th>
<th>80</th>
<th>160</th>
<th>320</th>
</tr>
</thead>
<tbody>
<tr>
<td>$|w^k|_\infty$</td>
<td>0.882497</td>
<td>0.778801</td>
<td>0.606531</td>
<td>0.367877</td>
</tr>
<tr>
<td>$(\frac{2}{N} F^k)^{1/2} + l(\frac{l}{T^2} F^k)^{1/2}$</td>
<td>24.6687</td>
<td>31.6094</td>
<td>38.0807</td>
<td>41.87903</td>
</tr>
</tbody>
</table>

Using the data in Tables 3, 4 and with the help of MATLAB, we obtain the linear fitting functions of $w$ and $\varphi$:

$- \log \|w - w_{ht}\|_\infty \approx 0.5501 + 1.9931(- \log h),$

$- \log \|\varphi - \varphi_{ht}\|_\infty \approx -1.2860 + 1.9944(- \log h).$

That is to say, $p_1 = 1.9931$ and $p_2 = 1.9944$, respectively.
Table 6
The numerical solutions of $\phi$ in maximal norm and their respective bounds in inequalities (3.74) when $N = M = 320$

<table>
<thead>
<tr>
<th>$k$</th>
<th>40</th>
<th>80</th>
<th>160</th>
<th>320</th>
</tr>
</thead>
<tbody>
<tr>
<td>$|\phi^k|_\infty$</td>
<td>0.829023</td>
<td>0.687268</td>
<td>0.472331</td>
<td>0.223143</td>
</tr>
<tr>
<td>$(\frac{2}{EI} F^k)^{1/2}$</td>
<td>79.8297</td>
<td>102.29</td>
<td>123.232</td>
<td>135.5234</td>
</tr>
</tbody>
</table>

Table 7
The maximum errors of the numerical solutions of $w$

<table>
<thead>
<tr>
<th>$M$</th>
<th>40</th>
<th>80</th>
<th>160</th>
<th>320</th>
<th>640</th>
</tr>
</thead>
<tbody>
<tr>
<td>$|w - w_h|_\infty$</td>
<td>3.6787e − 04</td>
<td>9.3358e − 05</td>
<td>2.3410e − 05</td>
<td>5.8657e − 06</td>
<td>1.4679e − 06</td>
</tr>
</tbody>
</table>

Table 8
The maximum errors of the numerical solutions of $\phi$

<table>
<thead>
<tr>
<th>$M$</th>
<th>40</th>
<th>80</th>
<th>160</th>
<th>320</th>
<th>640</th>
</tr>
</thead>
<tbody>
<tr>
<td>$|\phi - \phi_h|_\infty$</td>
<td>2.3030e − 03</td>
<td>5.8009e − 04</td>
<td>1.4573e − 04</td>
<td>3.6517e − 05</td>
<td>9.1389e − 06</td>
</tr>
</tbody>
</table>

5. Conclusion

In this study, we develop a finite difference scheme by the method of reduction of order. Scheme (1.15)–(1.21) is a three-level scheme in time. It is shown by the discrete energy method that the scheme is uniquely solvable, unconditionally convergent and stable. Numerical results demonstrate the theoretical results.

The following Timoshenko beam with boundary feedback [5,16]:

\[
\rho \frac{\partial^2 w(x,t)}{\partial t^2} - K \left[ \frac{\partial^2 w(x,t)}{\partial x^2} - \frac{\partial \varphi(x,t)}{\partial x} \right] = 0, \quad 0 < x < l, \quad t > 0,
\]

\[
I \rho \frac{\partial^2 \varphi(x,t)}{\partial t^2} - E I \frac{\partial^2 \varphi(x,t)}{\partial x^2} - K \left[ \frac{\partial w(x,t)}{\partial x} - \varphi(x,t) \right] = 0, \quad 0 < x < l, \quad t > 0,
\]

\[
w(0,t) = 0, \quad \varphi(0,t) = 0, \quad t > 0,
\]

\[
\frac{\partial w(l,t)}{\partial x} - \varphi(l,t) = 0, \quad \frac{\partial \varphi(l,t)}{\partial x} = 0, \quad t > 0,
\]

\[
w(x,0) = g_1(x), \quad \frac{\partial w(x,0)}{\partial t} = g_2(x), \quad \varphi(x,0) = g_3(x), \quad \frac{\partial \varphi(x,0)}{\partial t} = g_4(x), \quad 0 \leq x \leq l
\]

can be considered similarly.

Acknowledgement

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References


