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A coding theoretic approach to extending designs

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Abstract

We introduce the study of designs in a coset of a binary code which can be held by vectors of a fixed weight. If C is a binary $[2n, n, d]$ code with n odd and the words of weights $n-1$ and $n+1$ hold complementary t-designs, then we show that the vectors of weight n in a coset of weight 1 also hold a t-design. We also show how to "extend" these designs. We then consider designs in cosets of type I self-dual codes, in particular in the shadow. If the vectors of a fixed weight in the code hold t-designs then so do the vectors of a fixed weight in the shadow. For $[24k-2, 12k-1, 2+4k]$ type I codes, these designs extend to designs in the type II parent code.

1. Introduction

A key problem in the theory of designs is the existence of a design with parameters t, v, k, and λ, denoted t-(v, k, λ), when the necessary arithmetic conditions are satisfied. We are interested in the subsequent problem of extending an existing design and how this extension might be realized. We recall that a $(t+1)$-$(v+1, k+1, \lambda_{t+1})$ design is an *extension* of a t-(v, k, λ_t) design if when we remove some point from the extended design and look at those blocks which contain that point, then we obtain the t-(v, k, λ_t) design, also called the *derived* design. Clearly, certain arithmetic conditions must be satisfied for an extension of a design to exist.

Lemma 1.1. *A necessary condition for a t-(v, k, λ) design to be extendible is that $k+1$ divide $(v+1)b$ where b is the number of blocks in the original design.*

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Coding theory has made many contributions to the theory of combinatorial designs. Codes generated by the incidence matrix of a design have been useful in either constructing the design or showing that the design does not exist, such as the projective plane of order 10. Designs have been found in the words of a fixed weight in a code. A method to determine if the words of a fixed weight "hold" a design is via the Assmus-Mattson theorem [11, Chap. 6], which we give below. As we are only concerned with binary codes in this paper, we give the following version of this theorem.

For brevity if all vectors of a fixed weight in either a code or a coset hold a t-design, we all them *t-vectors*. When we say the vectors in a code are t-vectors, we mean the vectors of each fixed weight are t-vectors.

Theorem 1.1. *Let C be an $[n,k,d]$ binary code. Let t be a positive integer $< d$. Let $s = |\{i\colon B_i \neq 0,\ 0 < i \leqslant n - t\}|$ where B_i is the number of vectors of weight i in $C^\perp$. If $s \leqslant d - t$, then the vectors in C are t-vectors and the vectors in $C^\perp$ are also t-vectors.*

We are interested in how coding theory might be used in order to extend designs. A natural place to look for vectors to extend a design held by the vectors in a code is in a coset of the code. This leads to two distinct problems. The weaker problem is to determine when vectors in a coset are t-vectors. The stronger problem is given a design in a coset, to determine when it can be used to extend a design in the code.

In Section 2 we look at the problem of when a coset holds a design. When the length of the code is $2n$ with n odd, then under certain conditions the words of weight n in a coset of weight 1 hold a t-design. This design can be used, in conjunction with a design in the code, to construct a t-design on $(v + 1)$ and $(v + 2)$ points. We give many examples of codes where these designs occur.

In the final section we find t-designs in a special coset, the shadow, of type I self-dual codes whenever vectors in the code are t-vectors. Based on parameters related to the shadow, we get conditions stronger than a generic application of the Assmus-Mattson theorem, determining that vectors in the code are t-vectors. We show that the baby Golay code G_{22} holds 3-designs without recourse to its automorphism group. We show how to extend these 3-designs to the 5-designs in the Golay code G_{24}, without resort to group theory. This procedure extends to the general class of extremal type II codes of length $24k$.

2. Codes, cosets and designs

We introduce some terminology associated with the parameters of a t-design. Let $P_1, \ldots, P_i$ be points associated with the design. Recall that a t-design is also a $(t - i)$ design for $0 < i \leqslant t - 1$ and that the number of blocks containing i points is denoted by λ_i. For $t \geqslant i \geqslant j$ we define the *block intersection numbers* λ_{ij} to be the number of blocks containing the points $P_1, \ldots, P_j$ but which do not contain $P_{j+1}, \ldots, P_i$. It is known that the λ_{ij} are independent of the points chosen [13, Theorem 86]. If $j = 0$, then λ_{i0} is the number of blocks that do not contain the points $P_1, \ldots, P_i$. One sees that $\lambda_{ii} = \lambda_i$, but more generally we have that

$$\lambda_{ij} = \sum_{k=0}^{i-j} (-1)^k \binom{i-j}{k} \lambda_{j+k} \tag{1}$$

as well as

$$\lambda_0 = \sum_{k=0}^{i} \binom{i}{k} \lambda_{ik} \quad \forall i, \tag{2}$$

$$\lambda_i = \prod_{j=0}^{i-1} [(k-j)/(v-j)] \cdot \lambda_0 \quad \forall i. \tag{3}$$

Formula (1) follows from the inclusion–exclusion principle. Formula (3) is the usual condition relating the parameters of a design.

The block intersection numbers fit together to form a Pascal triangle. In other words the following relation holds:

$$\lambda_{ij} = \lambda_{i+1,j} + \lambda_{i+1,j+1}. \tag{4}$$

The entries of a Pascal triangle can be the block intersection numbers of a design if and only if the λ_{ii} satisfy formula (3). Also, note that the Pascal triangle associated to the derived design sits within the Pascal triangle of the original design and has its apex at the node λ_{11}.

To say that one can extend the parameters of a design means that the associated Pascal triangle for the extension has the original Pascal triangle embedded with its apex at λ_{11}. A Pascal triangle is *symmetric* if $\lambda_{ij} = \lambda_{i,i-j}$ for all i, j. Our main result is a consequence of properties of the block intersection numbers of t-$(2n, n, \lambda_t)$ designs. We show first that the associated Pascal triangle is symmetric.

Recall that the complements of the blocks of a t-design constitute the blocks of a t-design called the *complementary design* [13, Theorem 91].

Lemma 2.1. *Let D be a t-$(2n, n, \lambda_t)$ design. Then the associated Pascal triangle is symmetric.*

Proof. Since the complementary design of a t-$(2n, n, \lambda_t)$ design is also a t-$(2n, n, \lambda_t)$ design [13, Theorem 91], the Pascal triangle of a t-$(2n, n, \lambda_t)$ design is symmetric. □

It is interesting that we are able to derive an equation among certain block intersection numbers from the symmetry of a Pascal subtriangle.

Theorem 2.1. *Let D be a t-$(2n, n-1, \lambda_t)$ design. Then the following relation holds among the block intersection numbers: $\lambda_{i,i-1} = \lambda_{i+1,0} + \lambda_{i+1,i}$ for $1 \leqslant i < t$.*

Proof. It suffices to show that the Pascal subtriangle with apex at λ_{20} is symmetric. This is the Pascal triangle of the second derived design of the complements of the blocks and is a $(t-2)$-$(2n-2, n-1, \lambda_{t-2})$ design which is symmetric by Lemma 2.1. □

If all the weights occurring in the code are divisible by 2 then the code is called *even*. We consider only binary even codes which contain the all-one vector. We suppose that the vectors of a fixed weight in a code C hold a t-design. Clearly, when we puncture these vectors on the coordinate given by the coset leader, then these vectors constitute the blocks of the derived design. However, we are interested in extending designs, so we ask when cosets of weight one hold t-designs. We will need the following theorem due to Alltop [1] for our construction.

Theorem 2.2. *A* $2t$-$(2n, n, \lambda_{2t})$ *self-complementary design is necessarily a* $(2t+1)$-$(2n, n, \lambda_{2t+1})$ *design.*

The following theorem tells us when the "middle" weight vectors in a coset of weight one are t-vectors. Block intersection numbers in Theorems 2.4 and 2.6 refer to the t-$(2n, n-1, \lambda_t)$ design and not to its complementary design.

Theorem 2.3. *Let C be a $[2n, k]$ even code with n odd such that the vectors of weights $n-1$ and $n+1$ hold complementary t-designs. Then the vectors of weight n in a coset of weight 1 hold a t-design when t is odd and they hold a $(t-1)$-design when t is even.*

Proof. Let E be a coset of weight one in C. Without loss of generality, we can say that the weight one vector in E has a one in the first coordinate. Note that vectors of weight n in E arise from vectors of weight $n-1$ in C which have a 0 in their first position or from vectors of weight $n+1$ in C which have a 1 in their first position. We will first show that these vectors hold a $t-1$-design. If a set of size $t-1$ contains the first coordinate then $\lambda_{t-1,t-2}$ vectors in C of weight $n-1$ cover all but the first of these positions, hence $\lambda_{t-1,t-2}$ vectors of weight n in E cover it. If the $t-1$ ones do not cover the first position then there are $\lambda_{t,t-1}$ vectors of weight $n-1$ in C which cover these $t-1$ positions but not the first position and there also are $\lambda_{t,0}$ vectors of weight $n+1$ in C which cover these $t-1$ positions and the first position. By Theorem 2.1 we know that $\lambda_{t-1,t-2} = \lambda_{t,0} + \lambda_{t,t-1}$, so that these sums are the same and the vectors of weight n in E are $t-1$ vectors. If t is odd, then $t-1$ is even and Alltop's theorem applies because the vectors of weight n are self-complementary so that we may conclude that these vectors are actually t-vectors. □

There are two kinds of binary self-dual codes; those of *type I* contain vectors whose weights are $\equiv 2 \pmod 4$. All vectors in a *type II* code have weights divisible by 4. There are bounds on the largest minimum weight possible for both types of codes [13, Corollary to Theorem 84]. A self-dual code whose minimum weight attains this bound is called *extremal*.

Consider the unique, extremal [22, 11, 6] self-dual code which we refer to as the baby Golay code. We will show later that all vectors in this code are 3-vectors. Therefore by Theorem 2.3 the words of weight 11 in a coset of weight 1 hold a 3-(22, 11, 72) design.

We now show that the t-design that we have constructed on $2n$ points can be combined with the original design to contruct a t-design on $2n + 1$ points. We have not extended the design in the traditional sense as then we would have a $t + 1$ design on $2n + 1$ points. In these constructions the λ in the designs is given in terms of the parameters of the original t-$(2n, n - 1, \lambda_t)$ design.

Theorem 2.4. *Let C be a $[2n, k]$ code with n odd such that the vectors of weights $n - 1$ and $n + 1$ hold complementary t-designs. Adjoin an additional coordinate equal to one to the vectors of weight $n - 1$ in C and add a zero coordinate to the vectors of weight n in a coset of weight 1. Then the vectors of weight $n - 1$ in C extended in this way together with the extended vectors of weight n in a coset of weight 1 hold a t-$(2n + 1, n, \lambda_{t-1})$ design when t is odd and a $(t - 1)$-$(2n + 1, n, \lambda_{t-2})$ design when t is even.*

Proof. If t is odd, then Theorem 2.3 gives a t-$(2n, n, \lambda_{t-1} - \lambda_t)$ design along with the design with parameters t-$(2n, n - 1, \lambda_t)$ that come from the words of weight $n - 1$ in the code. We must show that the extended design is a t-design, that is that any t points are contained in the same number of vectors of weight n. Call the new point ∞. If the t points are among the original points then $\lambda_t + \lambda_{t-1} - \lambda_t = \lambda_{t-1}$ blocks contain them. If one of the points is ∞, then the only blocks containing them are the λ_{t-1} blocks arising from vectors of weight $n - 1$.

If t is even, then Theorem 2.3 gives only a $(t - 1)$-$(2n, n, \lambda_{t-2} - \lambda_{t-1})$ design. Therefore, when we extend we only get a $(t - 1)$-$(2n + 1, n, \lambda_{t-2})$ design. □

To extend the design further we need the following result of Alltop [1].

Theorem 2.5. *Any t-$(2n + 1, n, \lambda_t)$ design with t even is extendible to a $t + 1$-$(2n + 2, n + 1, \lambda_{t+1})$ design by extending the blocks and adjoining complements.*

We now extend the design to $2n + 2$ points.

Theorem 2.6. *Under the assumptions of Theorems 2.3 and 2.4, if t is odd there exists a t-$(2n + 2, n + 1, \lambda_{t-2})$ design; if t is even there exists a $(t - 1)$-$(2n + 2, n + 1, \lambda_{t-3})$ design.*

Proof. If t is even, then Theorem 2.4 gives a $t - 1$ design on $2n + 1$ points. Then $t - 2$ is even and applying Theorem 2.5 we get a $(t - 1)$-$(2n + 2, n + 1, \lambda_{t-3})$ design. If t is odd, then Theorem 2.4 yields a t-design. Then $t - 1$ is even and we can apply Theorem 2.5 to get a t-$(2n + 2, n + 1, \lambda_{t-2})$ design. □

The above construction uses a coset of weight one to extend designs which are held by the support of words of a fixed weight in a code. However, we can apply these theorems to designs contained in vectors of a fixed weight in $C \cup C^{\perp}$ when C is a formally self-dual code. A code C is called *formally self-dual* (*f.s.d.*) if C and $C^{\perp}$ have the same weight distribution. Here we use cosets of weight one in the code and the corresponding coset in the dual. This construction is quite formal and we can restate the previous theorem without relying on properties of codes and cosets.

Corollary 2.1. *Let D be a t-$(2n, n-1, \lambda_t)$ design. It t is even then the following three designs exist.*
1. *a $(t-1)$-$(2n, n, \lambda_{t-2} - \lambda_{t-1})$ design,*
2. *a $(t-1)$-$(2n+1, n, \lambda_{t-2})$ design and*
3. *a $(t-1)$-$(2n+2, n+1, \lambda_{t-3})$ design.*

If t is odd then the following three designs exist
1. *a t-$(2n, n, \lambda_{t-1} - \lambda_t)$ design,*
2. *a t-$(2n+1, n, \lambda_{t-1})$ design and*
3. *a t-$(2n+2, n+1, \lambda_{t-2})$ design.*

Some of our most interesting examples occur in formally self-dual even codes. If $n \equiv 1 \pmod 4$, then the union of words of any fixed weight in extremal *f.s.d.* even codes C and $C^{\perp}$ hold 3-designs [10]. These occur at lengths 10 and 18. Hence the above theorems apply and we get new 3-designs.

As stated previously the words of any fixed weight in the baby Golay code hold 3-designs so that our theorems give new 3-designs on 22, 23 and 24 points. There is a known 3-(22, 11, 72) design which is a twice derived design from the Steiner system $S(5, 12, 24)$. However, this design is not isomorphic to the design constructed above which will be shown in the next section.

Magliveras and Leavitt [12] have constructed a 6-(20, 9, 112) design. This leads to new 5-designs on 20, 21 and 22 points. Also, generalized quadratic residue codes often hold 3-designs because of the action of a 3-homogeneous group [14]. If the length of the code is $\equiv 2 \pmod 4$, then the above theorems apply and we obtain additional 3-designs. In particular, there is a *f.s.d.* even [26, 13, 6] code that is a generalized quadratic residue code that has $PGL(2, 25)$ acting on it. The weight enumerator is $1 + 65x^6 + 325x^8 + 1430x^{10} + 2275x^{12} + \cdots$. This yields 3-designs on 26, 27 and 28 points.

We give a small table of designs (see Table 1) constructed as above:

Table 1
Designs from cosets

$(2n, n-1, \lambda)$	$(2n, n, \lambda)$	$(2n+1, n, \lambda)$	$(2n+2, n+1, \lambda)$	Comments
6-(20, 9, 112)	5-(20, 10, 924)	5-(21, 20, 1344)	5-(22, 11, 3808)	Magliveras [12]
3-(18, 8, 21)	3-(18, 9, 35)	3-(19, 9, 56)	3-(20, 10, 136)	f.s.d. code [10]
3-(26, 12, 385)	3-(26, 13, 539)	3-(27, 13, 924)	3-(28, 14, 2100)	f.s.d. code [10]
3-(22, 10, 48)	3-(22, 11, 72)	3-(23, 101, 120)	3-(24, 12, 280)	Baby Golay [4]

3. Designs in cosets of self-dual codes

In the first section we found t-designs held by vectors of weight n in a coset of weight 1 of a code of length $2n$ (n odd) whenever vectors of weights $n-1$ and $n+1$ in the code held t-designs. We noted that this was particularly applicable to formally self-dual codes. In this section we will determine when cosets of *type I* self-dual codes hold designs. These codes have the property that a certain coset is distinguished.

If C is a *type I* code, we let C_0 denote the unique subcode consisting of all vectors in C whose weights are divisible by 4. Clearly, C_0 has codimension one in C. Furthermore, $C = C_0 \cup C_2$ where C_2 is a coset of C_0 in $C_0^\perp$. The distinguished coset of C in the whole space called the *shadow* is $S = C_1 \cup C_3$ where $C_0^\perp = C_0 \cup C_1 \cup C_2 \cup C_3$ with C_1 and C_3 cosets of C_0 in $C_0^\perp$ [7, 9]. If C has length $2n$ then it is known that all the weights in the shadow are congruent to $n \pmod 4$. The following theorem allows us to find designs in C.

Theorem 3.1. *Let C be a type I $[2n, n, d]$ code where $n \equiv 0, 1, 3 \pmod 4$. Let t be a positive integer with $t < d$. Assume that C_0 has s distinct non-zero weights $\leqslant 2n - t$. Let $\bar{d} = wt(S)$ and let $d' = \text{minimum}(\bar{d}, d)$. If $s \leqslant d' - t$, then the vectors in C are t-vectors.*

If n is odd, then the vectors in C_0 are t-vectors as are the vectors in $C_2 \cup S$.

Proof. We apply the Assmus-Mattson theorem to the code $C_0^\perp = C \cup S$. As d' is the minimum weight of $C_0^\perp$, the Assmus-Mattson theorem states that the vectors of any fixed weight in C_0 hold t-designs as well as the vectors of a fixed weight in $C_0^\perp$. As n is not congruent to $2 \pmod 4$, the vectors in C_2 are the only vectors in $C_0^\perp$ of weight $\equiv 2 \pmod 4$ and so hold t-designs. If n is odd, then the vectors in $C_0^\perp = C_0 \cup C_2 \cup S$ are t-vectors by the Assmus-Mattson theorem, but $C_0 \cap (C_2 \cup S) = 0$, so the result follows. □

The restriction on n in the previous theorem is necessary as there exists a [28, 14, 6] self-dual code with weight enumerator

$$1 + 42x^6 + 378x^8 + 1624x^{10} + 3717x^{12} + 4680x^{14} + \cdots$$

such that the words in C_0 are 2-vectors but the words in C_2 hold only a 1-design, even though the words in $C_2 \cup S$ hold a 2-design. Also, the previous proof leads to the following theorem about designs held in the words of a fixed weight in the shadow.

Theorem 3.2. *Let C be a type I $[2n, n, d]$ code whose vectors are t-vectors. Then the vectors in the shadow are also t-vectors.*

Proof. As C_0 is the unique subcode of C consisting of vectors whose weights are $\equiv 0 \pmod 4$, the vectors in C_0 are t-vectors. Hence, by the Assmus-Mattson theorem the vectors in $C_0^\perp$ are also t-vectors. Recall that $C_0^\perp = C_0 \cup C_2 \cup S$. We consider separately the three cases; n is odd, $n \equiv 0 \pmod 4$ and $n \equiv 2 \pmod 4$. If n is odd, then the vectors in S are t-vectors as these are the only odd weight vectors in $C_0^\perp$. If $n \equiv 0 \pmod 4$, then both vectors in C_0 and vectors in S have weights $\equiv 0 \pmod 4$. Since vectors in C_0 hold t-designs and vectors in $C_0^\perp$ hold t-designs, the vectors in S hold t-designs. If $n \equiv 2 \pmod 4$, a similar argument holds as we can show that the vectors in C_2 must hold t-designs. □

Often when vectors in the shadow hold a t-design, then so do the vectors in C_1 and C_3 separately.

Theorem 3.3. *Let C be a type I $[2n, n, d]$ code and let t be a positive integer with $t < d$. Assume that the vectors in C_0 are t-vectors. Let d_i be the minimum weight of C_i, $i = 1, 3$ and let s_i be the number of weights $\leqslant 2n - t$ in C_i. If n is odd and either $s_1 \leqslant d_3 - t$ or $s_3 \leqslant d_1 - t$ then the vectors in either C_1 or C_3 are t-vectors. If n is even and $s_i \leqslant d_i - t$, $i = 1, 3$, then the vectors in either C_1 or C_3 are t-vectors.*

Proof. As the vectors in C_0 are t-vectors, so are the vectors in $C_0^\perp = C_0 \cup C_2 \cup C_1 \cup C_3$. If n is odd, we note that $C_0 \cup C_1$ and $C_0 \cup C_3$ are dual codes. Suppose $s_3 \leqslant d_1 - t$. Let D be the $[2n - t, n, d_1 - t]$ code obtained by puncturing $C_0 \cup C_1$ on a fixed set T of t coordinate positions. Then $D^\perp$ is the $[2n - t, n - t, d_3]$ code gotten by cutting t coordinates off the subcode of $C_0 \cup C_3$ which is zero on these t coordinates. Since vectors in C_0 contain t-designs and the only weights $\equiv 0 \pmod 4$ in $D^\perp$ arise from vectors in C_0 with zeros on these t positions, all weights in $D^\perp \equiv 0 \pmod 4$ are uniquely determined. As $s_3 \leqslant d_1 - t$, using the power moment identities [13, Section 8.3] we can determine all the remaining weights in $D^\perp$, hence in D. Thus the vectors of a fixed weight in C_1 hold a t-design. If $s_1 \leqslant d_3 - t$, then we interchange the roles of $C_0 \cup C_1$ and $C_0 \cup C_3$.

If n is even, then $C_0 \cup C_1$ and $C_0 \cup C_3$ are each self-dual and the conditions that $s_i \leqslant d_i - t$ tells us that vectors in either C_1 or C_3 are t-vectors as this known for vectors in C_0. □

Theorem 3.4. *Let $n \equiv 0 \pmod 4$ and suppose that C satisfies all the other assumptions of Theorem 3.1. If one of the type II codes $C_0 \cup C_1$ or $C_0 \cup C_3$ hold t-designs, then the vectors in C_1 and in C_3 hold t-designs.*

Proof. By assumption the vectors in C_0 are t-vectors. By Theorem 3.2 we know that the shadow also holds t-designs. If either $C_0 \cup C_1$ or $C_0 \cup C_3$ hold a t-design, then so must C_1 and C_3 separately. □

Consider the "odd Golay code" which is a $[24, 12, 6]$ self-dual code [9]. Its weight distribution $W(x) = 1 + 64x^6 + 375x^8 + 960x^{10} + \cdots$ and the weight distribution of the shadow $S(x) = 6x^4 + 744x^8 + 2596x^{12} + \cdots$. Since $s = 3$ and $d' = 4$, Theorem 3.1 shows that C holds a 1-design. Theorem 3.2 says that the vectors in the shadow also are 1-vectors. As $C_0 \cup C_3$ is the Golay code, by Theorem 3.4, C_1 and C_3 hold 1-designs.

Table 2
The distribution of weights in the cosets of the [22, 11, 6] bary Golay code

Weight	0	1	2	3	4	5	6	7	8	9	10	11	Number
0	1						77		330		616		1
1		1				21		176		490		672	22
2			1		5		72		320		626		231
3				2		24		168		488		684	770
4					8		72		312		632		770
5						32		160		480		704	231
6							112		240		672		22
7								352				1344	1

We will apply these theorems to the baby Golay code which actually inspired them. By Theorem 3.1 the vectors in any [22, 11, 6] self-dual code are 3-vectors. This was known previously since the baby Golay code G_{22} is the unique [22, 11, 6] self-dual code and the three design property follows from its triply transitive automorphism group. However, our proof is independent of any group action. By Theorem 3.2, we get the new result that the vectors in the shadow are 3-vectors. We show that these pieces of information about the baby Golay code determine its complete coset weight distribution, and these cosets exhibit a remarkable structure [4] (see Table 2).

We note first that there is a unique coset weight distribution for each coset of a given weight. The fact that the vectors of each weight in G_{22} hold 3-designs determines the weight distributions of any coset of G_{22} of weight one, two or three. The number of cosets of these weights is also determined. The argument for the other cosets is more subtle. Consider the coset of weight 7. By Corollary 1 of [3], the weight distribution of a coset of weight 7 is uniquely determined. Any coset with these gaps in its weight distribution must be orthogonal to the unique codimension one subcode whose weights are all divisible by four. Hence, the only weight 7 coset is the shadow. As the vectors of weight 7 in the shadow hold 3-designs, there are 112 vectors of weight 6 in a coset of weight 6 with a zero in a fixed position. The vectors of weight 8 in such a coset arise from vectors of weight 7 in the shadow with a zero in that position, of which there are 240. This determines the weight distribution of the 22 cosets of weight 6. The 2-designs held by the vectors of weight 7 in the shadow determine $\binom{22}{2} = 231$ cosets of weight 5 with 32 vectors of weight 5. These vectors all have zeros in two fixed positions. As all other vectors in a weight 5 coset have odd weight and all other odd weight vectors in the space have been determined, the entire weight distribution of a weight 5 coset can be calculated. There are 8 weight 7 vectors in the shadow which have zeros in 3 fixed positions. This gives 8 for the number of vectors of weight 4 in a coset of weight 4. The number of such cosets is 770 as there is no room for anymore. In this case weight 4 vectors (which are covered by a weight 7 vector in the shadow) with 0's on 3 different positions can be in one coset if the two sets of 3 positions constitute a weight 6 vector in G_{22}. The rest of the weight distribution of a weight 4 coset is now completely determined as all other even weight vectors have

Table 3
The distribution of weights in the cosets of the [6, 3, 2] self-dual code

Weight	0	1	2	3	4	5	6	Number
0	1		3		3		1	1
1		2		4		2		3
2			4		4			3
3				6				1

been accounted for. We note the pairing between weight i and weight $7 - i$ cosets for $i = 1, 2, 3$. The leaders in paired cosets have ones (weight i) where the leaders in the corresponding pair have zeros (weight $7 - i$).

As in [5], we define a partial ordering on the cosets of a binary code. If C_1 and C_2 are cosets of C, we say that $C_1 \prec C_2$ if there exists a coset leader of C_1 which is covered by a coset leader of C_2. In other words the support of one is contained in the support of the other. The set of all cosets of a code form a partially ordered set under this order. An *orphan* is a maximal element in a chain. We note that the shadow is the unique orphan of G_{22}. Thus, every coset of G_{22} is $\prec$ the shadow. We can define the *rank of a coset* to be its weight. One can show that the cosets of a binary code form a ranked coset under this rank function [2].

Let N_i denote the number of cosets of a given rank, i.e. weight. Then for G_{22} the N_i are unimodal [2] and symmetric, i.e. $N_i = N_{7-i}$. We know of only one other example of this phenomenon, namely a [6, 3, 2] self-dual code, which is a child of the [8, 4, 4] Hamming code. Both of these codes are children of distinguished type II self-dual codes, the Golay code G_{24} and the Hamming code E_8. We give the complete coset weight distribution of this child of E_8 (see Table 3).

Once again there is a unique orphan, the maximal weight coset. By Theorem 3.2, vectors in this coset hold a 1-design.

Proposition 3.1. *Let C be a binary code with minimum distance at least 3. If the minimum weight vectors in any coset of C hold a 1-design, then that coset is an orphan.*

Proof. By [5] a coset of a code as described above is an orphan if and only if the coset leaders of weight w or vectors of weight $w + 1$ cover all the coordinate positions. □

Under the conditions of the next proposition if a coset of weight s exists, the code has covering radius s and so this coset must be an orphan.

Proposition 3.2. *Suppose C is an even weight binary code and $C^{\perp}$ has s distinct non-zero weights, then the vectors in a coset of weights s (if it exists) are 1-vectors.*

Proof. We suppose that a coset of weight s exists. Then, the theorem follows as a corollary of Delsarte's theorem which says that the weight distribution of a coset of weight $s-1$ is unique. See [3]. □

If C is a *type II* code of length $2n \equiv 0 \pmod 8$, then a *type I child* [8] C' of C has length $2n-2 \equiv 6 \pmod 8$. The vectors in C' are those vectors in C with 11 or 00 in two fixed positions with those positions removed. Every *type I* code of length $\equiv 6 \pmod 8$ is a child of a *type II* code.

Let C be a type II code of length $2n \equiv 0 \pmod 8$ and let C' be its type I child of length $2n-2 \equiv 6 \pmod 8$. If $C' = C_0 \cup C_2$ has shadow $S = C_1 \cup C_3$, then its parent can be constructed by adjoining 00 to vectors in C_0, 11 to vectors in C_2, 01 to vectors in C_1 and 10 to vectors in C_3. If the parent, C is an extremal type II code, then the vectors of any fixed weight in C hold a 1-design. Hence, the weight distribution W_1 of C_1 and W_3 of C_3 must be the same. We have demonstrated the following theorem.

Theorem 3.5. *If C is a type I child of an extremal type II parent with shadow $S = C_1 \cup C_3$, then $W_1 = W_3$.*

By Theorem 3.4, we get that $W_1 = W_3$ for the type I [54, 27, 10] code with shadow of weight 11, and for the type I [38, 19, 8] code with shadow of weight 7. These decompositions are not listed in [9].

One of our interests in finding designs in cosets is in order to extend designs. We show first how to construct the 5-designs associated to the [24, 12, 8] Golay code, G_{24}, from the 3-designs in G_{22} and its cosets without relying on its automorphism group. One reason for doing this is that a similar situation occurs for $[24k, 12k, 4k+4]$ extremal type II codes where such highly transitive automorphism groups do not exist.

We construct a [23, 12, 7] self-orthogonal code D from G_{22} as follows: the vectors in D are of the following type:

1. $(c, 0)$ where c is in $C_0 \cup C_1$,
2. $(c, 1)$ where c is in $C_2 \cup C_3$.

Here C_i refers to the decomposition of G_{22} and its shadow. By Theorem 3.5 we know that $W_1 = W_3$. By the Assmus-Mattson theorem the vectors in D or $D^\perp$ are 4-vectors. Hence D is the well-known [23, 12, 7] Golay code G_{23}.

By Theorem 2.3 we know that the 672 vectors of weight 11 in a coset of weight 1 of G_{22} hold a 3-design. Call this set E. By Theorem 3.3 we know that the 672 vectors of weight 11 in either C_1 or C_3 hold 3-designs. We note that even though these 3-designs have the same parameters they cannot be equivalent as they have different automorphism groups. The group of E is the stabilizer of a point in M_{22} while the group of C_1 or C_3 is all of M_{22}. We cannot use E to extend the 3-designs held by vectors of weight 10 in G_{22} to 4-designs held by vectors of length 23 and weight 11.

Extending G_{23} to a [24, 12, 8] code C by adding an overall parity check gives the same code as adjoining the all-one vector to $G_{23}^\perp$ extended. This amounts to adjoining complements to the vectors of weight 11 in $G_{23}^{\perp}$ extended, which is the content of Theorem 2.5. By the Assmus-Mattson theorem, the vectors of any weight in C hold 5-designs. Hence C is the well-known extended Golay G_{24}. We can treat the general case in an analogous way.

Theorem 3.6. *Let C be a $[24k-2, 12k-1, 2+4k]$ self-dual binary code whose shadow has minimum weight $3+4k$. Then C is a child of an extremal type II $[24k, 12k, 4+4k]$ code. Furthermore, the vectors of any weight in C and its shadow hold 3-designs and these 3-designs can be extended to 5-designs in its extremal parent as described above.*

Proof. The general proof is similar to the proof for the last example. □

There exists a [46, 23, 10] self-dual type I code C whose shadow has weight 11 [9]. By the last theorem the vectors of any weight in C and its shadow hold 3-designs and these designs can be extended to the 5-designs in an extremal [48, 24, 12] type II code. A long-standing open problem is the existence of an extremal [72, 36, 16] code. If it exists, so must its children, in particular a [70, 35, 14] self-dual code whose shadow has weight 15. We conclude by giving the weight enumerator of this code and its shadow since its existence is unknown. Unfortunately, the weight enumerator of the shadow has the necessary divisibility properties for holding 3-designs.

Recall that the weight enumerator of any self-dual binary code can be represented as an integral combination of Gleason polynomials and that the coefficients in this combination also determine the weight distribution of the shadow. See [9] for a proof of the following.

Theorem 3.7. *Let C be a binary self-dual code such that*

$$W(y) = \sum_{j=0}^{[n/8]} a_j \cdot (1+y^2)^{n/2-4j}\{y^2(1-y^2)^2\}^{2j}$$

then

$$S(y) = \sum_{j=0}^{[n/8]} (-1)^j a_j \cdot 2^{n/2-6j} y^{n/2-4j}(1-y^4)^{2j}$$

If C is a [70, 35, 14] self-dual code, then to compute its weight distribution we need to determine $a_1, a_2, \ldots, a_8$, as $a_0 = 1$. Since $d = 14$, $a_1, \ldots, a_6$ are determined and a_7 and a_8 are arbitrary. Since C is a child of an extremal [72, 36, 16] type II code, then its shadow must have minimum weight 15, which implies that $a_6 = a_7 = a_8 = 0$. Thus a child of a [72, 36, 16] code has a unique weight distribution. We give the weight distribution of this [70, 35, 14] code along with the weight distribution of its shadow see Tables 4 and 5.

Table 4
Weight distribution of a [70, 35, 14] child of an extremal [72, 36, 16] code

Weight	Number	Factorization														
		2	3	5	7	11	13	17	19	23	31	43	47	281	863	7853
14	11 730	1	1	1				1		1						
16	15035			1	1	1		1		1						
18	1 345 960	3		1	1	1			1	1						
20	9 393 384	3	1		1	1	1	1		1						
22	49 991 305			1	1		1	1		1				1		
24	204 312 290	1		1	1		1	1					1	1		
26	650 311 200	5	3	2	1	1		1		1						
28	1 627 498 400	5		2		2		1		1		1				
30	3 221 810 284	2			1	1		1		1	1				1	
32	5 066 556 495		1	1	1	1	1	1		1					1	
34	6 348 487 600	4		2	1		1			1						1

Table 5
Weight distribution of its shadow

Weight	Number	Factorization											
		2	3	5	7	11	13	17	23	281	503	863	1201
15	87 584	5			1			1	1				
19	2 524 480	6		1	3				1				
23	208 659 360	5	1	1	1		1	1		1			
27	1 762 190 080	8		1	1			1	1		1		
31	8 314 349 400	6		1	1	1		1	1			1	
35	12 728 678 400	11	2	2					1				1

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