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Densing Sets

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Let $\mathscr{H}$ be a family of "large" (in various senses, e.g., of positive Hausdorff dimension or Lebesgue measure) subsets of $\mathbf{R}$. We study sets D of real numbers which are $\mathscr{H}$-densing, namely have the property that, given any set $H \in \mathscr{H}$ and $\varepsilon > 0$, there exist an $a \in \mathscr{D}$ for which the set aH is ε-dense modulo 1. In the special case, where $\mathscr{H}$ consists of all subsets of $\mathbf{R}$ having a finite accumulations point, $\mathscr{H}$-densing sets are simply Glasner sets, studied earlier.

1. INTRODUCTION

Denote by $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ the circle group. A subset of $\mathbf{T}$ is *ε-dense* in $\mathbf{T}$ if it meets every interval of length ε. A subset of $\mathbf{R}$ is *ε-dense modulo* 1 if its projection in $\mathbf{T}$ is ε-dense in $\mathbf{T}$. Glasner, dealing with some generalizations of Kronecker's theorem, proved the following.

PROPOSITION A [G]. *Let A be an infinite subset of $\mathbf{T}$. Then, for any $\varepsilon > 0$ there exists an integer n such that the set $nA = \{na : a \in A\}$ is ε-dense in $\mathbf{T}$.*

In [BP] the notion of a Glasner set of integers was introduced. A set $S \subseteq \mathbf{Z}$ is a *Glasner set* if, for A and ε as in the proposition, an appropriate n can be chosen from S. With this terminilogy, Proposition A amounts to the assertion that $\mathbf{Z}$ is a Glasner set. It turns out [BP] that there are many quite small sets of integers which are Glasner. For example, any set with positive (Banach) density is such, and so is the set of values assumed by any non-constant integer polynomial. On the other hand, lacunary

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sequences (and finite unions thereof), as well as some other families of sets, fail to have the property. It will follow from Proposition 3.1 that "most" sublacunary sequences, however, are Glasner sequences (see the discussion following Proposition 3.7).

We mention in passing that Proposition A also admits a quantitative version, which provides ε-dense dilations for sufficiently large (as a function of ε) finite sets A. This has been the starting point of [AP], where analogous results were obtained with low discrepancy replacing small gaps. This quantitative result was also imperative in proving that a sublacunary sequence has only a Hausdorff dimension 0 set of dilations which are not dense modulo 1 [Bo2, Theorem 1.3] (as opposed to the situation for lacunary sequences [M, P1]).

Our goal here is to study the analogue of a Glasner set if we only require the property of having arbitrarily dense dilations to hold for sets satisfying some "largeness" property. As dilations of sets and of their closures are "equally dense", we shall consider dilations of closed sets only. Denoting by $\mathscr{F}$ the family of all closed subsets of **R**, we start with the following.

DEFINITION 1.1. A family $\mathscr{S}$ of closed subsets of **R** is a *family of slim sets*, and its complement $\mathscr{H}$ in $\mathscr{F}$ is a *family of hefty sets*, if it satisfies the following properties:

(1) If $A \in \mathscr{S}$ and $B \subseteq A$ then $B \in \mathscr{S}$.

(2) If $A \in \mathscr{S}$ then $rA + s \in \mathscr{S}$ for every $r, s \in \mathbf{R}$.

(4) If $A \cap J \in \mathscr{S}$ for every closed finite interval $J \subset \mathbf{R}$, then $A \in \mathscr{S}$.

(4) $\mathscr{S}$ is closed under finite unions.

(5) $\{0\} \in \mathscr{S}$, $\mathbf{R} \notin \mathscr{S}$.

The complement in $\mathscr{F}$ of a family of slim sets is a *family of hefty sets*. It will be actually more convenient to state all our results in terms of families of hefty sets. Throughout the paper, $\mathscr{H}$ will denote an arbitrary family of hefty sets and $\mathscr{S}$ the corresponding family of slim sets. A closed set A is $\mathscr{H}$-*hefty* (or simply *hefty*) if $E \in \mathscr{H}$ and $\mathscr{H}$-*slim* if $E \in \mathscr{S}$.

We shall now present a few "natural" examples of families of hefty sets; these will be treated more carefully later.

EXAMPLE 1.1. The following are families of hefty sets (in each case, only closed sets are considered):

(1) $\mathscr{N}\mathscr{D}$—the family of all non-discrete sets. This is the largest family of hefty sets. For, by properties (2) and (5) in Definition 1.1, $\mathscr{S}$ must contain every finite set and therefore, by property (3), every discrete set as well.

(2) $\mathscr{U}$—the family of all uncountable sets.

(3) $\mathscr{HD}_{>d}$—the family of all sets whose Hausdorff dimension exceeds d (for any fixed $0 \leqslant d < 1$).

(4) $\mathscr{HD}_1$—the family of all sets intersecting some finite interval on a set of Hausdorff dimension 1. (Note that the family of all sets of Hausdorff dimension 1 is not a family of hefty sets according to our definition.)

(5) $\mathscr{LM}_{>0}$—the family of all set of positive Lebesgue measure.

(6) $\mathscr{CI}$—the family of all sets containing an interval (of positive length). This is the smallest family of hefty sets. In fact, if there exists an $S \in \mathscr{S}$ containing an interval, then, by properties (1) and (2) in Definition 1.1, every closed finite interval belongs to $\mathscr{S}$. But then (3) implies that $\mathbf{R} \in \mathscr{S}$, which is impossible.

DEFINITION 1.2. A set $D \subseteq \mathbf{R}$ is *$\mathscr{H}$-densing* if for every $H \in \mathscr{H}$ and $\varepsilon > 0$ there exists an $a \in D$ such that the set aH is ε-dense modulo 1.

With this terminology, Glasner sets are $\mathscr{IS}$-densing sets (of integers).

In Section 2 we formulate a few equivalent conditions for a set to be $\mathscr{H}$-densing and some results for general families of hefty sets. It turns out, for example, that a bounded perturbation of an $\mathscr{H}$-densing set is such as well (Theorem 2.4), and, in particular, the property of being a Glasner set is invariant under such perturbations. Section 3 deals with the special families defined in Example 1.1. Clearly, as these families become smaller and smaller, a densing set for one of them is necessarily densing for its successor. We show that for each such pair of families (with the exception of the last two), there exist sets which are non-densing with respect to the first family, yet they are densing with respect to the second. Section 4 is devoted to proving the theorems presented in Section 2 and Section 3. In Section 5 we deal with sets satisfying a condition even stronger than being $\mathscr{IS}$-densing—analogous to the quantitative version of Proposition A mentioned earlier. While this condition is not equivalent to being $\mathscr{H}$-densing for any family of hefty sets, it is nevertheless very related and turns out to enjoy similar properties.

2. THE MAIN RESULTS

The following two lemmas are routine. The proofs of the other results will be deferred to Section 4.

LEMMA 2.1. *Every $\mathscr{H}$-densing set is unbounded.*

LEMMA 2.2. *If the set* $D \subset \mathbf{R}$ *is* $\mathscr{H}$*-densing then so is* $rD+s$ *for any* r, $s \in \mathbf{R}$, $r \neq 0$.

Denote by $|J|$ the length of a finite interval J.

THEOREM 2.1. *The following conditions on a set* $D \subseteq \mathbf{R}$ *are equivalent*:

(1) D *is* $\mathscr{H}$*-densing*.

(2) *For every mapping* $\psi: D \to \mathbf{R}$ *and every interval* J *with* $|J|>0$ *the set*

$$V = V(\psi, J) = \{x \in \mathbf{R} : ax + \psi(a) \notin J \ (\text{mod } 1),\ a \in D\} \tag{2.1}$$

is slim.

(3) *For every* $\varepsilon > 0$ *and every mapping* $\psi: D \to \mathbf{R}$ *the set of all* $x \in \mathbf{R}$, *for which the set* $\{ax + \psi(a) : a \in D\}$ *is not* ε*-dense modulo* 1, *is slim*.

(4) *There exists an interval* I *with* $|I| > 0$ *such that, for every mapping* $\psi: D \to \mathbf{R}$ *and every interval* J *with* $|J| > 0$, *the set*

$$W(\psi, J, I) = \{x \in I : ax + \psi(a) \notin J \ (\text{mod } 1),\ a \in D\} \tag{2.2}$$

is slim.

(5) *There exists an interval* I *with* $|I| > 0$ *such that, for every* $\varepsilon > 0$ *and every mapping* $\psi: D \to \mathbf{R}$, *the set of all* $x \in I$, *for which the set* $\{ax + \psi(a) : a \in D\}$ *is not* ε*-dense modulo* 1, *is slim*.

EXAMPLE 2.1. In Theorem 2.1 it is not enough to test conditions (2)–(5) for $\psi \equiv 0$. In fact, the set $\{2^m 6^n : m, n \geqslant 0\}$ is not $\mathscr{NG}$-densing [BP, Theorem 1.5], yet it satisfies the conditions in the theorem for $\psi \equiv 0$ (as follows from [F1, Chap. IV]).

A set $E \subseteq \mathbf{R}$ is *σ-slim* if it is contained in a countable union of slim sets. The implication (1) ⇒ (3) in Theorem 2.1, applied to $\psi \equiv 0$, yields the following.

THEOREM 2.2. *If set* $D \subseteq \mathbf{R}$ *is* $\mathscr{H}$*-densing, then the set of all* $\alpha \in \mathbf{R}$ *for which the set* $D\alpha$ *is not dense modulo* 1 *is* σ*-slim*.

As there are many results pertaining to the number of dilations of various sets which are not dense modulo 1, Theorem 2.2 is particularly useful for showing that certain sets fail to be $\mathscr{H}$-densing.

COROLLARY 2.1. *If* (a_n) *is a Glasner sequence, then for every* $\varepsilon > 0$ *and every sequence* (β_n) *the set of all* $\alpha \in \mathbf{R}$, *for which the sequence* $a_n\alpha - \beta_n$ *is not* ε*-dense modulo* 1, *is finite. Moreover, under the same conditions (or,* *more generally, if (a_n) is only assumed to be a $\mathscr{H}$-density sequence), for every sequence (β_n) the set of all $\alpha \in \mathbf{R}$, for which the sequence $a_n\alpha - \beta_n$ is not dense modulo 1, is denumerable.*

THEOREM 2.3. *Let $D \subseteq \mathbf{R}$ an $\mathscr{H}$-densing set, $K \subseteq \mathbf{R}$ any set, and $F = \{f_a : a \in D\}$ a family of functions $f_a : K \to \mathbf{R}$. Suppose F is equicontinuous on every bounded subset of K. Then:*

(1) *For any interval J with $|J| > 0$, the set*

$$U = \{x \in K : ax + f_a(x) \notin J \pmod 1, \ a \in D\} \tag{2.3}$$

is slim.

(2) *The set E of all $x \in \mathbf{R}$ for which the set $\{ax + f_a(x) : a \in D\}$ is not dense modulo 1 is σ-slim.*

A set $A \subseteq \mathbf{R}$ is a *bounded perturbation* of a set $B \subseteq \mathbf{R}$ if there exists a constant M such that every interval of length M centered at a point of $A \cup B$ contains both a point of A and a point of B.

THEOREM 2.4. *The property of being $\mathscr{H}$-densing is invariant under bounded perturbations.*

The theorem follows as special case from Theorem 2.3. It implies in particular that, in studying $\mathscr{H}$-densing sets, one may restrict his attention to sets of integers.

The next theorem provides a sufficient condition for a set to be $\mathscr{H}$-densing. It utilizes the notion of a uniformly distributed modulo 1 (henceforward u.d. mod 1) sequence (for the definition and basic properties of uniform distribution modulo 1, in $\mathbf{R}$ as well as in $\mathbf{R}^l$; see [KN]).

THEOREM 2.5. *Let $D = \{r_n : n \in \mathbf{N}\}$. Denote by E the set of all $\alpha \in \mathbf{R}$ for which the sequence $(r_n\alpha)$ is not u.d.* mod 1. *Suppose no countable union of translates of E contains an $\mathscr{H}$-hefty set. Then D is $\mathscr{H}$-densing.*

3. DENSING SETS FOR SPECIFIC FAMILIES

We shall investigate here densing sets for the families of hefty sets presented in Example 1.1. In particular, it will follow that, for each of the first four families in that example, there exists a set which is not densing for that family but is densing for the next family. For the last two families, being densing is equivalent.

A sequence (r_n) in $\mathbf{R}$ is *homogeneously distributed* if the sequence $(r_n\alpha)$ is u.d. mod 1 for every $\alpha \neq 0$. A sequence (r_n) in $\mathbf{Z}$ is *homogeneously* *distributed* if the sequence $(r_n \alpha)$ is u.d. mod 1 for every irrational α and, moreover, (r_n) is u.d. modulo every integer $m \neq 0$. (See [Bo1], where different, but equivalent, definitions were given. In general, a sequence in a locally compact abelian group G is *homogeneously distributed* if it is uniformly distributed in the Bohr compactification of the group relative to the Haar measure.)

PROPOSITION 3.1. *A homogeneously distributed sequence of reals or of integers is* $\mathcal{NS}$*-densing.*

Given an increasing sequence $(n_k)_{k=1}^{\infty}$ of non-negative integers, consider the set:

$$D = \left\{ \sum_{k=1}^{L} \xi_k n_k : L \in \mathbf{N},\ \xi_k = 0, 1 \right\}. \tag{3.1}$$

(This set is the so-called IP-set generated by the sequence (n_k) [F2, Definition 2.3].)

PROPOSITION 3.2. *Suppose n_k is a proper divisor of n_{k+1} for each k. Then D is* $\mathcal{U}$*-densing if and only if the sequence (n_{k+1}/n_k) is bounded.*

Now if, say, $n_k = 2^{m_k}$, where (m_k) is an increasing sequence of integers, then, according to the proof of Proposition 4.1 in [BP], the set D defined in (3.1) is not $\mathcal{NS}$-densing unless (m_k) contains all integers from some place on. Since, by Proposition 3.2, D is $\mathcal{U}$-densing in this case if and only if the sequence $(m_{k+1} - m_k)$ is bounded, this proves the following.

COROLLARY 3.1. *There exists a* $\mathcal{U}$*-densing set which is not* $\mathcal{NS}$*-densing.*

The following strengthened version of [BP, Proposition 4.1] is another consequence of the "only if" part of Proposition 3.2.

PROPOSITION 3.3. *For any sequence (δ_n) satisfying $\delta_n \leqslant 1$ for each n and $\delta_n \to 0$, there exists a non-*$\mathcal{U}$*-densing set $P \subseteq \mathbf{N}$ such that*

$$\#(P \cap [1, n]) \geqslant \delta_n \cdot n, \qquad n \geqslant 1.$$

In fact, let D be again as in (3.1), with $n_k = 2^{m_k}$, and observe that $(m_{k+1} - m_k)$ can be unbounded while $m_k - k$ diverges to infinity arbitrarily slowly.

PROPOSITION 3.4. *A finitely generated multiplicative semigroup of integers, whose generators have a non-trivial common divisor, is not* $\mathcal{U}$*-densing.*

A sequence (r_n) of positive numbers is *lacunary* if $r_{n+1} > \lambda r_n$ for some $\lambda > 1$. The sequence is *sub-lacunary* if $r_n \to \infty$ and $r_{n+1}/r_n \to 1$. It is *super-lacunary* if $r_{n+1}/r_n \to \infty$.

PROPOSITION 3.5. *A sub-lacunary sequence is $\mathcal{HS}_{>0}$-densing.*

In view of [F1, Lemma IV.1], a multiplicative semigroup of positive integers, not contained in the set of powers of a single integer, is sub-lacunary. It follows that the semigroup $\{2^m 6^n : m, n \geqslant 0\}$, for example, satisfies the conditions of both Propositions 3.4 and 3.5, which implies the following.

COROLLARY 3.2. *There exists a $\mathcal{HS}_{>0}$-densing set which is not $\mathcal{U}$-densing.*

PROPOSITION 3.6. *Let (r_n) be an increasing sequence of integers satisfying $r_n = O(n^t)$ for a certain real number $t \geqslant 1$. Then (r_n) is $\mathcal{HS}_{>1-1/t}$-densing.*

PROPOSITION 3.7. *Let (r_n) be a finite union of lacunary sequences of positive numbers. Then (r_n) is not $\mathcal{HS}_{>d}$-densing for any $d < 1$.*

Note that the proposition is best possible in the following sense. Given any sequence growing slower than exponentially, a random "small" perturbation of the sequenced almost surely forms a homogeneously distributed sequence [AHK; Bo1] (see also [Bou] for a different random construction) and is, therefore, $\mathcal{NS}$-densing.

PROPOSITION 3.8. *A sequence (r_n) of positive real numbers, satisfying $r_n \to \infty$ and $r_{n+1} < Cr_n$ for some constant C is $\mathcal{HS}_1$-densing.*

The proposition follows direct from [Bo2, Propositions 3.4 and 5.4]. A straightforward consequence of Propositions 3.7 and 3.8 is the following.

COROLLARY 3.3. *There exists a $\mathcal{HS}_1$-densing set which is not $\mathcal{HS}_{>d}$-densing for any $d < 1$.*

PROPOSITION 3.9. *Super-lacunary sequences are not $\mathcal{HS}_1$-densing.*

The proposition follows immediately from the following result, which we prefer to state separately.

PROPOSITION 3.10. *If (r_n) is super-lacunary, then for every $\varepsilon > 0$ the set $\{\alpha \in \mathbf{R} : \|r_n \alpha\| \leqslant \varepsilon, n \in \mathbf{N}\}$ is of Hausdorff dimension 1.*

The proof is a simple application of [ET, Theorem A] (see also [ET, Theorem 8A] for a related result).

PROPOSITION 3.11. *Any unbounded set is* $\mathscr{L}\mathscr{M}_{>0}$*-densing.*

The last two propositions yield the following.

COROLLARY 3.4. *There exists an* $\mathscr{L}\mathscr{M}_{>0}$*-densing set which is not* $\mathscr{H}\mathscr{S}_1$*-densing.*

Also, by Lemma 2.1 and Proposition 3.11 we obtain the following.

COROLLARY 3.5. *Let* $D \subseteq \mathbf{R}$. *The following conditions are equivalent:*

(1) *D is* $\mathscr{L}\mathscr{M}_{>0}$*-densing.*

(2) *D is* $\mathscr{S}\mathscr{S}$*-densing.*

(3) *D is unbounded.*

4. PROOFS

In this section we prove all the results stated, but not proven, in the preceding two sections.

Proof of Theorem 2.1. We establish the theorem by proving the chain of implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (5) \Rightarrow (4) \Rightarrow (2) \Rightarrow (1)$.

$(1) \Rightarrow (2)$ Put $\varepsilon = |J|$. By the definition of V, for any $a \in D$ we have $ax \notin J - \psi(a) \pmod 1$, $x \in V$. In particular, aV is not ε-dense modulo 1, which means that V is slim.

$(2) \Rightarrow (3)$ Follows straightforwardly from the fact that for every $\varepsilon > 0$, $\mathbf{T}$ is a finite union of intervals of length $\varepsilon/2$.

$(3) \Rightarrow (5)$ Obvious.

$(5) \Rightarrow (4)$ Obvious.

$(4) \Rightarrow (2)$ Let G denote the family of all sets $K \subseteq \mathbf{R}$ having the property that, for every mapping $\psi: D \to \mathbf{R}$ and interval J with $|J| > 0$, the set $W(\psi, J, K)$ (defined as in (2.2), except that K is not necessarily an interval), is slim. We first claim that G is closed under translations. In fact, let $K \in G$ and $c \in \mathbf{R}$. Given $\psi: D \to \mathbf{R}$ and interval J, we readily observe that $W(\psi, J, K + c) = W(\psi_1, J, K)$, where $\psi_1: D \to \mathbf{R}$ is defined by $\psi_1(a) = \psi(a) + ca$. Consequently, $W(\psi, J, K + c)$ is slim, so that $K + c \in G$. Since a finite union of slim sets is itself slim, G is closed under finite unions, and it is certainly closed under passage to subsets. Thus every finite interval belongs to G, and (by the definition of a family of slim sets) $\mathbf{R} \in G$, which means that $V(\psi, J) = W(\psi, J, \mathbf{R})$ is slim.

(2) $\Rightarrow$ (1) Suppose we have a set E such that aE is not ε-dense modulo 1 for any $a \in D$. Let J be an arbitrary interval of length ε. Then for every $a \in D$ there exists a $\psi(a) \in \mathbf{R}$ for which the sets $aE + \psi(a)$ and $J + \mathbf{Z}$ do not intersect. Let V be as in (2.1). Then V is slim and, since $E \subseteq V$, so is E. This proves the theorem.

Proof of Theorem 2.3. (1) We may assume K to be bounded. Write $J = (a, b)$, and choose ε with $0 < \varepsilon < (b-a)/2$. Putting $L = (a+\varepsilon, b-\varepsilon)$ we have $|L| > 0$. Since F is equi-continuous on K, there exists a $\delta > 0$ such that $|f_a(y) - f_a(x)| < \varepsilon$ for every $a \in D$ and $x, y \in K$ with $|y - x| < \delta$. Take a finite set B which is δ-dense in K. Denote

$$V(b) = \{x \in K : ax + f_a(b) \notin L \ (\mathrm{mod}\ 1), a \in L\}, \qquad b \in B.$$

In view of Theorem 2.1, each $V(b)$ is slim, whence so is the set $V = \bigcup_{b \in B} V(b)$. It remains to show that $U \subseteq V$; in other words, that $x \in K$, $x \notin V \Rightarrow x \notin U$.

In fact, choose $b_0 \in B$ with $|x - b_0| < \delta$. Since $x \notin V(b_0)$, there exists an $a_0 \in D$ such that $a_0 x + f_{a_0}(b_0) \in L$ (mod 1). Then $|f_{a_0}(x) - f_{a_0}(b_0)| < \varepsilon$ and, therefore, $a_0 x + f_{a_0}(x) \in J$ (mod 1). Thus $x \notin U$.

(2) The proof of this part is analogous to that of Theorem 2.2.

This completes the proof.

Proof of Theorem 2.4. We have to show that, if $D \subseteq \mathbf{R}$ is $\mathscr{H}$-densing and $D_1 + (-c, c) \supseteq D$ for some $c > 0$, then D_1 is $\mathscr{H}$-densing as well. In fact, take a function $g: D \to D_1$ such that $|g(a) - a| < c$, $a \in D$. As $g(D) \supset D_1$, it suffices to show that $g(D)$ is $\mathscr{H}$-densing, and we proceed to accomplish this by using the criterion for being $\mathscr{H}$-densing provided by Theorem 2.1. To this end, given an arbitrary mapping $\psi: D \to \mathbf{R}$, define a family of functions $F = \{f_a : a \in D\}$ by

$$f_a(x) = (g(a) - a)x + \psi(g(a)), \qquad a \in D, \quad x \in \mathbf{R}.$$

Obviously, F forms an equi-continuous family of functions from $\mathbf{R}$ into itself, and consequently, applying Theorem 2.2.(1) with $K = \mathbf{R}$, we see that for every interval J of positive length, the set U defined in (2.3) is slim. Now $ax + f_a(x) = g(a)x + \psi(g(a))$, whence by Theorem 2.1 the set $g(D)$ is $\mathscr{H}$-densing. This proves the theorem.

For the proof of Theorem 2.5 we need the following.

PROPOSITION 4.1. *Let* (r_n) *be a sequence of reals. Denote by* E *the set of all* $\alpha \in \mathbf{R}$ *for which the sequence* $(r_n \alpha)$ *is not u.d.* mod 1. *Let* $A \subseteq \mathbf{R}$ *be a set* *not contained in any countable union of translates of E. Then there exists a sequence* (α_k) *in A such that for every* $l \in \mathbf{N}$ *the sequence* $(r_n(\alpha_1, \alpha_2, ..., \alpha_l))$ *is u.d.* mod 1 *in* $\mathbf{R}^l$.

Proof. We construct the sequence (α_k) inductively. Take α_1 as any element of $A - E$. Suppose $\alpha_1, \alpha_2, ..., \alpha_{l-1}$ have been chosen in such a way that $(r_n(\alpha_1, \alpha_2, ..., \alpha_{l-1}))$ is u.d. mod 1 in $\mathbf{R}^l$. According to Weyl's equidistribution criterion, given any $\beta \in \mathbf{R}$, the sequence $(r_n(\alpha_1, \alpha_2, ..., \alpha_{l-1}, \beta))$ is u.d. mod 1 in $\mathbf{R}^l$ if and only if for any integers $h_1, h_2, ..., h_{l-1}, h$, not all 0, we have

$$\frac{1}{N} \sum_{n=1}^{N} e^{2\pi i r_n (h_1 \alpha_1 + h_2 \alpha_2 + \cdots + h_{l-1} \alpha_{l-1} + h\beta)} \xrightarrow[N \to \infty]{} 0. \tag{4.1}$$

By the induction hypothesis, we only need to find a $\beta \in A$ such that (4.1) is satisfied for l-tuples $(h_1, h_2, ..., h_{l-1}, h)$ with $h \neq 0$. For such a fixed l-tuple, the set of all $\beta \in A$ for which (4.1) does not hold is contained in the set

$$\{\beta \in \mathbf{R} : h\beta \notin E - h_1 \alpha_1 - h_2 \alpha_2 - \cdots - h_{l-1} \alpha_{l-1}\}. \tag{4.2}$$

Obviously, $(1/h) E \subseteq E$, whence the set in (4.2) is contained in some dilation of E. Thus, the set of all β which cannot be adjoined to our $(l-1)$-tuples to give an l-tuple as required is contained in a countable union of translates of E. Hence we can find a $\beta = \alpha_l \in A$ having the property sought for, which proves the proposition.

Proof of Theorem 2.5. Let H be a hefty set and $\varepsilon > 0$. Take an integer $l > 1/\varepsilon$. According to Proposition 4.1, we can find $\alpha_1, \alpha_2, ..., \alpha_l \in H$ such that, denoting $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_l)$, the sequence $(r_n \boldsymbol{\alpha})$ is u.d. mod 1 in $\mathbf{R}^l$. In particular, the vector $r_n \boldsymbol{\alpha}$ can be made arbitrarily close to $(0, 1/l, 2/l, ..., (l-1)/l)$ modulo 1 by an appropriate choice of n. Hence $r_n H$ can be made ε-dense modulo 1. This proves the theorem.

Proof of Proposition 3.1. By [N, Theorem 1], if (r_n) is a homogeneously distributed sequence of reals, then $([r_n])$ is a homogeneously distributed sequence of integers. In view of Theorem 2.4 it suffices therefore to deal with the latter case. Let (r_n) be such a sequence and $\alpha_1, \alpha_2, ..., \alpha_l$ any distinct elements of $\mathbf{T}$. Set $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_l) \in \mathbf{T}^l$. Consider the rotation $R: \mathbf{T} \to \mathbf{T}$ defined by $R(x) = x + \boldsymbol{\alpha}$, $x \in \mathbf{T}^l$. According to [Bo1, Theorem 1.9], for almost every $x \in \mathbf{T}^l$ the closures of the sets $\{R^{r_n}(x) : n \in \mathbf{N}\}$ and $\{R^n(x) : n \in \mathbf{N}\}$ coincide. Consequently, the closures of the sets $\{r_n \boldsymbol{\alpha} : n \in \mathbf{N}\}$ and $\{n \boldsymbol{\alpha} : n \in \mathbf{N}\}$ coincide as well. The quantitative version of Glasner's theorem [BP, Theorem 1.1] guarantees that, given $\varepsilon > 0$, we can find an n such that the set $\{n\alpha_1, n\alpha_2, ..., n\alpha_l\}$ is ε-dense in $\mathbf{T}$. Hence there exists an n such that the set $\{r_n\alpha_1, r_n\alpha_2, ..., r_n\alpha_l\}$ is ε-dense in $\mathbf{T}$. This completes the proof.

Proof of Proposition 3.2. Write $D = \{d_m : m \in \mathbf{N}\}$, where the sequence (d_m) is increasing. If (n_{k+1}/n_k) is bounded, then, in view of [Be, Corollary 4.1], for every irrational α the sequence $(d_m\alpha)$ is u.d. mod 1. Hence D is $\mathscr{U}$-densing by Theorem 2.5.

Now suppose (n_{k+1}/n_k) is unbounded. In view of the proof of [Be, Theorem 4.1], the set of all $\alpha \in \mathbf{R}$, for which the sequence $(d_m\alpha)$ is not dense modulo 1, is uncountable, or, in other words, is not σ-$\mathscr{U}$-slim. According to Theorem 2.2, this means that D is not $\mathscr{U}$-densing.

Proof of Proposition 3.4. Let S be a semigroup satisfying the conditions of the theorem. As in the proof of [BP, Theorem 1.5], we may assume S to be generated by the numbers $pa_1, pa_2, ..., pa_l$, where p is a prime and $1 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_l$ are relatively prime to it. Let (n_k) be an increasing sequence in $\mathbf{N}$. It is readily verified that, choosing (n_k) to grow sufficiently fast, we can ensure that, if $s \in S$ and $k = k(s)$ is the least integer for which sp^{-n_k} is not an integer, then $sp^{-n_l} < 1/p^{l-k+2}$, $l > k$. Put $H = \{\sum_{k=1}^{\infty} \xi_k p^{-n_k} : \xi_k = 0, 1\}$. Given $s \in S$, letting $k = k(s)$ be as before, we obtain

$$\sum_{l \neq k} \|sp^{-n_l}\| < \sum_{l=k+1}^{\infty} \frac{1}{p^{l-k+2}} \leqslant \frac{1}{p^2}.$$

Consequently, for every $s \in S$ the set sH is contained modulo 1 in the union of two intervals of length $1/p^2$ each, whence it is not 1/4-dense. H being uncountable, this proves the proposition.

Proof of Proposition 3.5. Let (r_n) be sub-lacunary. By [Bo, Theorem 7.7], for any sequence (b_n), the set of all $\alpha \in \mathbf{R}$ for which $\{r_n\alpha - b_n : n \in \mathbf{N}\}$ is not dense modulo 1 is of Hausdorff dimension 0 or, in other words, is $\mathscr{HS}_{>0}$-slim. The proposition then follows from Theorem 2.1.

Proof of Proposition 3.6. Denote by E the set of all $\alpha \in \mathbf{R}$ for which the sequence $(r_n\alpha)$ is not u.d. mod 1. According to [ET, Theorem 13], $\dim_H E \leqslant 1 - 1/t$, and therefore no countable union of translates of E contains an $\mathscr{HS}_{>1-1/t}$-hefty set. The proposition follows from Theorem 2.3.

Proof of Proposition 3.7. Denote by E the set of all points for which the sequence $(r_n\alpha)$ is not dense modulo 1. By [M; P1], if (r_n) is lacunary, then $\dim_H E = 1$. The same is true also if (r_n) is only assumed to be a finite union of lacunary sequences [P2]. Thus E is not σ-$\mathscr{HS}_{>d}$-slim for any $d < 1$. The proposition follows now from Theorem 2.2.

Proof of Proposition 3.11. Let D be an unbounded set and A a set with $\mu(A) > 0$, where μ denotes the Lebesgue measure. By the Lebesgue density theorem, given $\varepsilon > 0$ there exists a finite interval $J = [a, b) \subset \mathbf{R}$ such that $\mu(A \cap J) > (1-\varepsilon)\mu(J)$. For $t \in \mathbf{R}$, put $K_t = [\mu(tJ)] = [t(b-a)]$ and $J_t = [ta, ta + K_t)$. Clearly,

$$\mu(tA \cap J_t) \geqslant \mu(t(A \cap J)) - 1 > (1-\varepsilon)\, |t|\, (b-a) - 1.$$

Subdivide J_t into K_t disjoint intervals of length 1. For at least one of these intervals, say $J' \subset J_t$, we have

$$\mu(tA \cap J') > (1-\varepsilon)\, \frac{|t|\,(b-a)}{K_t} - \frac{1}{K_t}.$$

Denoting by π the natural projection of $\mathbf{R}$ onto $\mathbf{T}$ we obtain

$$\liminf_{|t| \to \infty} \mu(\pi(tA)) \geqslant \liminf_{|t| \to \infty} \mu(\pi(tA \cap J')) \geqslant 1 - \varepsilon,$$

since $\mu(J') = 1$ and $K_t \to \infty$ as $t \to \infty$. It follows that, if $t \in D$ is chosen with $|t|$ sufficiently large, then $\pi(tA)$ is 2ε-dense. This proves the proposition.

5. Concluding Remarks and Open Questions

Denote by $\#(F)$ the cardinality of a finite set F.

Definition 5.1. A set $D \subseteq \mathbf{R}$ is *effectively densing* (or simply an *ED-set*) if for every $\varepsilon > 0$ there exists an $n = n(\varepsilon)$ such that, if $H \subseteq [0, 1]$ with $\#(H) \geqslant n$, then there exists an $a \in D$ such that the set aH is ε-dense modulo 1.

It turns out that, if in Definition 5.1 the unit interval $[0, 1]$ is replaced by any interval J of positive length, the ensuing class of ED-sets is unchanged. Theorems 2.1, 2.3.(1), and 2.4 admit analogous for the family of ED-sets. For example, an analogue of condition (2) of Theorem 2.1 is: For every interval J with $|J| > 0$ there exists a positive integer $n = n(J)$ such that for everyh mapping $\psi\colon D \to \mathbf{R}$ the set

$$\{x \in [0, 1] : ax + \psi(a) \notin J \ (\mathrm{mod}\ 1),\ a \in D\}$$

is of cardinality not exceeding n. (In fact, this is also equivalent to the same condition with n required to depend only on $|J|$.) Although this family does not coincide (at least not by its definition) with the family of $\mathscr{H}$-densing sets for any family $\mathscr{H}$ of hefty sets, the proofs are quite similar to those given for the above-mentioned theorems.

Some of the results of [BP; AP] can be reformulated to assert that certain sets of integers are ED-sets. For example, this is the case with sets of positive upper (Banach) density [BP, Proposition 2.1], images of (non-constant) polynomials and the set of all primes [AP, Theorem 6.3]. Also, homogeneously distributed sequences of reals or of integers form ED-sets (this strengthens Proposition 3.1; the proof is the same). Moreover, the quantitative estimates obtained in [AP] for the sequence 1, 2, 3, ... are valid for general homogeneously distributed sequences.

Somewhat surprisingly, there is no known example of an $\mathscr{N}\mathscr{D}$-densing set which is not an ED-set. This naturally raises the question whether in fact the two notions are equivalent. Another question of interest, mentioned already in [BP], is whether the family of all Glasner sets has the Ramsey property, namely whether ($D_1 \cup D_2$ is $\mathscr{N}\mathscr{D}$-densing) $\Rightarrow$ (either D_1 or D_2 is $\mathscr{N}\mathscr{D}$-densing). Of course, the same question may be asked for the family of $\mathscr{H}$-densing sets for any family $\mathscr{H}$ of hefty sets. (Note that, in view of Corollary 3.5, the answer is trivially affirmative for $\mathscr{H} = \mathscr{L}\mathscr{M}_{>0}$ and $\mathscr{H} = \mathscr{S}\mathscr{D}$).

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