A Beilinson-type Theorem for Coherent Sheaves on Weighted Projective Spaces

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INTRODUCTION

In the study of projective varieties it is customary to consider embeddings as projective spectra of graded rings not necessarily generated in degree 1 (cf., for instance, [5, 6, 9]). Therefore one considers coherent sheaves on weighted projective spaces and their classification.

The aim of this paper is to extend to weighted projective spaces an important theorem by Beilinson, which describes the bounded derived category $D^b(Coh(P^n))$ of coherent sheaves on $P^n$ (see [1–3]). In its more explicit form the theorem gives, for every coherent sheaf $\mathcal{F}$ on $P^n$, two bounded complexes of vector bundles $E^\bullet_S$ and $E^\bullet_3$ which are “two-sided resolutions” of $\mathcal{F}$ (that is, their cohomology is $\mathcal{F}$, concentrated in degree zero). Each term of $E^\bullet_S$ (respectively, of $E^\bullet_3$) is a finite direct sum of locally free sheaves of the form $\mathcal{O}_{P^n}(j)$ (respectively, $\Omega^1_{P^n}(j)$) for $0 \leq j \leq n$, the number of copies of each such sheaf being given by the dimension of a suitable cohomology group of $\mathcal{F}$.

If $Q = (q_0, \ldots, q_n)$ is an $(n + 1)$-uple of positive integers (we assume that each of the $q_i$ is prime to the characteristic of the field $\mathbb{K}$, if this is not zero), the weighted projective space of type $Q$ is the projective algebraic variety $P(Q) = \text{Proj} \mathbb{K}[x_0^{q_0}, \ldots, x_n^{q_n}]$: an important property of $P(Q)$, which is essential for our purpose, is that $P(Q)$ is isomorphic to the quotient of $P^n$ by an action of the finite abelian group $\mu_Q = \mathbb{Z}/q_0\mathbb{Z} \times \cdots \times \mathbb{Z}/q_n\mathbb{Z}$.

The way we obtain a version of Beilinson’s theorem on $P(Q)$ is based on the following idea: given a coherent sheaf $\mathcal{F}$ on $P(Q)$, we apply Beilinson’s

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theorem to the coherent sheaf on $\mathbb{P}^n$ ($\pi: \mathbb{P}^n \to \mathbb{P}(O)$ is the projection), thus getting two resolutions $E_\star^*\pi$ and $E_\lambda^*\pi$ of $\pi^*\mathcal{F}$, and then applying the invariant direct image functor $\pi_*^\mu\pi^*$ yields the desired resolutions of $\mathcal{F}$. The main part of the present paper is dedicated to the proof that this procedure actually produces a similar result: in Section 1 we establish some properties of the functors $\pi^*$ and $\pi_*^\mu\pi^*$ (in particular, we prove that $\pi_*^\mu\pi^*$ is exact and that $F_D\pi$), and in Section 3 we verify that the resolutions $E_\star^*\pi$ and $E_\lambda^*\pi$ are compatible with the action of $\mu_O$ on $\pi^*\mathcal{F}$ (to this purpose, we have to examine some parts of the proof of Beilinson’s theorem). Moreover, since each term in the resolutions of $\mathcal{F}$ is a finite direct sum of sheaves of the form $\pi_*^*\pi^*(-j)$ or $\pi_*^*\Omega^i\pi^*(j)$ (where $\chi \in \mu_O^*$ is a character of $\mu_O$), in Sections 1 and 2 we obtain a description of these two types of sheaves, which allows us to formulate our main result (Theorem 4.1).

We were not able to find (as in the case of $\mathbb{P}^n$) categories equivalent to $D^b(Coh(\mathbb{P}(O)))$ (see Remark 4.3). In a forthcoming paper we will give applications of our main result to the canonical rings of algebraic surfaces (cf. [7]).

1. WEIGHTED PROJECTIVE SPACES

Let $\mathbb{K}$ be an algebraically closed field. Given an $(n+1)$-uple of positive integers $Q = (q_0, \ldots, q_n)$, the weighted projective space (over $\mathbb{K}$) of type $Q$ is the scheme $\mathbb{P}(O) = \text{Proj} S(O)$, where $S(O)$ is the graded ring $\mathbb{K}[t_0, \ldots, t_n]$ with $\deg t_i = q_i \forall i = 0, \ldots, n$ (in particular, for $Q = (1, \ldots, 1)$ we obtain the usual projective space $\mathbb{P}^n = \text{Proj} S$, where $S = \mathbb{K}[x_0, \ldots, x_n]$ with the usual gradation). We refer to [4, 8, 9, 12] for a complete treatment of weighted projective spaces; here we will recall only those aspects that we will need later.

From now on we will assume that $Q$ is reduced (that is, G.C.D. $(q_0, \ldots, q_n) = 1$) and that $\forall i = 0, \ldots, n \text{ char } \mathbb{K} | q_i$: the first hypothesis can be made without loss of generality (see [4, Lemma 3A.3 and Proposition 3C.1]), whereas the second is absolutely necessary to make our approach work.

It is not difficult to prove (see [4, Theorem 3A.1]) that $\mathbb{P}(O)$ is isomorphic to the quotient of $\mathbb{P}^n$ by the following action of the finite abelian group $\mu_Q = \mu_{q_0} \times \cdots \times \mu_{q_n}$ (here $\mu_{q_i} = \{\xi \in \mathbb{K} | \xi^{q_i} = 1\}$ is the cyclic group of order $q_i$):

$$\begin{align*}
\mu_Q \times \mathbb{P}^n &\rightarrow \mathbb{P}^n \\
((\xi_0, \ldots, \xi_n), [a_0, \ldots, a_n]) &\mapsto [\xi_0 a_0, \ldots, \xi_n a_n].
\end{align*}$$

From this we see that $\mathbb{P}(O)$ is a normal irreducible projective algebraic variety (of dimension $n$), because these properties are preserved by taking
the quotient under the action of a finite group; we will denote by \( \pi: \mathbb{P}^n \to \mathbb{P}(Q) \) the canonical projection, which is a finite morphism. The group \( \mu_Q \) acts accordingly on \( S \) by

\[ \mu_Q \times S \to S \]
\[ ((\xi_0, \ldots, \xi_n), p(x_0, \ldots, x_n)) \mapsto p(\xi_0 x_0, \ldots, \xi_n x_n). \]

Let \( S^{\mu_Q} \) be the subring of invariants of \( S \); clearly \( S^{\mu_Q} \cong \mathbb{K}[x_0^{q_0}, \ldots, x_n^{q_n}] \), which is isomorphic (as graded \( \mathbb{K} \)-algebra) to \( S(Q) \), and the morphism \( \pi: \text{Proj} S \to \text{Proj} S(Q) \) is induced by the inclusion of graded rings \( S \subset S(Q) \).

Let \( \mu^*_Q := \text{Hom}(\mu_Q, \mathbb{K} \setminus \{0\}) \) be the group of characters of \( \mu_Q \); since \( \mu^*_Q \cong \bigoplus_{i=0}^n \mu^*_{q_i} \), and \( \mu^*_{q_i} \cong \mathbb{Z}/q_i \mathbb{Z} \) corresponds to the homomorphism \( \mu_{q_i} \to \mathbb{K} \setminus \{0\} \) defined by \( \xi \mapsto \xi^{q_i} \), \( \forall \chi \in \mu^*_Q \), there exist unique numbers \( m_i(\chi) \) (for \( i = 0, \ldots, n \)) such that

\[ 0 \leq m_i(\chi) < q_i, \quad \text{and} \quad \chi = \sum_{i=0}^n m_i(\chi) \chi_i \]

(where \( \chi_i \) is the character defined by \( \chi_i((\xi_0, \ldots, \xi_n)) = \xi_i \)). We also define the weight \( |\chi| \) of a character \( \chi \) by

\[ |\chi| := \sum_{i=0}^n m_i(\chi). \]

Since the order of \( \mu_Q \) is \( \prod_{i=0}^n q_i \), which is invertible in \( \mathbb{K} \), every representation of \( \mu_Q \), that is every action (by automorphisms) of \( \mu_Q \) on a \( \mathbb{K} \)-vector space \( V' \), decomposes as \( V = \bigoplus_{\chi \in \mu^*_Q} V^\chi \), where \( V^\chi = \{ v \in V \mid \xi v = \chi(\xi) v, \forall \xi \in \mu_Q \} \) is the subspace on which \( \mu_Q \) acts by the character \( \chi \). A similar result holds in the case of an action of \( \mu_Q \) (compatible with the one on \( \mathbb{P}^n \)) on a sheaf \( F \) of \( \mathcal{O}_{\mathbb{P}^n} \)-modules (we will call \( F \) a \( \mu_Q \)-sheaf, cf. [13, Chap. II, Sect. 7]); \( \forall U \subset \mathbb{P}(Q) \) open, \( \pi_* F(U) = \#(\pi^{-1}(U)) \) is a \( \mathbb{K} \)-vector space on which \( \mu_Q \) acts, so that we can define (\( \forall \chi \in \mu^*_Q \)) the sheaf \( \pi^* F \) by \( \pi^*_\chi F(U) = F(\pi^{-1}(U))^\chi \) (if \( \chi \) is the null character, we will denote \( \pi^* F \) by \( \pi^* F \)), and it is clear that

\[ \pi_* F = \bigoplus_{\chi \in \mu^*_Q} \pi^*_\chi F. \]

In particular, denoting by \( \mu_Q \text{-Coh} (\mathbb{P}^n) \) (respectively, \( \mu_Q \text{-Qcoh} (\mathbb{P}^n) \)) the (abelian) category of coherent (respectively, quasi-coherent) \( \mu_Q \)-sheaves (morphisms are, of course, morphisms of sheaves preserving the action of \( \mu_Q \)), we have \( \forall \chi \in \mu^*_Q \) exact functors

\[ \pi^* : \mu_Q \text{-Coh} (\mathbb{P}^n) \to \text{Coh}(\mathbb{P}(Q)), \quad \pi_* : \mu_Q \text{-Qcoh} (\mathbb{P}^n) \to \text{Qcoh}(\mathbb{P}(Q)). \]
(as \( \pi \) is a finite morphism, \( \pi_* = \bigoplus_{k \in \mathbb{Z}} \pi_*^k \) is an exact functor and \( \pi_\ast \mathcal{F} \) is coherent if \( \mathcal{F} \) is coherent). As \( \pi \) is not flat, \( \pi_*^1 \) of a vector bundle is not a vector bundle in general; however, we have the following result.

**Proposition 1.1.** If \( E \) is a vector bundle on \( \mathbb{P}^n \), then \( \forall \chi \in \mu^\ast_{\mathbb{Q}} \) the coherent sheaf \( \pi^* \mathcal{E} \) on \( \mathbb{P}(Q) \) is reflexive.

**Proof.** Let \( W \subseteq \mathbb{P}(Q) \) be the open subset of regular points, let \( W' := \pi^{-1}(W) \subseteq \mathbb{P}^n \), and denote by \( \hat{\pi} : W' \rightarrow W \) the restriction of \( \pi \) (\( W \) is the quotient of \( W' \) under the action of \( \mu_Q \) restricted to \( W' \)) and by \( i : W \rightarrow \mathbb{P}(Q) \), \( i' : W' \rightarrow \mathbb{P}^n \) the inclusions. As \( \mathbb{P}(Q) \) is normal, \( \text{codim}(\mathbb{P}(Q) \setminus W) \geq 2 \), and then also \( \text{codim}(\mathbb{P}^n \setminus W') \geq 2 \), whence \( \mathcal{E} \cong i'_* (\mathcal{E}|_{W'}) \). Since \( \pi_*^1 \circ i_* \cong \pi_* \circ \hat{\pi}_* \), we obtain

\[
\pi^* \mathcal{E} \cong \pi_*^1 \hat{i}_* (\mathcal{E}|_{W'}) \cong i_* \pi_*^1 (\mathcal{E}|_{W'}).
\]

Now, \( \pi_*^1 (\mathcal{E}|_{W'}) \) is locally free (in particular reflexive) because \( \hat{\pi} \) is flat (see [15, Sect. 1]), and so \( \pi^* \mathcal{E} \) is reflexive, too (because \( \text{codim}(\mathbb{P}(Q) \setminus W) \geq 2 \)). \( \square \)

**Proposition 1.2.** If \( \mathcal{F} \in \mu_{\mathbb{Q}} \cdot \text{Coh}(\mathbb{P}^n) \), then \( \mu_{\mathbb{Q}} \) acts naturally on \( H^i(\mathbb{P}^n, \mathcal{F}) \) \( \forall i \geq 0 \) and

\[
H^i(\mathbb{P}^n, \mathcal{F}^\chi) \cong H^i(\mathbb{P}(Q), \pi^* \mathcal{F}) \quad \forall \chi \in \mu_{\mathbb{Q}}^\ast.
\]

**Proof.** Let \( \mathcal{U} = \{ U_i \}_{i=0}^n \) be the standard open affine cover of \( \mathbb{P}^n \) \( (U_i = D_{+}(x_i)) \) and consider the Čech complex

\[
0 \rightarrow C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{d^0} C^1(\mathcal{U}, \mathcal{F}) \rightarrow \cdots \rightarrow C^n(\mathcal{U}, \mathcal{F}) \rightarrow 0.
\]

Since each of the \( U_i \) (and then also any intersection of them) is a \( \mu_{\mathbb{Q}}^\ast \)-invariant open affine, it is immediate from the definition that \( \forall i \geq 0 \ C^i(\mathcal{U}, \mathcal{F}) \) is a \( \mu_{\mathbb{Q}}^\ast \)-vector space and \( d^i \) is a \( \mu_{\mathbb{Q}}^\ast \)-morphism. From this it follows that \( \mu_{\mathbb{Q}} \) acts naturally on the cohomology groups of the complex, which are, on the other hand, isomorphic to \( H^i(\mathbb{P}^n, \mathcal{F}) \) ([10, Chap. III, Theorem 4.5]). As for the second statement, let \( \mathcal{V} = \{ V_i \}_{i=0}^n \) be the open affine cover of \( \mathbb{P}(Q) \) given by \( V_i = D_{+}(t_i) \); since \( \pi^{-1}(V_i) = U_i \), the above complex can be identified with the Čech complex of \( \pi_* \mathcal{F} \) with respect to the cover \( \mathcal{V} \), whence

\[
H^i(\mathbb{P}^n, \mathcal{F}) \cong H^i(\mathbb{P}(Q), \pi_\ast \mathcal{F}) \quad \text{and}
\]

\[
H^i(\mathbb{P}^n, \mathcal{F}^\chi) \cong H^i(\mathbb{P}(Q), \pi_* \mathcal{F}^\chi) \quad \forall \chi \in \mu_{\mathbb{Q}}^\ast.
\]

\( \square \)

For every graded ring \( R \), let \( \text{mod}(R) \) be the category of finitely generated graded \( R \)-modules (with morphisms preserving degrees); if \( M = \bigoplus_{i} M_i \),
is an object of \( \text{mod}(R) \), \( M(j) \) will denote the module obtained by shifting degrees, that is, \( M(j) := M_{j^+} \). Given \( M \in \text{mod}(R) \), let \( M^\vee \) be the coherent sheaf on \( \text{Proj} \ R \) associated to \( M \) by the Serre correspondence (if \( M = Rj \)), we obtain the twisted sheaves \( c_{\text{Proj} \ R}(j) := R(j) \). Moreover, let \( \mu_\mathbb{Q} \text{-mod}(S) \) be the category of finitely generated \( \mu_\mathbb{Q} \text{-}S \)-modules (a \( \mu_\mathbb{Q} \text{-}S \)-module is an \( S \)-module together with an action of \( \mu_\mathbb{Q} \) on it compatible with the action on \( S \), and the morphisms in \( \mu_\mathbb{Q} \text{-mod}(S) \) are the morphisms in \( \text{mod}(S) \) that preserve the action); then it is clear that \( M^\vee \in \mu_\mathbb{Q} \text{-Coh}(\mathbb{P}^n) \) if \( M \in \mu_\mathbb{Q} \text{-mod}(S) \), and we have the following result.

**Lemma 1.3.** \( \forall M \in \mu_\mathbb{Q} \text{-mod}(S) \) and \( \forall y \in \mu_\mathbb{Q}^* \), \( \pi_y^x(M^\vee) \cong (M^x)^\vee \) in \( \text{Coh}(\mathbb{P}(Q)) \) (note that \( M^x \) is an \( S^\mu_\mathbb{Q} \cong S(Q) \)-module).

**Proof.** Let \( U \subset \mathbb{P}(Q) \) be an affine open subset of the form \( U = D_+(f) \cong \text{Spec} \ S_F^\mu_\mathbb{Q} \) for some homogeneous \( f \in S_F^\mu_\mathbb{Q} \). As \( V := \pi^{-1}(U) \cong \text{Spec} \ S_F(j) \), we obtain (denoting by \( \bar{\pi} \) the restriction of \( \pi \) to \( V \) and using \( \bar{\pi} \) also for the sheaf associated to a module on an affine scheme)

\[
\pi_y^x(M^\vee)|_{U} \cong \bar{\pi}_y^x(M^\vee)|_{V} \cong \bar{\pi}_y^x(M_{U(j)}^\vee) \cong ((M_{U(j)})^x)^\vee \\
\cong ((M^x)_{(j)})^\vee \cong (M^x)^\vee|_{U},
\]

and it is clear that these identifications are compatible as \( f \) varies among the homogeneous elements of \( S^\mu_\mathbb{Q} \), thus yielding the desired isomorphism of sheaves. \( \blacksquare \)

Applying this result in the case \( M = S(j) \) (on which \( \mu_\mathbb{Q} \) acts as on \( S \); in general, if \( N \in \mu_\mathbb{Q} \text{-mod}(S) \) then \( N(j) \) will be considered a \( \mu_\mathbb{Q} \text{-}S \)-module with the same action of \( \mu_\mathbb{Q} \) as on \( N \)), we obtain an explicit description of the sheaves of the form \( \pi_y^x(c_{\mathbb{P}(Q)}(j)) \), which will play a central role in our theorem.

**Corollary 1.4.** \( \pi_y^x(c_{\mathbb{P}(Q)}(j)) \cong c_{\mathbb{P}(Q)}(j - |\chi|) \forall \chi \in \mu_\mathbb{Q}^* \) and \( \forall j \in \mathbb{Z} \).

**Proof.** It is enough to show that \( S(j)^\vee \cong S^{\mu_\mathbb{Q}}(j - |\chi|) \), because then

\[
\pi_y^x(c_{\mathbb{P}(Q)}(j)) = \pi_y^x(S(j)^\vee) \cong (S(j)^\vee)^\vee \cong S^{\mu_\mathbb{Q}}(j - |\chi|)^\vee = c_{\mathbb{P}(Q)}(j - |\chi|).
\]

Since the action is always the same, we can suppose that \( j = 0 \). Now, a monomial \( s = x_0^l \cdots x_n^l \in S \) is in \( S^x \) if and only if \( \forall \xi = (\xi_0, \ldots, \xi_n) \in \mu_\mathbb{Q}, \xi s = \xi_0^l \cdots \xi_n^l s = \chi(\xi)s := \xi_0^{m_0(\chi)} \cdots \xi_n^{m_n(\chi)} s, \)

that is if and only if \( i_j \equiv m_\chi(j) \text{ (mod } q_j) \forall j = 0, \ldots, n. \) From this we see that multiplication by \( x_0^{m_0(\chi)} \cdots x_n^{m_n(\chi)} \) induces an isomorphism (of degree zero) of \( S^{\mu_\mathbb{Q}}(-|\chi|) \) onto \( S^x. \) \( \blacksquare \)
If \( \mathcal{F} \) is a sheaf of \( \mathcal{O}(Q) \)-modules, then \( \pi^* \mathcal{F} = \pi^{-1} \mathcal{F} \otimes_{\mathcal{O}(Q)} \mathcal{O}_\mathcal{P} \) is in a natural way a \( \mu_Q \)-sheaf of \( \mathcal{O}_\mathcal{P} \)-modules (\( \mu_Q \) acts on \( \mathcal{O}_\mathcal{P} \)), and so there is a functor
\[
\pi^* : \text{Coh}(\mathcal{P}(Q)) \to \mu_Q \cdot \text{Coh}(\mathcal{P}^n)
\]
(which is right exact but not exact, because \( \pi \) is not a flat morphism).

**Proposition 1.5.** \( \forall \mathcal{F} \in \text{Coh}(\mathcal{P}(Q)), \forall \mathcal{G} \in \mu_Q \cdot \text{Coh}(\mathcal{P}^n) \) and \( \forall \chi \in \mu_Q^* \), there is an isomorphism of sheaves on \( \mathcal{P}(Q) \),
\[
\pi^*_\chi(\pi^* \mathcal{F} \otimes \mathcal{G}) \cong \mathcal{F} \otimes \pi^*_\chi \mathcal{G}.
\]

**Proof.** Let \( U \cong \text{Spec } A \) be an open affine subset of \( \mathcal{P}(Q) \); since \( \pi \) is an affine morphism, \( V := \pi^{-1}(U) \cong \text{Spec } B \) is also affine and \( U \) is the quotient of \( V \) by the action of \( \mu_Q \) on \( \mathcal{P}^n \) restricted to \( V \); moreover, the morphism of affine schemes \( \tilde{\pi} := \pi|_V : V \to U \) is induced by the morphism of rings \( A \cong B^{\mu_Q} \hookrightarrow B \). Now, let \( M = \mathcal{F}(U) \) and \( N = \mathcal{G}(V) \) (\( M \) is an \( A \)-module and \( N \) is a \( \mu_Q \)-\( B \)-module), so that \( \mathcal{F}|_U \cong M^* \) and \( \mathcal{G}|_U \cong N^* \). Then we have natural isomorphisms:
\[
\pi^*_\chi(\pi^* \mathcal{F} \otimes \mathcal{G})|_U \cong \tilde{\pi}^*_\chi(\tilde{\pi}^*(\mathcal{F}|_U) \otimes \mathcal{G}|_V) \cong \tilde{\pi}^*_\chi(\tilde{\pi}^*(M^*) \otimes N^*)
\]
\[
\cong \tilde{\pi}^*_\chi((M \otimes_A B)^* \otimes N^*)
\]
\[
\cong \tilde{\pi}^*_\chi(M \otimes_A B \otimes_B N)^* \cong \tilde{\pi}^*_\chi(M \otimes_A N^*)
\]
\[
\cong ((M \otimes_A N^*)^*) \cong (M \otimes_A N^*)^* \cong M^* \otimes (N^*)^*
\]
\[
\cong \mathcal{F}|_U \otimes \tilde{\pi}^*_\chi(\mathcal{G}|_V) \cong (\mathcal{F} \otimes \pi^*_\chi \mathcal{G})|_U.
\]

It is easy to verify that these isomorphisms (as \( U \) varies among the open affine subsets of \( \mathcal{P}(Q) \)) glue to give the required isomorphism of sheaves.

**Corollary 1.6.** \( \pi_*^{\mu_Q} \circ \pi^* \cong \text{id} : \text{Coh}(\mathcal{P}(Q)) \to \text{Coh}(\mathcal{P}(Q)). \)

**Proof.** \( \forall \mathcal{F} \in \text{Coh}(\mathcal{P}(Q)) \) we have (by Proposition 1.5 and Corollary 1.4)
\[
\pi_*^{\mu_Q}(\pi^* \mathcal{F}) \cong \pi_*^{\mu_Q}(\pi^* \mathcal{F} \otimes \mathcal{O}_{\mathcal{P}^n}) \cong \mathcal{F} \otimes \pi_*^{\mu_Q} \mathcal{O}_{\mathcal{P}^n} \cong \mathcal{F} \otimes \mathcal{O}_{\mathcal{P}(Q)} \cong \mathcal{F}.
\]

**Remark 1.7.** It is not true, however, that
\[
\pi_*^{\mu_Q} : \mu_Q \cdot \text{Coh}(\mathcal{P}^n) \to \text{Coh}(\mathcal{P}(Q)) \quad \text{and} \quad \pi^* : \text{Coh}(\mathcal{P}(Q)) \to \mu_Q \cdot \text{Coh}(\mathcal{P}^n)
\]
are equivalences of categories (this would be the case if the action of \( \mu_Q \) on \( \mathcal{P}^n \) were free (see [13, Chap. II, Sect. 7, Proposition 2])).
2. SHEAVES OF DIFFERENTIALS

Let \( \Omega_{S(Q)} := \Omega_{S(Q)/\mathbb{R}} \) be the \( S(Q) \)-module of \( \mathbb{R} \)-differentials of \( S(Q) \) and \( \forall j \geq 0 \) let \( \Omega^j_{S(Q)} := \Lambda^j(\Omega_{S(Q)}) \); denoting by \( d : S(Q) \to \Omega_{S(Q)} \) the universal derivation, \( \Omega^0_{S(Q)} \) is a free \( S(Q) \)-module with basis

\[
\{ dt_i \wedge \cdots \wedge dt_j \mid 0 \leq i_1 < \cdots < i_j \leq n \},
\]

which is naturally graded by \( \text{deg}(dt_i \wedge \cdots \wedge dt_j) = q_i + \cdots + q_j \). For \( j > 0 \) the morphisms (of degree zero)

\[
\Delta_j : \Omega^j_{S(Q)} \to \Omega^{j-1}_{S(Q)}
\]

\[
dt_i \wedge \cdots \wedge dt_j \mapsto \sum_{k=1}^j (-1)^{k-1} q_k t_k dt_i \wedge \cdots \wedge \hat{dt}_k \wedge \cdots \wedge dt_j
\]
yield a complex in \( \text{mod}(S(Q)) \),

\[
0 \to \Omega^{n+1}_{S(Q)} \xrightarrow{\Delta_{n+1}} \Omega^n_{S(Q)} \to \cdots \to \Omega^1_{S(Q)} \xrightarrow{\Delta_1} \Omega^0_{S(Q)} = S(Q) \to 0,
\]

which is just the Koszul complex of the regular sequence \( (q_0 t_0, \ldots, q_n t_n) \), and hence is exact everywhere, except that \( \text{coker} \Delta_1 = S(Q)/(q_0 t_0, \ldots, q_n t_n) \cong \mathbb{R} \) (in degree zero). Setting \( \hat{\Omega}^i_{S(Q)} := \text{im} \Delta_{i+1} \) (with the induced grading), \( \hat{\Omega}^i_{\text{P}(Q)} := (\hat{\Omega}^i_{S(Q)})^* \) is called the sheaf of regular differential \( j \)-forms; one defines more generally \( \hat{\Omega}^i_{\text{P}(Q)}(k) := (\hat{\Omega}^i_{S(Q)}(k))^* \forall k \in \mathbb{Z} \).

Remark 2.1. Unlike the case of \( \mathbb{P}^n \), it is not true in general that, given \( M \in \text{mod}(S(Q)) \) and \( k \in \mathbb{Z} \), the coherent sheaves on \( \mathbb{P}(Q) \) \( M \otimes \mathcal{O}_Q(k) \) and \( M(k) \) are isomorphic (see [9, 1.5]). Besides, even if for every \( \mathcal{F} \in \text{Coh}(\mathbb{P}(Q)) \) there is \( M \in \text{mod}(S(Q)) \) such that \( \mathcal{F} \cong M^* \) (see [8, Proposition 4.3]), it may happen that \( M^* \cong N^* \) but \( M(k) \not\cong N(k) \), whence there is not a natural way to define \( \mathcal{F}(k) \) in general.

These definitions apply in particular to \( \mathbb{P}^n \); in this case \( \mu_O \) acts on \( \Omega^j_{\mathbb{P}} \) by

\[
((\xi_0, \ldots, \xi_n), dx_{i_1} \wedge \cdots \wedge dx_{i_j}) \mapsto \xi_i \cdots \xi_j dx_{i_1} \wedge \cdots \wedge dx_{i_j},
\]

and then the \( \Delta_j \) are \( \mu_O \)-morphisms, so that \( \hat{\Omega}^i_{\mathbb{P}} \in \mu_O \cdot \text{mod}(S) \) and \( \hat{\Omega}^j_{\mathbb{P}}(k) \in \mu_O \cdot \text{Coh}(\mathbb{P}^n) \). The reason the \( \hat{\Omega}^j_{\text{P}(Q)} \) are called sheaves of regular differentials and are preferred to the usual \( \Omega^j_{\text{P}(Q)} \) is that, denoting by \( W \) the open subset of regular points in \( \mathbb{P}(Q) \) and by \( i : W \to \mathbb{P}(Q) \) the inclusion, one has (see [9, 2.2.4]) \( \hat{\Omega}^j_{\text{P}(Q)} \cong i_* \Omega^j_W \) (in particular, \( \hat{\Omega}^j_{\mathbb{P}} \cong \Omega^j_W \)). Moreover, \( \hat{\Omega}^j_{\text{P}(Q)} \cong \pi^*_{\mathbb{P}} \Omega^j_{\mathbb{P}} \) (see [9, 2.2.3]); however, for our theorem we need to know more generally what the sheaves of the form \( \pi^*_{\mathbb{P}}(\Omega^j_{\mathbb{P}}(k)) \) \( \forall \xi \in \mu_O \) and \( \forall k \in \mathbb{Z} \) are. For this reason we are led to introduce certain sheaves of “logarithmic differentials.”
Definition 2.2. Let $I$ be a subset of $\{0, \ldots, n\}$ and let $1_I$ be the characteristic function of $I$. $\forall j \geq 0$ $\Omega^j_{S(Q)}(\log t')$ is the free $S(Q)$-module with basis

$\left\{ \frac{dt_i}{t_i^{1_{j}(t_i)}} \wedge \cdots \wedge \frac{dt_j}{t_j^{1_{j}(t_j)}} \mid 0 \leq i_1 < \cdots < i_j \leq n \right\},$

graded by

$$\deg\left( \frac{dt_i}{t_i^{1_{j}(t_i)}} \wedge \cdots \wedge \frac{dt_j}{t_j^{1_{j}(t_j)}} \right) = q_i^{1-1_{j}(i)} + \cdots + q_j^{1-1_{j}(j)}.$$ 

$\Delta^j: \Omega^j_{S(Q)}(\log t') \to \Omega^{j-1}_{S(Q)}(\log t')$ is the morphism (of degree zero) defined by

$$\Delta^j\left( \frac{dt_i}{t_i^{1_{j}(t_i)}} \wedge \cdots \wedge \frac{dt_j}{t_j^{1_{j}(t_j)}} \right) = \sum_{k=1}^{j} (-1)^{k-1} q_{i_k}^{1-1_{j}(i_k)} \frac{dt_i}{t_i^{1_{j}(t_i)}} \wedge \cdots \wedge \frac{dt_{i_k}}{t_{i_k}^{1_{j}(t_{i_k})}} \wedge \cdots \wedge \frac{dt_j}{t_j^{1_{j}(t_j)}},$$

and $\tilde{\Omega}^j_{S(Q)}(\log t') := \text{im}\Delta^j_{j+1}$ (with the induced graduation).

Remark 2.3. The definitions given at the beginning of this section are just the case $I = \emptyset$ of Definition 2.2: $\Omega_j^{'} = \Omega_j^{'}(\log r')$, $\Delta_j = \Delta_j^{'}$ and $\tilde{\Omega}_j^{'} = \tilde{\Omega}_j^{'}(\log r')$.

Remark 2.4. The complex in mod$(S(Q))$

$$0 \to \Omega^j_{S(Q)}(\log t') \xrightarrow{\Delta^j} \Omega_{S(Q)}(\log t') \to \cdots \to \Omega^1_{S(Q)}(\log t') \xrightarrow{\Delta^1} \Omega^0_{S(Q)}(\log t') = S(Q) \to 0$$

is just the Koszul complex of the sequence $(q_0^{1-1_{j}(0)}, \ldots, q_n^{1-1_{j}(n)})$; if $I \neq \emptyset$ it is exact everywhere, because in that case the ideal generated by the sequence is $S(Q)$.

Definition 2.5. The sheaf of logarithmic differentials of type $I$ $\tilde{\Omega}^j_{P(Q)}(\log t')$ is the sheaf associated to the graded $S(Q)$-module $\tilde{\Omega}^j_{S(Q)}(\log t')$. More generally, for $k \in \mathbb{Z}$, we set

$$\tilde{\Omega}^j_{P(Q)}(\log t')(k) := (\tilde{\Omega}^j_{S(Q)}(\log t')(k)).$$
We will now prove that the sheaves of the form $\pi^*(\Omega^j_{\log}(k))$ are (up to twist) sheaves of logarithmic differentials. For this we need a notation: 
\[
\forall \chi = \sum_{i=1}^k m_i(\chi)x_i \in \mu^*_Q, \text{ let }
I(\chi) := \{i \in \{0, \ldots, n\} | m_i(\chi) > 0\}.
\]

**Lemma 2.6.** $\forall \chi \in \mu^*_Q$ and $\forall k \in \mathbb{Z}$, there are natural isomorphisms in mod($S(Q)$)
\[
(\Omega^j_\log(k))^x \cong \Omega^j_{S(Q)}(\log t^{j(x)})(k-|\chi|) \quad \text{and} \quad (\tilde{\Omega}^j_\log(k))^x \cong \tilde{\Omega}^j_{S(Q)}(\log t^{j(x)})(k-|\chi|).
\]

**Proof.** Since the action of $\mu_Q$ is the same, we can suppose $k = 0$.
Now, a monomial $\omega = x_0^m \cdots x_n^m dx_{i_1} \cdots \cdots dx_{i_l} \in \Omega^j_S$ is in $(\Omega^j_S)^x$ if and only if $\forall i = (\xi_0, \ldots, \xi_n) \in \mu_Q$, we have $\xi_i \omega = \chi(\xi)\omega$, that is, if and only if $\xi_0 \cdots \xi_n \xi_{i_1} \cdots \xi_{i_l} \omega = \xi_0^m(x) \cdots \xi_n^m(x) \omega$,
whence we obtain
\[
\omega \in (\Omega^j_S)^x \Leftrightarrow r_l \equiv m_l(\chi) - 1_{(i_1, \ldots, i_l)}(l) (\mod q_l) \quad \forall l = 0, \ldots, n.
\]
From this it follows that the morphism
\[
\alpha: \Omega^j_{S(Q)}(\log t^{j(x)}) \to \Omega^j_S
\]
\[
\left(\frac{dt_{i_1}}{t_{i_1}^{(s_1(l))}} \wedge \cdots \wedge \frac{dt_{i_l}}{t_{i_l}^{(s_l(l))}}\right) \mapsto x_0^{m_0(x)} \cdots x_n^{m_n(x)} \frac{q_{s_1}^{-1}}{t_{i_1}^{(s_1(l))}} dx_{i_1} \wedge \cdots \wedge \frac{q_{s_l}^{-1}}{t_{i_l}^{(s_l(l))}} dx_{i_l}
\]
gives an isomorphism of $\Omega^j_{S(Q)}(\log t^{j(x)})$ onto $(\Omega^j_S)^x$. Indeed,
\[
\alpha_l\left(\frac{dt_{i_1}}{t_{i_1}^{(s_1(l))}} \wedge \cdots \wedge \frac{dt_{i_l}}{t_{i_l}^{(s_l(l))}}\right) = q_1 \cdots q_l x_0 \cdots x_n dx_{i_1} \wedge \cdots \wedge dx_{i_l},
\]
where $s_l = m_l(\chi) - 1_{(i_1, \ldots, i_l)}(l) + q_l 1_{(i_1, \ldots, i_l) - t(l)}(l)$; then we have $\im \alpha_l \subset (\Omega^j_S)^x$ because $s_l \equiv m_l(\chi) - 1_{(i_1, \ldots, i_l)}(l) (\mod q_l)$ and $(\Omega^j_S)^x \subset \im \alpha_l$ because $0 \leq s_l < q_l$ ($\alpha_l$ is obviously injective). As $\alpha_l$ shifts the degrees by $|\chi|$, we obtain the desired graded isomorphism of $S(Q)$-modules $(\Omega^j_S)^x \cong \Omega^j_{S(Q)}(\log t^{j(x)})(-|\chi|)$. Moreover, it is straightforward to verify that these isomorphisms are compatible, in the sense that the diagram
\[
\begin{array}{ccc}
(\Omega^j_S)^x & \xrightarrow{\cong} & \Omega^j_{S(Q)}(\log t^{j(x)})(-|\chi|) \\
\downarrow{\cong} & & \downarrow{\cong} \\
(\Omega^j_S^{-1})^x & \xrightarrow{\cong} & \Omega^j_{S(Q)}^{-1}(\log t^{j(x)})(-|\chi|)
\end{array}
\]
is commutative ($\Delta^i_j$ is the restriction of $\Delta_j$ to $(\Omega^{i}_j)^*$), and clearly this implies that $(\check{\Omega}^{i}_j)^* \cong \check{\Omega}^{i}_{S(Q)}(\log t^{i(x)})(-|\chi|)$.

**Corollary 2.7.** $\forall \chi \in \mu_Q^*$ and $\forall k \in \mathbb{Z}$, there are isomorphisms in $\text{Coh}(\mathbb{P}(Q))$,

$$\pi^*_k(\Omega^i_{\mathbb{P}^n}(k)) \cong \check{\Omega}^{i}_{\mathbb{P}(Q)}(\log t^{i(x)})(k - |\chi|).$$

**Proof.** This follows immediately from Lemmas 1.3 and 2.6, since $(\check{\Omega}^{i}_{Q}(k))^* \cong \check{\Omega}^{i}_{\mathbb{P}^n}(k)$.

### 3. BEILINSON'S THEOREM FOR $\mu_Q$-SHEAVES

First we recall the statement of Beilinson's theorem (see [1–3]) and fix the notation.

Let $V$ be a $\mathbb{K}$-vector space of dimension $n + 1$, let $\mathbb{P}^n = \mathbb{P}^n(V)$, and consider the symmetric and exterior algebras $S = S(V^*)$ ($V^*$ is the dual of $V$, thus $\mathbb{P}^n = \text{Proj}S$) and $\Lambda = \Lambda(V)$. Given a graded ring $R$ ($S$ and $\Lambda$ in our case), $\text{mod}_{0,1}(R)$ will be the full subcategory of $\text{mod}(R)$, whose objects are isomorphic to finite direct sums of $R$-modules of the form $R(-j)$ for $0 \leq j \leq n$.

Since (by [2, Lemma 2])

$$\text{Hom}_{\text{Coh}(\mathbb{P}^n)}(\Omega^i_{\mathbb{P}^n}(i), \Omega^j_{\mathbb{P}^n}(j)) \cong \Lambda^{i-j}(V),$$

$$\text{Hom}_{\text{Coh}(\mathbb{P}^n)}(\omega_{\mathbb{P}^n}(-i), \omega_{\mathbb{P}^n}(-j)) \cong \Lambda^{i-j}(V^*),$$

it is easy to see that there exist additive functors,

$$\tilde{F}_S: \text{mod}_{0,1}(S) \rightarrow \text{Coh}(\mathbb{P}^n) \quad \text{and} \quad \tilde{F}_\Lambda: \text{mod}_{0,1}(\Lambda) \rightarrow \text{Coh}(\mathbb{P}^n),$$

uniquely determined (up to isomorphism) by the conditions

$$\tilde{F}_S(S(-j)) = \omega_{\mathbb{P}^n}(-j) \quad \text{and} \quad \tilde{F}_\Lambda(\Lambda(-j)) = \Omega^i_{\mathbb{P}^n}(j) \quad (0 \leq j \leq n).$$

If $A$ is an abelian category or, more generally, an additive subcategory of an abelian category (such as $\text{mod}_{0,1}(R)$), $K(A)$ and $D(A)$ will denote the homotopy and the derived category of $A$, respectively; we recall shortly the definitions (see [11, Chap. I, and 16] for more details).

The objects of $K(A)$ are just the complexes of $A$ (by a complex we always mean a cohomological complex, that is, differentials have degree $+1$). A complex

$$\ldots \rightarrow X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \rightarrow \ldots$$

will be denoted by \((X^\bullet, d^\bullet)\) or simply by \(X^\bullet\); similarly, a morphism of complexes from \((X^\bullet, d^\bullet)\) to \((Y^\bullet, e^\bullet)\) given by morphisms \(f^i: X^i \to Y^i\) (with \(e^{i+1} \circ f^i = f^{i+1} \circ d^i\)) will be denoted by \(f^\bullet\). The morphisms of \(K(A)\) are the homotopy equivalence classes of morphisms of complexes (two morphisms of complexes \(f^\bullet, g^\bullet: (X^\bullet, d^\bullet) \to (Y^\bullet, e^\bullet)\) are homotopic if there exist morphisms \(k^i: X^i \to Y^{i+1}\) such that \(f^i - g^i = e^{i+1} \circ k^i + k^{i+1} \circ d^i\)). Note that if \(f^\bullet: X^\bullet \to Y^\bullet\) is a morphism of \(K(A)\), then the morphisms in cohomology \(H^i(f^\bullet): H^i(X^\bullet) \to H^i(Y^\bullet)\) are well defined; in particular, \(f^\bullet\) is said to be a quasi-isomorphism if \(H^i(f^\bullet)\) is an isomorphism \(\forall i \in \mathbb{Z}\).

The objects of \(D(A)\) are always the complexes of \(A\), but the morphisms are obtained from those of \(K(A)\) by inverting formally the quasi-isomorphisms; more precisely, a morphism in \(D(A)\) from \(X^\bullet\) to \(Y^\bullet\) is represented by a couple of morphisms \((f^\bullet, s^\bullet): (X^\bullet, d^\bullet) \to (Y^\bullet, e^\bullet)\), where \(s^\bullet\) is a quasi-isomorphism and \(Z^\bullet\) is another object of \(D(A)\); on these couples one introduces the equivalence relation generated by the direct relation which identifies two couples \((f^\bullet, s^\bullet): (X^\bullet, d^\bullet) \to (Y^\bullet, e^\bullet)\) and \((g^\bullet, t^\bullet): (W^\bullet, f^\bullet) \to (Y^\bullet, Z^\bullet)\) if there exists a morphism \(a^\bullet: W^\bullet \to Z^\bullet\) such that \(f^\bullet = a^\bullet \circ g^\bullet\) and \(s^\bullet = a^\bullet \circ t^\bullet\).

\(K^b(A)\) and \(D^b(A)\) are the full subcategories of \(K(A)\) and \(D(A)\) having as objects the bounded complexes (a complex \(X^\bullet\) is bounded if there exists an integer \(n\) such that \(X^i = 0\) for \(|i| > n\)). We recall that \(K(A), K^b(A), D(A),\) and \(D^b(A)\) are triangulated categories and that an additive functor between two triangulated categories is said to be exact if it preserves the triangulated structure (see [11, chap. I, and 16] for the definition and the main properties of triangulated categories).

Finally, if \(F\) is an exact functor between two homotopy categories (usually coming from an additive functor between the corresponding abelian categories), then \(RF\) (respectively, \(LF\)) will denote the right (respectively, left) derived functor of \(F\) between the corresponding derived categories (if it exists).

**Beilinson’s Theorem.** The functors \(\hat{F}_S\) and \(\hat{F}_\Lambda\) introduced above extend naturally to exact functors of triangulated categories,

\[
\begin{align*}
F_S: K^b(\text{mod}_{[0, n]}(S)) &\to D^b(\text{Coh}(\mathbb{P}^n)) \\
F_\Lambda: K^b(\text{mod}_{[0, n]}(\Lambda)) &\to D^b(\text{Coh}(\mathbb{P}^n)).
\end{align*}
\]

\(F_S\) and \(F_\Lambda\) are equivalences of triangulated categories.

Explicitly, for every bounded complex \(\mathcal{F}^\bullet\) of coherent sheaves on \(\mathbb{P}^n\), there are two bounded complexes \(E_S^\bullet = E_S^\bullet(\mathcal{F}^\bullet)\) and \(E_\Lambda^\bullet = E_\Lambda^\bullet(\mathcal{F}^\bullet)\) of vector bundles
such that

\[ E^k_S = \bigoplus_{j=0}^n \mathcal{E}(-j) \otimes \kappa H^{i+k}(\mathbb{P}^n, \mathcal{F} \otimes \Omega^j(j)), \]

\[ E^k_\lambda = \bigoplus_{j=0}^n \Omega^j(j) \otimes \kappa H^{i+k}(\mathbb{P}^n, \mathcal{F} \otimes \mathcal{E}(-j)) \]

and \( \mathcal{F}^* \cong E^*_S \cong E^*_\lambda \) in \( D^b(\text{Coh}(\mathbb{P}^n)) \).

Remark 3.1. In the particular case (which is actually the most interesting) of a coherent sheaf \( \mathcal{F} \) (considered as a complex concentrated in position 0), the isomorphisms \( \mathcal{F}^* \cong E^*_S(\mathcal{F}) \cong E^*_\lambda(\mathcal{F}) \) in \( D^b(\text{Coh}(\mathbb{P}^n)) \) just mean that

\[ H^k(E^*_S) \cong H^k(E^*_\lambda) \cong \begin{cases} \mathcal{F} & \text{if } k = 0 \\ 0 & \text{if } k \neq 0. \end{cases} \]

Now we want to study the case where \( \mathcal{F} \) is a \( \mu_\mathcal{O} \)-sheaf in the above situation. To this purpose we introduce some terminology. A complex of sheaves \((\mathcal{G}^*, d^*)\) will be called a \( \mu_\mathcal{O} \)-complex if the \( \mathcal{G}^i \) are \( \mu_\mathcal{O} \)-sheaves and the \( d^i \) are \( \mu_\mathcal{O} \)-morphisms; similarly, a morphism \( f^* : \mathcal{G}^1_1 \rightarrow \mathcal{G}^2_2 \) of \( \mu_\mathcal{O} \)-complexes will be called a \( \mu_\mathcal{O} \)-morphism if the \( f^i \) are \( \mu_\mathcal{O} \)-morphisms of sheaves (note that in this case \( H^i(f^*) : H^i(\mathcal{G}^1_1) \rightarrow H^i(\mathcal{G}^2_2) \) are \( \mu_\mathcal{O} \)-morphisms). Then the main result of this section will be the following generalization of Beilinson’s theorem.

Main proposition 3.2. \( \forall \mathcal{F} \in \mu_\mathcal{O} \cdot \text{Coh}(\mathbb{P}^n) \), there exist two bounded \( \mu_\mathcal{O} \)-complexes \((E^*_S, e^*_S)\) and \((E^*_\lambda, e^*_\lambda)\) of vector bundles such that

\[ E^k_S = \bigoplus_{j=0}^n \mathcal{E}(-j) \otimes \kappa H^{i+k}(\mathbb{P}^n, \mathcal{F} \otimes \Omega^j(j)), \]

\[ E^k_\lambda = \bigoplus_{j=0}^n \Omega^j(j) \otimes \kappa H^{i+k}(\mathbb{P}^n, \mathcal{F} \otimes \mathcal{E}(-j)) \]

and

\[ H^k(E^*_S) \cong H^k(E^*_\lambda) \cong \begin{cases} \mathcal{F} & \text{if } k = 0 \\ 0 & \text{if } k \neq 0. \end{cases} \]

in \( \mu_\mathcal{O} \cdot \text{Coh}(\mathbb{P}^n) \).

Since the two versions of the theorem are perfectly dual to each other, we can set \( A_j := \mathcal{E}(-j), B_j := \Omega^j(j) \) in the case of \( E^*_S \), and \( A_j := \Omega^j(j) \),
First we recall that the $A_j$ and the $B_j$ (whence also $\mathcal{F} \otimes B_j$) are $\mathcal{O}_\Delta$-sheaves and that $\mathcal{O}_\Delta$ also acts naturally on the $\kappa$-vector spaces $H^{i+k}(\mathbb{P}^n, \mathcal{F} \otimes B_j)$ (see Sections 1 and 2).

Let $p, q: \mathbb{P}^n \times \mathbb{P}^n \rightarrow \mathbb{P}^n$ be the two projections, and observe that (letting $\mathcal{O}_\Delta$ act on each component of $\mathbb{P}^n \times \mathbb{P}^n$) they are $\mathcal{O}_\Delta$-morphisms; from this it follows that we have functors

$$p_*: \mathcal{O}_\Delta\text{-Coh}(\mathbb{P}^n \times \mathbb{P}^n) \rightarrow \mathcal{O}_\Delta\text{-Coh}(\mathbb{P}^n)$$

$$q_*: \mathcal{O}_\Delta\text{-Coh}(\mathbb{P}^n) \rightarrow \mathcal{O}_\Delta\text{-Coh}(\mathbb{P}^n \times \mathbb{P}^n)$$

(and similarly for quasi-coherent sheaves).

The starting point in the proof of Beilinson’s theorem is that $\mathcal{F} \cong R p_* (\mathcal{O}_\Delta \otimes q^* \mathcal{F})$ (here $\mathcal{O}_\Delta$ is the structure sheaf of the diagonal $\Delta \subset \mathbb{P}^n \times \mathbb{P}^n$), and then one must show that $R p_* (\mathcal{O}_\Delta \otimes q^* \mathcal{F}) \cong \mathcal{E}^*$. To compute $\mathcal{E}^*$, let $C_\Delta$ be a resolution of $\mathcal{O}_\Delta$ given by the resolution of the diagonal (if $\mathcal{G}_1$ and $\mathcal{G}_2$ are sheaves of $\mathcal{O}_\mathbb{P}^n$-modules, then $\mathcal{G}_1 \otimes \mathcal{G}_2$ denotes the sheaf of $\mathcal{O}_\mathbb{P}^n$-modules $p^* \mathcal{G}_1 \otimes q^* \mathcal{G}_2$),

$$0 \rightarrow A_n \otimes B_n \xrightarrow{d^n} A_{n-1} \otimes B_{n-1} \rightarrow \cdots \rightarrow A_1 \otimes B_1 \xrightarrow{d^1} A_0 \otimes B_0$$

$$= \mathcal{O}_\mathbb{P}^{n+1} \rightarrow \mathcal{O}_\Delta \rightarrow 0$$

(see [14, Chap. II, Sect. 3.1]). Setting $D^j := A_j \otimes B_j$, it is clear that the $D^j$ are $\mathcal{O}_\Delta$-sheaves, and it is straightforward to verify that the $d^j$ are $\mathcal{O}_\Delta$-morphisms, so that $(D^*, d^*)$ is a $\mathcal{O}_\Delta$-complex isomorphic to $\mathcal{O}_\Delta$ in the derived category, whence $\mathcal{O}_\Delta \otimes q^* \mathcal{F} \cong D^* \otimes q^* \mathcal{F}$ in $D^b(\text{Coh}(\mathbb{P}^n \times \mathbb{P}^n))$, and $D^* \otimes q^* \mathcal{F}$ is a $\mathcal{O}_\Delta$-complex.

For the computation of $R p_*$ we need a $p_*$-acyclic resolution: for this reason we consider the sheafified Čech complex $\check{\mathcal{C}}^*(u, \mathcal{F})$ (see the proof of 1.2 for the definition of $\check{\mathcal{C}}$), which is a resolution of $\mathcal{F}$ (we recall that $\Gamma(\mathbb{P}^n, \check{\mathcal{C}}^*(u, \mathcal{F})) = C^*(u, \mathcal{F})$ by definition). As usual, the fact that $\mathcal{F}$ is a $\mathcal{O}_\Delta$-sheaf implies that $\check{\mathcal{C}}^*(u, \mathcal{F})$ is a $\mathcal{O}_\Delta$-complex and the natural morphism $i: \mathcal{F} \rightarrow \check{\mathcal{C}}^*(u, \mathcal{F})$ is a $\mathcal{O}_\Delta$-quasi-isomorphism of complexes. More generally, we set (for $0 \leq j \leq n$)

$$\left(\check{\mathcal{E}}_j^*, \gamma_j^*\right) := \check{\mathcal{C}}^*(u, \mathcal{F} \otimes B_j) \quad \text{and} \quad \left(C_j^*, c_j^*\right) := C^*(u, \mathcal{F} \otimes B_j)$$

and observe that $\check{\mathcal{E}}_j^* \cong \check{\mathcal{E}}_0^* \otimes B_j$ (if $\mathcal{E}$ is a quasi-coherent sheaf and $\check{\mathcal{E}}$ is a vector bundle, it is easy to verify that there is an isomorphism of complexes
Now we obtain $O_1 L q F D q C_0$ in $Db \cong Qcoh \cong n$. Setting $G k D n M j D 0$, we have $G$ is a $Q$-complex. Next we note that $D j q C j C k 0 D p A j q B j q C j C k 0 D p A j C j C k 0$, which is $p^*$-acyclic (see [1, Lemma 3.2]), whence we obtain the following isomorphisms in $D^b(\text{Qcoh}(\mathbb{P}^n))$:

$$\mathcal{F} \cong R p_* (\mathcal{F} \otimes q^* \mathcal{F}) \cong R p_* (G^*) \cong p_*(G^*) := (F^*, f^*).$$

Clearly $F^*$ is a $\mu_Q$-complex and $F^k = \bigoplus_{j=0}^n X_j^{i+k}$, where (by K"unneth's formula)

$$X_j^i = p_*(D^{-i} \otimes q^* \mathcal{C}_j^i) \cong p_*(A_j \otimes \mathcal{C}_j^i) \cong A_j \otimes \mathcal{C}_j^i.$$  

We denote by $\alpha_j^i: X_j^i \to X_{j-1}^i$ and $\beta_j^i: X_j^i \to X_{j+1}^i$ the only non zero terms of $f^*$ and observe that $\beta_j^i$ is (up to a sign) $\text{id}_{A_j} \otimes c_j^i$. Moreover, it is easy to verify that the isomorphism in $D^b(\text{Qcoh}(\mathbb{P}^n))$ $\mathcal{F} \cong F^*$ can be represented by $(i: \mathcal{F} \to \mathcal{C}_0^i, s^*: F^* \to \mathcal{C}_0^i)$, where the only non zero term of $s^k: F^k \to \mathcal{C}_0^k$ is the natural morphism

$$X_0^k \cong A_0 \otimes \mathcal{C}_0^k \cong \mathcal{O} \otimes \Gamma(\mathbb{P}^n, \mathcal{C}_0^k) \to \mathcal{C}_0^k.$$  

From this we see that both $i$ and $s^*$ are $\mu_Q$-quasi-isomorphisms, and so $H^0(F^*)$ is $\mu_Q$-isomorphic to $\mathcal{F}$ (and obviously $H^k(F^*) = 0$, $\forall k \neq 0$).

To complete the proof of 3.2 we need a technical lemma.
Lemma 3.3. Let $\mathcal{A}$ be an abelian category and let $(V^\bullet, v^\bullet) \in K(\mathcal{A})$ be such that

$$V^i = R^i \oplus Y^i \oplus H^i \oplus Y^{i+1} \quad \text{and} \quad v^i = \begin{pmatrix}
  r^i & t_1^i & t_2^i & t_3^i \\
  s_1^i & 0 & 0 & \text{id}_{Y^{i+1}} \\
  s_2^i & 0 & 0 & 0 \\
  s_3^i & 0 & 0 & 0
\end{pmatrix}.$$ 

Then $(V^\bullet, v^\bullet)$ is isomorphic (in $K(\mathcal{A})$) to $e((V^\bullet, v^\bullet)) := (W^\bullet, w^\bullet)$, where

$$W^i = R^i \oplus H^i \quad \text{and} \quad w^i = \begin{pmatrix}
  r^i - t_3^i \circ s_1^i & t_2^i \\
  s_2^i & 0
\end{pmatrix}.$$ 

Proof. One verifies directly (using the hypothesis that $v^\bullet$ is a differential) that $w^\bullet$ is a differential, too, and that the morphisms of complexes $a^\bullet: (V^\bullet, v^\bullet) \to (W^\bullet, w^\bullet)$ and $b^\bullet: (W^\bullet, w^\bullet) \to (V^\bullet, v^\bullet)$, defined by

$$a^i = \begin{pmatrix}
  \text{id}_R & -t_3^i -1 & 0 & 0 \\
  0 & 0 & \text{id}_H & 0
\end{pmatrix} \quad \text{and} \quad b^i = \begin{pmatrix}
  \text{id}_R & 0 \\
  0 & 0 \\
  0 & \text{id}_H \\
  -s_3^i & 0
\end{pmatrix},$$

yield the required isomorphism: $a^i \circ b^i = \text{id}_{W^\bullet}$ and $b^i \circ a^i = l^{i+1} \circ v^i + v^{i-1} \circ l^i$, where the homotopy operator $l^i: V^i \to V^{i-1}$ is given by

$$l^i = \begin{pmatrix}
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & \text{id}_{Y^i} & 0 & 0
\end{pmatrix}.$$ 

Proof of 3.2. It remains to prove the existence of the $\mu_Q$-complex $(E^\bullet, e^\bullet)$ isomorphic to $(F^\bullet, f^\bullet)$ in $D^b(\text{Qcoh}(\mathbb{P}^n))$ (because then $\mathcal{F} \cong F^\bullet \cong E^\bullet$ in $D^b(\text{Qcoh}(\mathbb{P}^n))$, from which it follows that $\mathcal{F} \cong E^\bullet$ in $D^b(\text{Coh}(\mathbb{P}^n))$, too) with $H^0(E^\bullet) \mu_Q$-isomorphic to $H^0(F^\bullet)$; clearly it is enough to show that $(F^\bullet, f^\bullet) \cong (E^\bullet, e^\bullet)$ in $K^b(\mu_Q \cdot \text{Qcoh}(\mathbb{P}^n))$.

Since for $0 \leq j \leq n$ the complex $(X^\bullet_j, \beta^\bullet_j)$ is just a $\mu_Q$-complex of $\mathbb{K}$-vector spaces tensored by $A_j$, it follows that, setting

$$Y_j^i := \text{im } \beta_j^{i-1}, \quad H_j^i := H^i(X^\bullet_j) \cong H^i(A_j \otimes_{\mathbb{K}} \mathcal{C}_j) \cong A_j \otimes_{\mathbb{K}} H^i(\mathbb{P}^n, \mathcal{F} \otimes B_j),$$
$X_j$ is $\mu_Q$-isomorphic to $Y_j \oplus H_j \oplus Y_{j+1}$, and, with this identification, the only non zero term of $\beta_j$ is $id_{Y_{j+1}}$. Let us define, moreover (for $0 \leq r \leq n+1$),

$$R^k_r = \bigoplus_{0 \leq j < r} H^{l+k}_{j} \oplus X^{l+k}_j.$$ 

Then each term of $(F^*_0, f^*_0) := (F^*, f^*)$ can be decomposed as $F^*_0 = R^k_0 \oplus Y^0_0 \oplus H^{0+1}_0$, and we are going to construct inductively for $0 \leq r \leq n+1$ complexes $(F^*_r, f^*_r)$ isomorphic to $(F^*, f^*)$ in $K^b(\mu_Q \cdot Qcoh(\mathbb{P}^n))$, with $F^*_r = R^k_r \oplus Y^r_{r+k} \oplus H^{r+k}_r \oplus Y^{r+k+1}_r$ (the last three terms are defined to be zero if $r = n+1$).

Suppose that $0 \leq r \leq n$ and that $(F^*_r, f^*_r)$ is a complex with the following properties:

1. $F^*_r = R^k_r \oplus Y^r_{r+k} \oplus H^{r+k}_r \oplus Y^{r+k+1}_r$,
2. For $r \leq l \leq n$ the only non zero term of the component $f^*_r(l, l): X^{l+k}_l = Y^{l+k}_l \oplus H^{l+k}_l \oplus Y^{l+k+1}_l \rightarrow X^{l+k+1}_l = Y^{l+k+1}_l \oplus H^{l+k+1}_l \oplus Y^{l+k+2}_l$

of $f^*_r$ is $id_{Y^{l+k+1}_l}$.

The case $l = r$ of property 2 just says that $(F^*_r, f^*_r)$ (with the decomposition given by property 1) satisfies the hypothesis of Lemma 3.3, and we claim that the complex $(F^*_r, f^*_r) := e((F^*_r, f^*_r))$ (which is isomorphic in $K^b(\mu_Q \cdot Qcoh(\mathbb{P}^n))$ to $(F^*_r, f^*_r)$ by the lemma) also satisfies properties 1 and 2. Indeed, by definition of $e$,

$$F^*_{r+1} = R^k_{r+1} \oplus H^{r+k+1}_{r+1} \oplus Y^{r+k+2}_{r+1} \oplus Y^{r+k+1}_{r+1}.$$ 

and $f^*_{r+1}(l, l) = f^*_r(l, l)$ for $r + 1 \leq l \leq n$. To see this last fact, observe that, again by definition of $e$, $f^*_{r+1}(l, l) = f^*_r(l, l) - s^*_l(l) \circ t^*_r(l)$ for suitable morphisms

$$s^*_l(l): X^{l+k}_l \rightarrow Y^{r+k+1}_l \quad \text{and} \quad t^*_r(l): Y^{r+k+1}_r \rightarrow X^{l+k+1}_l,$$

and that necessarily $t^*_r(l) = 0$, because

$$\text{Hom}_{\mathbb{Q}coh(\mathbb{P}^n)}(A_j, A_{j'}) = 0 \quad \text{if} \quad j < j'.$$

Eventually, since $(F^*_r, f^*_r)$ clearly satisfies properties 1 and 2, we obtain by induction that $(F^*, f^*)$ is isomorphic in $K^b(\mu_Q \cdot Qcoh(\mathbb{P}^n))$ to $(E^*, e^*) := (F^*_n, f^*_n)$, and (as required)

$$E^* = R^k_{n+1} = \bigoplus_{j=0}^n H^{j+k}_j = \bigoplus_{j=0}^n A_j \otimes \mu_n H^{j+k}(\mathbb{P}^n, \mathbb{F} \otimes B_j).$$
4. BEILINSON’S THEOREM ON WEIGHTED PROJECTIVE SPACES

Using the notation introduced in the previous sections, we have the following version of Beilinson’s theorem on weighted projective spaces.

**Theorem 4.1.** \( \forall \mathcal{F} \in \text{Coh}(\mathbb{P}(Q)) \), there are two bounded complexes \( \mathcal{G}^* \) and \( \mathcal{H}^* \) of coherent sheaves on \( \mathbb{P}(Q) \) such that

\[
H^k(\mathcal{G}^*) \cong H^k(\mathcal{H}^*) \begin{cases} \mathcal{F} & \text{if } k = 0 \\ 0 & \text{if } k \neq 0, \end{cases}
\]

and \( \mathcal{G}^k = \bigoplus_{j=0}^n V_j^{j+k} \), \( \mathcal{H}^k = \bigoplus_{j=0}^n W_j^{j+k} \), where

\[
V_j^i = \bigoplus_{x \in \mu_0^i} \mathcal{O}_{\mathbb{P}(Q)}(-j - |x|) \otimes \mathcal{H}^i(\mathbb{P}(Q), \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}(Q)}(\log t_1(-x))(j - |x|))
\]

\[
W_j^i = \bigoplus_{x \in \mu_0^i} \mathcal{O}_{\mathbb{P}(Q)}^j(\log t_1(-x))(j - |x|) \otimes \mathcal{H}^i(\mathbb{P}(Q), \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}(Q)}(-j - |x|)).
\]

**Proof.** By Proposition 3.2 applied to \( \pi^* \mathcal{F} \in \mu_Q^0 \text{Coh}(\mathbb{P}^n) \) there are two bounded \( \mu_Q^0 \)-complexes \( E^*_S = E^*_S(\pi^* \mathcal{F}) \) and \( E^*_A = E^*_A(\pi^* \mathcal{F}) \) with cohomology \( \mu_Q^0 \)-isomorphic to \( \pi^* \mathcal{F} \) and

\[
E^k_S = \bigoplus_{j=0}^n \mathcal{O}_{\mathbb{P}^n}(-j) \otimes \mathcal{H}^{i+k}(\mathbb{P}^n, \pi^* \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n}(j)) := \bigoplus_{j=0}^n P_j^{i+k},
\]

\[
E^k_A = \bigoplus_{j=0}^n \mathcal{O}_{\mathbb{P}^n}^j(\log t^{i+k}(j - |x|)) \otimes \mathcal{H}^i(\mathbb{P}(Q), \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}(Q)}(-j - |x|)).
\]

Setting \( \mathcal{G}^* := \pi^* \mu_Q^0 E^*_S \) and \( \mathcal{H}^* := \pi^* \mu_Q^0 E^*_A \), we have (since \( \pi^* \mu_Q^0 \) is an exact functor)

\[
H^k(\mathcal{G}^*) \cong H^k(\mathcal{H}^*) \cong \pi^* \mu_Q^0 H^k(E^*_S) \cong \pi^* \mu_Q^0 H^k(E^*_A)
\]

\[
\cong \begin{cases} \mu_Q^0 \mathcal{F} & \text{if } k = 0 \\ 0 & \text{if } k \neq 0, \end{cases}
\]

by Corollary 1.6. Moreover, \( \mathcal{G}^k = \bigoplus_{j=0}^n V_j^{j+k} \) and \( \mathcal{H}^k = \bigoplus_{j=0}^n W_j^{j+k} \), where

\[
V_j^i := \pi^* \mu_Q^0 P_j^i \cong \bigoplus_{x \in \mu_0^i} \pi^* \mathcal{O}_{\mathbb{P}^n}(-j) \otimes \mathcal{H}^i(\mathbb{P}^n, \pi^* \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n}(j))^{-x},
\]

\[
W_j^i := \pi^* \mu_Q^0 Q_j^i \cong \bigoplus_{x \in \mu_0^i} \pi^* \mathcal{O}_{\mathbb{P}^n}^j(\log t^{i+k}(j - |x|)) \otimes \mathcal{H}^i(\mathbb{P}(Q), \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}(Q)}(-j))^{-x}.
\]
On the other hand, by Corollaries 1.4 and 2.7,
\[ \pi^\chi_{\mathbb{P}^n}(-j) \cong \epsilon_{\mathbb{P}(Q)}(-j - \lvert \chi \rvert), \quad \pi^\chi \Omega^j_{\mathbb{P}^n}(j) \cong \tilde{\Omega}^j_{\mathbb{P}(Q)}(\log t^{(x)})(j - \lvert \chi \rvert)), \]
and, by Propositions 1.2 and 1.5,
\[ H^i(\mathbb{P}^n, \pi^* T \otimes \Omega^j_{\mathbb{P}^n}(j))^{-\chi} \cong H^i(\mathbb{P}(Q), T \otimes \tilde{\Omega}^j_{\mathbb{P}(Q)}(\log t^{(x)})(j - \lvert \chi \rvert)), \]
\[ H^i(\mathbb{P}^n, \pi^* T \otimes \epsilon_{\mathbb{P}^n}(-j))^{-\chi} \cong H^i(\mathbb{P}(Q), T \otimes \epsilon_{\mathbb{P}(Q)}(-j - \lvert \chi \rvert)). \]

Remark 4.2. The sheaves of the form
\[ \epsilon_{\mathbb{P}(Q)}(-j - \lvert \chi \rvert) \quad \text{and} \quad \tilde{\Omega}^j_{\mathbb{P}(Q)}(\log t^{(x)})(j - \lvert \chi \rvert)) \]
(which are not locally free in general) are reflexive; this follows from Proposition 1.1, since they are isomorphic, respectively, to \( \pi^\chi_{\mathbb{P}^n}(-j) \) and \( \pi^\chi \Omega^j_{\mathbb{P}^n}(j) \).

Remark 4.3. Clearly (setting \( |Q| := \sum_{i=0}^n q_i \)) the two sets of sheaves
\[ \{ \epsilon_{\mathbb{P}(Q)}(-j - \lvert \chi \rvert) \mid 0 \leq j \leq n, \ x \in \mu^*_Q \} = \{ \epsilon_{\mathbb{P}(Q)}(-i) \mid 0 \leq i < |Q| \}, \]

\[ \{ \tilde{\Omega}^j_{\mathbb{P}(Q)}(\log t^{(x)})(j - \lvert \chi \rvert)) \mid 0 \leq j \leq n, \ x \in \mu^*_Q \} \]
both generate \( D^b(\text{Coh}(\mathbb{P}(Q))) \) as a triangulated category, but we do not know if it is possible to find reasonable categories of modules equivalent to \( D^b(\text{Coh}(\mathbb{P}(Q))) \), as in the case of \( \mathbb{P}^n \). Indeed, we can define as before, at least if \( Q \) is normalized (that is, if \( G.C.D.(q_0, \ldots, \hat{q}_i, \ldots, q_n) = 1 \) for \( 0 \leq i \leq n \), a functor
\[ F_{S(Q)}; K^b(\text{mod}_{[0, |Q| - 1]}(S(Q))) \rightarrow D^b(\text{Coh}(\mathbb{P}(Q))) \]
by sending \( S(Q)(-j) \) to \( \epsilon_{\mathbb{P}(Q)}(-j) \); when \( Q \) is normalized
\[ \text{Hom}(\epsilon_{\mathbb{P}(Q)}(-i), \epsilon_{\mathbb{P}(Q)}(-j)) \cong \epsilon_{\mathbb{P}(Q)}(i - j) \]
(see [8, Lemma 4.1]), whence \( \text{Hom}_{\text{Coh}(\mathbb{P}(Q))}(\epsilon_{\mathbb{P}(Q)}(-i), \epsilon_{\mathbb{P}(Q)}(-j)) \cong S(Q)_{i-j} \), and this allows us to define a natural functor \( \text{mod}_{[0, |Q| - 1]}(S(Q)) \rightarrow \text{Coh}(\mathbb{P}(Q)) \), which extends to \( F_{S(Q)} \). However, \( F_{S(Q)} \) (which is essentially surjective, as we have already noted) is not fully faithful; this is a consequence of the fact that (unlike the case of \( \mathbb{P}^n \); see [2, Lemma 2]) \( \text{Ext}^l(\epsilon_{\mathbb{P}(Q)}(-i), \epsilon_{\mathbb{P}(Q)}(-j)) \neq 0 \) for \( l > 0 \) in general.
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