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Many Continuous Functions Have Many Proper Local Extrema

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Given a topological space X, let $M(X)$ (resp. $m(X)$) denote the set of all continuous real functions on X whose set of proper local maximum (resp. minimum) points is dense in X. We identify some classes of spaces X for which $M(X)$ is a dense subset of $C(X)$ endowed with the majorant topology. In particular, $M(X) \cap m(X)$ is dense in $C(X)$ with the majorant topology whenever X has a σ-discrete π-base and a dense subset whose points are G_δ-sets.

Also we show that $M(X) \cap m(X)$ is residual in $C(X)$ endowed with the topology of uniform convergence, provided that X has a σ-discrete π-base consisting of completely metrizable subspaces. This is true, in particular, for all completely metrizable spaces. © 1991 Academic Press, Inc.

INTRODUCTION

Let X be a topological space. Given a function $f\colon X \to \mathbb{R}$ we say that a point $x \in X$ is a proper local maximum (resp. minimum) point for f provided that there exists a neighbourhood U of x such that $f(y) < f(x)$ (resp. $f(y) > f(x)$) for every $y \in U \setminus \{x\}$. We denote by $M(f)$ (resp. $m(f)$) the set of all proper local maximum (resp. minimum) points for f. Of

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course $M(f) \cap m(f)$ consists of all the isolated points of X; also it is obvious that if f is a continuous function, then $\{x\}$ is a G_δ-set in X for every $x \in M(f) \cup m(f)$ (these easy remarks will explain the reason of some assumptions in the statements of Section 1).

Let $C(X)$ be the set of all continuous real functions on X. Also let $M(X)$ (resp. $m(X)$) denote the subset of $C(X)$ consisting of all functions f for which $M(f)$ (resp. $m(f)$) is a dense set in X. As the title suggests, the purpose of this paper is to show that if the space X satisfies some suitable assumptions, then $M(X)$, or even $M(X) \cap m(X)$, is (in a sense to be specified) a rather numerous subset of $C(X)$ with respect to some natural topology for $C(X)$.

To make clear the last point, there are two topologies that will be considered for $C(X)$, namely, the majorant topology, i.e., that topology in which basic neighborhoods of $f \in C(X)$ are the sets

$$\{g \in C(X) \colon |g(x) - f(x)| < \varepsilon(x) \text{ for every } x \in X\}$$

with $\varepsilon \in C(X)$, $\varepsilon > 0$ everywhere in X, and the topology of uniform convergence, i.e., the topology induced by the metric ρ of uniform convergence on X:

$$\rho(f, g) = \min\{1, \sup\{|f(x) - g(x)| \colon x \in X\}\}$$

for every $f, g \in C(X)$ (recall that with this metric $C(X)$ is a complete metric space). Of course the majorant topology is stronger than that of uniform convergence; moreover, the two topologies coincide if and only if the space X is pseudocompact (according to the definition given in [6, p. 126]).

Coming back to the subject of the paper, there are some recent results along the above indicated direction which have been obtained by A. Villani [5], who proved that $M(X)$ is a dense subset of $C(X)$ endowed with the majorant topology whenever the space X is metrizable, and by V. Drobot and M. Morayne [1], who showed that for $X = [0, 1]$, the set $M(X)$ is a residual Borel subset of $C(X)$; as a matter of fact, the argument given in [1] is easily realized to work with minor changes for any separable locally compact metrizable space, if the topology of uniform convergence is considered for $C(X)$. By the way, we recall here that a very strong result, implying that $M([0, 1]) \cap m([0, 1])$ is nonempty, is due to Z. Zalcwasser [7], who proved that for any two disjoint countable sets A, $B \subset [0, 1]$ there exists a differentiable function f: $[0, 1] \to \mathbb{R}$ such that $M(f) = A$, $m(f) = B$. We refer the reader also to [3] for a short proof of the result by Zalcwasser, to [4] for a nice elementary proof that $M(\mathbb{R})$ is nonempty, and to the literature cited in [1] for further references.

In this paper the above quoted results, namely those from [5, 1], are extended to more general classes of topological spaces. More precisely, in Section 1 we consider the density of $M(X)$, or $M(X) \cap m(X)$, in $C(X)$ endowed with the majorant topology, and show, among other things, that $M(X) \cap m(X)$ is dense in $C(X)$ provided that X is a completely regular space which has a σ-discrete π-base and a dense subset whose points are G_δ-sets in X. In Section 2 we consider the residuality of $M(X)$ in $C(X)$ endowed with the topology of uniform convergence. A simplified statement of our main result is that $M(X)$ (and hence $M(X) \cap m(X)$) is a residual set in $C(X)$ whenever X has a σ-discrete π-base consisting of completely metrizable subspaces. This is true in particular for all completely metrizable spaces.

It is worth recalling here, for the reader's convenience, that a family $\mathscr{U}$ of nonempty open subsets of a space X is said to be a *π-base* for X provided that every nonempty open set $A \subset X$ contains some $U \in \mathscr{U}$.

Throughout the paper, regular, completely regular and normal spaces need not be Hausdorff.

1. Some Density Results

As announced before, this section is devoted to establishing some sufficient conditions for a topological space X in order that $M(X)$ be a dense subset of $C(X)$ endowed with the majorant topology.

To simplify our exposition, it is convenient to set down formally the following definition.

Definition. A nonempty subset D of a space X is said to be *strongly discrete* provided that there exists a discrete family $\{V_s : s \in D\}$ of open subsets of X such that $s \in V_s$ for every $s \in D$. Also the empty set is considered to be strongly discrete, just by convention.

A set $S \subset X$ is said to be *σ-strongly discrete* if S is a countable union of strongly discrete sets.

Of course, a strongly discrete set D is also discrete; i.e., D is a discrete space with the relative topology, provided that D is nonempty. Also, in a T_1-space every strongly discrete set is closed. Conversely, if the space X is normal and collectionwise Hausdorff (i.e., for every nonempty discrete set $\Delta \subset X$ there is a family $\{W_t : t \in \Delta\}$ of pairwise disjoint open sets with $t \in W_t$ for each $t \in \Delta$), then every closed discrete set is strongly discrete. To see this, assume that Δ is a nonempty closed discrete set and take a family $\{W_t : t \in \Delta\}$ as before. Also, using normality, take an open set V with $\Delta \subset V \subset \bar{V} \subseteq \bigcup \{W_t : t \in \Delta\}$. Then $t \in V \cap W_t$ for every $t \in \Delta$, and it is easy to check that $\{V \cap W_t : t \in \Delta\}$ is a discrete family.

The following lemma will be applied repeatedly in the proof of forthcoming Theorem 1.

LEMMA. *Let X be a completely regular space. Let $D \subset X$ be a nonempty strongly discrete set such that every singleton $\{s\}$, with $s \in D$, is a G_δ-set in X. Also, let $H \subset X$ be a closed set disjoint from D.*

Then, for every φ, $\eta \in C(X)$ with $\eta > 0$ everywhere in X, there exist $\psi \in C(X)$ and $\{B_s : s \in D\}$, a discrete family of closed sets, disjoint from H, with $s \in \operatorname{Int}(B_s)$ for every $s \in D$, such that:

(i) $\varphi \leqslant \psi < \varphi + \eta$ *everywhere in* X,

(ii) $\psi = \varphi$ *in* $X \setminus \bigcup \{B_s : s \in D\}$,

(iii) $\psi(x) < \psi(s)$ *for every* $s \in D$ *and* $x \in B_s \setminus \{s\}$.

Proof. Let ϑ be a fixed number with $0 < \vartheta < 1$ and, for each $s \in D$, let $\lambda_s = \vartheta \eta(s)$.

As the space X is regular and the set D is strongly discrete, it is possible to find, for each $s \in D$, a closed neighbourhood B_s of s, disjoint from H, in such a way that $\{B_s : s \in D\}$ is a discrete family. Of course, it can be also assumed that $\varphi(B_s) \subset]-\infty, \varphi(s) + \lambda_s/2]$ for every $s \in D$.

Let, for every $s \in D$, $h_s : X \to [0, 1]$ be a continuous function such that $h_s^{-1}(1) = \{s\}$ and $h_s(X \setminus \operatorname{Int}(B_s)) = \{0\}$. The existence of such a function is proved in a standard way as follows. Since $\{s\}$ is a G_δ-set in X there exists $\{A_n\}$, a decreasing sequence of open sets, with $A_n \subset \operatorname{Int}(B_s)$ for all n, such that $\bigcap_{n=1}^{\infty} A_n = \{s\}$. Since X is a completely regular space there is, for each $n = 1, 2, \ldots$, a continuous function $g_n : X \to [0, 2^{-n}]$ such that $g_n(s) = 2^{-n}$ and $g_n(X \setminus A_n) = \{0\}$. Then, letting $h_s = \sum_{n=1}^{\infty} g_n$, we obtain a continuous function with the desired features.

Now, for each $s \in D$, let $\gamma_s \in C(X)$ be defined by

$$\gamma_s(x) = [1 - h_s(x)]\, \varphi(x) + h_s(x)[\varphi(s) + \lambda_s]$$

for every $x \in X$. We have $\gamma_s(s) = \varphi(s) + \lambda_s$. Also, if $x \in B_s \setminus \{s\}$ then $\gamma_s(x) < \gamma_s(s)$; indeed

$$\begin{aligned} \gamma_s(x) &\leqslant [1 - h_s(x)][\varphi(s) + \lambda_s/2] + h_s(x)[\varphi(s) + \lambda_s] \\ &= \varphi(s) + \lambda_s/2 + h_s(x)\lambda_s/2 < \varphi(s) + \lambda_s = \gamma_s(s). \end{aligned}$$

As the family $\{B_s : s \in D\}$ is discrete and for each $s \in D$ we have $\gamma_s = \varphi$ in $X \setminus \operatorname{Int}(B_s)$, then the function $\gamma : X \to \mathbb{R}$ defined by

$$\begin{aligned} \gamma(x) &= \varphi(x) && \text{if } x \in X \Big\backslash \bigcup_{s \in D} B_s, \\ \gamma(x) &= \gamma_s(x) && \text{if } x \in B_s,\ s \in D, \end{aligned}$$

is continuous. Also, it is apparent that γ satisfies conditions (ii) and (iii).

Moreover, we have $\gamma \geqslant \varphi$ everywhere in X. To see this, note that if $x \in B_s$ for some $s \in D$, then

$$\gamma(s) = \gamma_s(x) = \varphi(x) + [\varphi(s) + \lambda_s - \varphi(x)]\, h_s(x) \geqslant \varphi(x).$$

At this point, to obtain the function ψ that we are looking for, it is enough to put $\psi = \min\{\gamma, \varphi + \vartheta\eta\}$. Then conditions (i) and (ii) are obviously satisfied. To check (iii) note that for every $s \in D$ we have

$$\gamma(s) = \varphi(s) + \lambda_s < \varphi(s) + \vartheta\eta(s),$$

so that $\psi(s) = \gamma(s)$; hence, for every $x \in B_s \setminus \{s\}$ we have

$$\psi(x) \leqslant \gamma(x) < \gamma(s) = \psi(s). \quad \blacksquare$$

Now, we come to prove Theorem 1 below, the main result of this section. This theorem extends [5, Theorem 1] in two directions. First, a more general class of spaces is considered here (it should be observed that in a metrizable space every discrete set is a countable union of closed discrete sets, hence it is σ-strongly discrete according to the remark before the lemma; it follows that in a metrizable space a set is σ-discrete if and only if it is σ-strongly discrete). Secondly, it is possible here to prescribe for the approximating function g both maximum and minimum points simultaneously.

THEOREM 1. *Let X be a completely regular topological space. Let S, $T \subset X$ be two σ-strongly discrete disjoint sets such that each singleton $\{s\}$, with $s \in S \cup T$, is a G_δ-set in X.*

Then, for every f, $\varepsilon \in C(X)$, with $\varepsilon > 0$ everywhere in X, there exists $g \in C(X)$ such that:

(j) $|g - f| < \varepsilon$ *everywhere in X,*

(jj) $S \subset M(g)$, $T \subset m(g)$.

Also, if $K \subset X$ is any closed set disjoint from $S \cup T$, it is possible for g to satisfy

(jjj) $g|_K = f|_K$.

Proof. As the case $S \cup T = \emptyset$ is trivial, we can assume without loss of generality that S is nonempty. Then we have

$$S = D_0 \cup D_2 \cup D_4 \cup \cdots, \qquad T = D_1 \cup D_3 \cup D_5 \cup \cdots,$$

where D_0, D_1, D_2, ... are pairwise disjoint strongly discrete sets and $D_0 \neq \emptyset$. We set $E_n = \bigcup_{k=0}^{n} D_k$ for every $n = 0, 1, 2, \ldots$, and $E_{-1} = \emptyset$. Also, let $f_{-1} = f$.

We will show by induction that there exist $\{f_n : n = 0, 1, 2, \ldots\}$, a sequence in $C(X)$, and $\{B_s : s \in S \cup T\}$, a family of closed subsets of X, with

$s \in \mathrm{Int}(B_s)$ for every $s \in S \cup T$, such that the following conditions are satisfied for every $n = 0, 1, 2, \ldots$:

$(1)_n$ if $D_n \neq \emptyset$ then $\{B_s : s \in D_n\}$ is a discrete family,

$(2)_n$ $(\bigcup\{B_s : s \in D_n\}) \cap (K \cup E_{n-1}) = \emptyset$,

(here, of course, the convention $\bigcup\{B_s : s \in D_n\} = \emptyset$ whenever $D_n = \emptyset$ is observed; similar warnings will be omitted in the sequel),

$(3)_n$ $(-1)^n f_{n-1} \leqslant (-1)^n f_n < (-1)^n f_{n-1} + \varepsilon/2^{n+1}$ everywhere in X,

$(4)_n$ $f_n = f_{n-1}$ in $X \setminus \bigcup\{B_s : s \in D_n\}$,

$(5)_n$ $(-1)^k f_n(x) < (-1)^k f_n(s)$ whenever $k \leqslant n$, $s \in D_k$, and $x \in B_s \setminus \{s\}$,

$(6)_n$ if $B_u \cap B_v \neq \emptyset$ for some $v \in D_n$ and $u \in E_{n-1}$, then $v \in B_u$.

To prove this, we first note that applying the lemma with $D = D_0$, $H = K$, $\varphi = f_{-1}$, and $\eta = \varepsilon/2$, we obtain the existence of $f_0 \in C(X)$ and $\{B_s : s \in D_0\}$, a family of closed subsets of X, with $s \in \mathrm{Int}(B_s)$ for every $s \in D_0$, such that conditions $(1)_0$–$(6)_0$ are satisfied.

Next, let us assume that functions $f_0, \ldots, f_n$ and closed sets B_s, $s \in E_n$, have been found such that conditions $(1)_k$–$(6)_k$ are satisfied for $k = 0, \ldots, n$, and construct f_{n+1} and B_s, $s \in D_{n+1}$, such that $(1)_{n+1}$–$(6)_{n+1}$ hold.

Decompose D_{n+1} as $D_{n+1} = D'_{n+1} \cup D''_{n+1}$, with $D'_{n+1} = D_{n+1} \setminus \bigcup\{B_t : t \in E_n\}$ and $D''_{n+1} = D_{n+1} \setminus D'_{n+1}$.

If $D'_{n+1} = \emptyset$ we introduce the function h by setting $h = f_n$.

If $D'_{n+1} \neq \emptyset$ we apply the lemma with $D = D'_{n+1}$, $H = K \cup (\bigcup\{B_t : t \in E_n\})$ (this is a closed set by assumptions $(1)_k$, $k = 0, \ldots, n$), $\varphi = (-1)^{n+1} f_n$ and $\eta = \varepsilon/2^{n+3}$. Then there exist $h \in C(X)$ and, for each $s \in D'_{n+1}$, B_s, a closed neighbourhood of s, such that (replacing h with $-h$ if $n+1$ is an odd integer):

(a) $\{B_s : s \in D'_{n+1}\}$ is a discrete family,

(b) $(\bigcup\{B_s : s \in D'_{n+1}\}) \cap (K \cup (\bigcup\{B_t : t \in E_n\})) = \emptyset$,

(c) $(-1)^{n+1} f_n \leqslant (-1)^{n+1} h < (-1)^{n+1} f_n + \varepsilon/2^{n+3}$ everywhere in X,

(d) $h = f_n$ in $X \setminus \bigcup\{B_s : s \in D'_{n+1}\}$,

(e) $(-1)^{n+1} h(x) < (-1)^{n+1} h(s)$ for every $s \in D'_{n+1}$ and $x \in B_s \setminus \{s\}$.

Now, consider D''_{n+1}. If $D''_{n+1} = \emptyset$, then closed sets B_s, for $s \in D_{n+1}$, have already been constructed; moreover, letting $f_{n+1} = h$, it is clear, by (a)–(e) and $(5)_n$, that conditions $(1)_{n+1}$–$(6)_{n+1}$ are fulfilled.

Hence, suppose that $D''_{n+1} \neq \emptyset$ and let $\{U_s : s \in D''_{n+1}\}$ be any discrete family of open sets with $s \in U_s$ for every $s \in D''_{n+1}$.

For each $s \in D''_{n+1}$ denote by Δ_s the set of all points $t \in E_n$ such that $s \in B_t$. By conditions $(1)_k$, $k = 0, \ldots, n$, the set Δ_s is finite. Let $\tau_s =$

$\min\{|f_n(t)-f_n(s)|: t\in\Delta_s\}$; by condition $(5)_n$, τ_s is a positive number. Also, let C_s denote the union of all sets B_t with $t\in(E_n\setminus\Delta_s)\cup D'_{n+1}$; by conditions $(1)_k$, $k=0, ..., n$, and also (a) if $D'_{n+1}\neq\emptyset$, C_s is a closed set; moreover, using condition (b) if $D'_{n+1}\neq\emptyset$, we have that $s\notin C_s$. Then, applying the lemma again with $D=\{s\}$, $H=K\cup C_s\cup\Delta_s\cup(X\setminus U_s)$ (note that each $\{t\}$, $t\in\Delta_s$, is a closed set since it is a G_δ-set and the space X is regular), $\varphi=h$ and $\eta=\eta_s=\min\{\varepsilon/2^{n+3}, \tau_s\}$, we obtain a function $h_s\in C(X)$ and a closed neighbourhood B_s of s such that (replacing h_s with $-h_s$ if necessary):

$(\beta)_s$ $B_s\subset U_s\setminus(K\cup C_s\cup\Delta_s)$,

$(\gamma)_s$ $(-1)^{n+1}h\leqslant(-1)^{n+1}h_s<(-1)^{n+1}h+\eta_s$ everywhere in X,

$(\delta)_s$ $h_s=h$ in $X\setminus B_s$,

$(\varepsilon)_s$ $(-1)^{n+1}h_s(x)<(-1)^{n+1}h_s(s)$ for every $x\in B_s\setminus\{s\}$.

Do this for each $s\in D''_{n+1}$. Then closed sets B_s, $s\in D_{n+1}$, have been constructed. Moreover, having in mind conditions $(\beta)_s$, $s\in D''_{n+1}$, the discreteness of $\{U_s: s\in D''_{n+1}\}$, and also conditions (a)–(b) if $D'_{n+1}\neq\emptyset$, it is easy to check the validity of $(1)_{n+1}$, $(2)_{n+1}$, and $(6)_{n+1}$.

Let $f_{n+1}: X\to\mathbb{R}$ be defined as follows:

$$f_{n+1}(x)=h(x) \qquad \text{if} \quad x\in X\setminus\bigcup\{B_s: s\in D'_{n+1}\},$$

$$f_{n+1}(x)=h_s(x) \qquad \text{if} \quad x\in B_s,\ s\in D''_{n+1}.$$

By conditions $(1)_{n+1}$ and $(\delta)_s$, $s\in D''_{n+1}$, the function f_{n+1} is well defined and continuous. Also, using conditions $(\gamma)_s$, $s\in D''_{n+1}$, and as usual (c) and (d) if $D'_{n+1}\neq\emptyset$, it is easy to check that $(3)_{n+1}$ and $(4)_{n+1}$ are fulfilled.

Let us show that also $(5)_{n+1}$ is satisfied. This will conclude the inductive argument.

Let $k\leqslant n+1$, $s\in D_k$, and $x\in B_s\setminus\{s\}$. If $k=n+1$, then the inequality $(-1)^{n+1}f_{n+1}(x)<(-1)^{n+1}f_{n+1}(s)$ follows from (e) if $s\in D'_{n+1}$, or from $(\varepsilon)_s$ if $s\in D''_{n+1}$. If $k\leqslant n$ and $x\notin\bigcup\{B_t: t\in D_{n+1}\}$, then, using conditions $(4)_{n+1}$, $(5)_{n+1}$, and $(2)_{n+1}$, we have

$$(-1)^k f_{n+1}(x)=(-1)^k f_n(x)<(-1)^k f_n(s)=(-1)^k f_{n+1}(s).$$

Finally, let us consider the case $k\leqslant n$ and $x\in B_t$ for some $t\in D_{n+1}$. By condition $(6)_{n+1}$ we have $t\in B_s$, hence $t\in D''_{n+1}$ and $s\in\Delta_t$. We distinguish two subcases according to whether $(-1)^k=(-1)^{n+1}$ or $(-1)^k=(-1)^n$. In the first case we have

$$(-1)^k f_{n+1}(x)=(-1)^{n+1}h_t(x)\leqslant \qquad (\text{by } (\varepsilon)_t)$$

$$(-1)^{n+1}h_t(t)< \qquad (\text{by } (\gamma)_t)$$

$$(-1)^{n+1}h(t)+\eta_t\leqslant(-1)^{n+1}h(t)+|f_n(t)-f_n(s)|=$$

(using (d) if $D'_{n+1} \neq \varnothing$)

$$\begin{aligned}
&(-1)^{n+1} f_n(t) + |f_n(t) - f_n(s)| = \\
&(-1)^k f_n(t) + |(-1)^k f_n(t) - (-1)^k f_n(s)| = \qquad \text{(by } (5)_n\text{)} \\
&(-1)^k f_n(s) = \qquad \text{(by } (2)_{n+1} \text{ and } (4)_{n+1}\text{)} \\
&(-1)^k f_{n+1}(s).
\end{aligned}$$

If $(-1)^k = (-1)^n$, then we have

$$\begin{aligned}
&(-1)^k f_{n+1}(x) = (-1)^n h_t(x) \leqslant \qquad \text{(by } (\gamma)_t\text{)} \\
&(-1)^n h(x) = \qquad \text{(using (d) if } D'_{n+1} \neq \varnothing\text{)} \\
&(-1)^k f_n(x) < \qquad \text{(by } (5)_n\text{)} \\
&(-1)^k f_n(s) = \qquad \text{(by } (2)_{n+1} \text{ and } (4)_{n+1}\text{)} \\
&(-1)^k f_{n+1}(s).
\end{aligned}$$

At this point we are in a position to define the function g that we are looking for. Let

$$g = f_{-1} + \sum_{n=0}^{\infty} (f_n - f_{n-1}).$$

By conditions $(3)_n$, $n = 0, 1, 2, \ldots$, the function g is well defined, continuous, and satisfies

$$|g - f| = |g - f_{-1}| \leqslant \sum_{n=0}^{\infty} |f_n - f_{n-1}| < \varepsilon$$

everywhere in X, that is condition (j). Moreover, by conditions $(2)_n$ and $(4)_n$, $n = 0, 1, 2, \ldots$, it is clear that also (jjj) is satisfied.

Let us check (jj). We will show that $(-1)^k g(x) < (-1)^k g(s)$ whenever $s \in D_k$ and $x \in B_s \setminus \{s\}$. Of course, this implies (jj).

Since $g = \lim_n f_n$ and owing to conditions $(2)_n$ and $(4)_n$, $n = k+1, k+2, \ldots$, we have $f_k(s) = f_{k+1}(s) = \cdots = g(s)$. Denote by L the set of all integers $n = 0, 1, 2, \ldots$ such that $(-1)^n = (-1)^k$ and $x \in \bigcup \{B_t : t \in D_n\}$. If L is a finite set, then, letting $v = \max L$, by conditions $(2)_n$, $(4)_n$, and $(3)_n$, $n = v+1, v+2, \ldots$, we have $(-1)^k f_{n-1}(x) \geqslant (-1)^k f_n(x)$ for all $n \geqslant v$, hence by $(5)_v$,

$$(-1)^k g(x) \leqslant (-1)^k f_v(x) < (-1)^k f_v(s) = (-1)^k g(s).$$

If the set L is infinite, then fix any $m \in L$ with $m > k$ and denote by t the element of D_m for which $x \in B_t$. By $(6)_m$ we have $t \in B_s$ and so, by conditions $(5)_n$, $n = m, m+1, \ldots$,

$$\begin{aligned}(-1)^k f_n(x) = (-1)^m f_n(x) &\leqslant (-1)^m f_n(t) \\ &= (-1)^k f_n(t) < (-1)^k f_n(s)\end{aligned}$$

for every $n \geqslant m$. But, as above for s, we also have $f_m(t) = f_{m+1}(t) = \cdots = g(t)$. It follows $(-1)^k g(x) \leqslant (-1)^k g(t) < (-1)^k g(s)$. This accomplishes the proof. ▮

Now, we present some consequences of Theorem 1.

Henceforth, given a space X, we will denote by X' the derived set of X.

THEOREM 2. *Let X be a completely regular space.*

If there exists a σ-strongly discrete set F which is a dense subset of $\operatorname{Int}(X')$ *and such that $\{x\}$ is a G_δ-set in X for every $x \in F$, then both $M(X)$ and $m(X)$ are dense in $C(X)$ endowed with the majorant topology.*

If there exist two disjoint σ-strongly discrete sets F, G, both dense subsets of $\operatorname{Int}(X')$, *such that $\{x\}$ is a G_δ-set in X for every $x \in F \cup G$, then $M(X) \cap m(X)$ is dense in $C(X)$ endowed with the majorant topology.*

Proof. If a set F exists, then we apply Theorem 1 with $S = F$, $T = \varnothing$ and f, ε any two functions in $C(X)$, with $\varepsilon > 0$ everywhere in X. The corresponding function g belongs to $M(X)$. Indeed, we have $M(g) \supset F \cup (X \setminus X')$ and the last set is dense in X. This proves the density of $M(X)$. To prove that also $m(X)$ is dense, it is enough to interchange the roles of F and $\varnothing$, or to use the obvious remark that (regardless of which of two topologies is considered for $C(X)$) there exists a homeomorphism of $C(X)$ onto itself, namely $f \to -f$, which takes $M(X)$ onto $m(X)$.

If two sets F, G exist, then we apply Theorem 1 again with $S = F$, $T = G$. ▮

Theorem 2 yields in particular the following

COROLLARY 1. *Let X be a completely regular space.*

If $\operatorname{Int}(X')$ *has a countable dense subset whose points are G_δ-sets in X, then both $M(X)$ and $m(X)$ are dense in $C(X)$ endowed with the majorant topology.*

If $\operatorname{Int}(X')$ *has two disjoint countable dense subsets whose points are G_δ-sets in X, then $M(X) \cap m(X)$ is dense in $C(X)$ endowed with the majorant topology.*

Proof. Observe that any countable set is σ-strongly discrete. ▮

Finally, we prove

THEOREM 3. *Let X be a completely regular space with a σ-strongly discrete π-base and with a dense subset whose points are G_δ-sets in X. Then $M(X) \cap m(X)$ is dense in $C(X)$ endowed with the majorant topology.*

Proof. Of course, we can assume that $\mathrm{Int}(X') \neq \emptyset$, for if $\mathrm{Int}(X') = \emptyset$, then it is obvious that $M(X) = m(X) = C(X)$.

Let Y be a dense subset of X such that $\{y\}$ is a G_δ-set in X for every $y \in Y$. By the regularity of X, each singleton $\{y\}$, with $y \in Y$, is also a closed set, and consequently every nonempty open subset of $\mathrm{Int}(X')$ meets Y in infinitely many points.

Let $\mathscr{U}$ be a discrete π-base for X. Then $\mathscr{V} = \{V \in \mathscr{U}\colon V \subset \mathrm{Int}(X')\}$ is a π-base for the subspace $\mathrm{Int}(X')$; also we have $\mathscr{V} = \bigcup_{n=1}^{\infty} \mathscr{V}_n$, where each $\mathscr{V}_n$ is a discrete family in X.

We will construct by induction $\{F_n : n = 1, 2, ...\}$ and $\{G_n : n = 1, 2, ...\}$, two sequences of pairwise disjoint subsets of $Y \cap \mathrm{Int}(X')$, with $(\bigcup_{n=1}^{\infty} F_n) \cap (\bigcup_{n=1}^{\infty} G_n) = \emptyset$, such that, for every $n = 1, 2, ...,$ both F_n and G_n are contained in $\bigcup(V\colon V \in \mathscr{V}_n\}$ and meet each $V \in \mathscr{V}_n$ in one point. Then the theorem will follow from Theorem 2 taking $F = \bigcup_{n=1}^{\infty} F_n$ and $G = \bigcup_{n=1}^{\infty} G_n$.

To perform the construction, we first choose, for every $V \in \mathscr{V}_1$, two distinct points x_V, y_V in $Y \cap V$, and put $F_1 = \{x_V\colon V \in \mathscr{V}_1\}$, $G_1 = \{y_V\colon V \in \mathscr{V}_1\}$.

Next, let us assume that pairwise disjoint subsets $F_1, ..., F_n, G_1, ..., G_n$ of $Y \cap \mathrm{Int}(X')$ have been found such that, for every $i = 1, ..., n$, both F_i and G_i are contained in $\bigcup\{V\colon V \in \mathscr{V}_i\}$ and meet each $V \in \mathscr{V}_i$ in one point, and construct F_{n+1}, G_{n+1}.

Owing to the discreteness of $\mathscr{V}_1, ..., \mathscr{V}_n$, for each $V \in \mathscr{V}_{n+1}$ there is a nonempty open set $\Omega \subset V$ which meets $F_1 \cup \cdots \cup F_n \cup G_1 \cup \cdots \cup G_n$ in finitely many points. Hence, by the initial remark, two distinct points x_V, y_V can be chosen in $(Y \cap V) \backslash (F_1 \cup \cdots \cup F_n \cup G_1 \cup \cdots \cup G_n)$ for every $V \in \mathscr{V}_{n+1}$. At this point, to complete the inductive step, it is enough to take $F_{n+1} = \{x_V\colon V \in \mathscr{V}_{n+1}\}$, and $G_{n+1} = \{y_V\colon V \in \mathscr{V}_{n+1}\}$. ∎

Remark. By the above proof it is clear that in Theorem 3 the assumption that X has a σ-discrete π-base and a dense subset whose points are G_δ-sets in X can be weakened as follows: if $\mathrm{Int}(X') \neq \emptyset$, then $\mathrm{Int}(X')$, with the relative topology, has a π-base which is σ-discrete in X and a dense subset whose points are G_δ-sets.

2. A CATEGORY RESULT

In this section we are concerned with the residuality of $M(X)$ in $C(X)$ endowed with the topology of uniform convergence. Since $C(X)$ with the metric ρ of uniform convergence is a complete metric space, it is obvious that in this case residuality implies density. Also, by a previous remark (see the proof of Theorem 2), it is apparent that $M(X)$ is residual if and only if $m(X)$ is; hence $M(X)$ residual implies that also $M(X) \cap m(X)$ is residual.

THEOREM 4. *Let X be a topological space for which the following condition is satisfied: if* $\operatorname{Int}(X') \neq \emptyset$, *then* $\operatorname{Int}(X')$ *with the relative topology has a π-base $\mathscr{U}$ which consists of completely metrizable subspaces and which moreover is a σ-discrete family in X.*

Then $M(X) \cap m(X)$ is a residual set in $C(X)$ endowed with the topology of uniform convergence.

Proof. Of course we can assume that $\operatorname{Int}(X') \neq \emptyset$. Also, by the above remark, it is enough to prove that $M(X)$ is residual.

Let $\mathscr{U}$ be as in the statement and, for each $U \in \mathscr{U}$, let d_U be a complete compatible metric on U; we will denote by $\delta_U(V)$ the diameter of a set $V \subset U$ with respect to this metric. Also, let $\mathscr{U} = \bigcup_{n=1}^{\infty} \mathscr{U}_n$, where every $\mathscr{U}_n$ is a discrete family in X.

For every $n = 1, 2, \ldots$, let $\mathbb{V}_n$ be the set of all families $\mathscr{V}$ of the form $\mathscr{V} = \{V_U : U \in \mathscr{U}_n\}$, where V_U is a nonempty open subset of U for each $U \in \mathscr{U}_n$. Also, for every $n = 1, 2, \ldots$ and $k = 1, 2, \ldots$, let $\mathbb{W}_{n,k}$ be the set of all $\mathscr{V} \in \mathbb{V}_n$, with $\mathscr{V} = \{V_U : U \in \mathscr{U}_n\}$, such that $\delta_U(V_U) \leqslant 1/k$ for each $U \in \mathscr{U}_n$.

Now, for every $n = 1, 2, \ldots$ and $\mathscr{V} \in \mathbb{V}_n$, with $\mathscr{V} = \{V_U : U \in \mathscr{U}_n\}$, let $A_n(\mathscr{V})$ be the set of all functions $f \in C(X)$ with the property: there exists $\sigma > 0$ such that $\sup f(U) \geqslant \sup f(U \setminus V_U) + \sigma$ for every $U \in \mathscr{U}_n$ (here the convention $\sup \emptyset = -\infty$ is observed). We have that $A_n(\mathscr{V})$ is an open subset of $C(X)$. To check this, let $\bar{f}$ be any function in $A_n(\mathscr{V})$, so that there is a $\bar{\sigma} > 0$ such that $\sup \bar{f}(U) \geqslant \sup \bar{f}(U \setminus V_U) + \bar{\sigma}$ for every $U \in \mathscr{U}_n$. We claim that the open ball of $C(X)$ centered at $\bar{f}$ and with radius $r = \min\{\bar{\sigma}/2, 1\}$ is contained in $A_n(\mathscr{V})$. Indeed, from $\rho(f, \bar{f}) < r \leqslant 1$, it follows, for every $Y \subset X$,

$$\sup f(Y) \geqslant \sup \bar{f}(Y) - \rho(f, \bar{f}) \geqslant \sup f(Y) - 2\rho(f, \bar{f}),$$

and hence, for every $U \in \mathscr{U}_n$,

$$\begin{aligned} \sup f(U) &\geqslant \sup \bar{f}(U) - \rho(f, \bar{f}) \geqslant \sup \bar{f}(U \setminus V_U) + \bar{\sigma} - \rho(f, \bar{f}) \\ &\geqslant \sup f(U \setminus V_U) + \bar{\sigma} - 2\rho(f, \bar{f}), \end{aligned}$$

whence $f \in A_n(\mathscr{V})$ since $\bar{\sigma} - 2\rho(f, \bar{f}) > \bar{\sigma} - 2r \geqslant 0$.

Next, for every $n = 1, 2, \ldots$ and $k = 1, 2, \ldots$, let

$$B_{n,k} = U\{A_n(\mathscr{V}) : \mathscr{V} \in \mathbb{W}_{n,k}\}.$$

We will prove that the open set $B_{n,k}$ is dense in $C(X)$. To this aim, given

$f \in C(X)$ and $\varepsilon > 0$, we will construct a family $\tilde{\mathscr{V}} \in \mathbb{W}_{n,k}$ and a function $g \in A_n(\tilde{\mathscr{V}})$ such that $\rho(g, f) \leqslant \varepsilon$. Consider the family $\mathscr{U}'_n = \{U \in \mathscr{U}_n: \sup f(U) < +\infty\}$. If $\mathscr{U}'_n = \emptyset$, then $f \in A_n(\mathscr{V})$ for every $\mathscr{V} \in \mathbb{V}_n$, hence we can take $g = f$ and $\tilde{\mathscr{V}}$ any family in $\mathbb{W}_{n,k}$. If $\mathscr{U}'_n \neq \emptyset$ we proceed as follows. Let, for each $U \in \mathscr{U}'_n$, a point x_U and an open set W_U be fixed such that

$$x_U \in W_U, \overline{W_U} \subset U, \qquad \delta_U(W_U) \leqslant 1/k$$

and

$$\sup f(U) < f(x_U) + \varepsilon/2;$$

also, let $\varphi_U: U \to \mathbb{R}$ be a continuous function such that

$$(*) \quad f \leqslant \varphi_U \leqslant f + \varepsilon \qquad \text{everywhere in } U,$$

$$(**) \quad \varphi_U = f \qquad \text{in} \quad U \backslash W_U,$$

and $\varphi_U(x_U) = f(x_U) + \varepsilon$. Then, owing to the discreteness of $\mathscr{U}'_n$, to condition (**) and to the fact that $\overline{W_U} \subset U$, the function $g: X \to \mathbb{R}$, defined by

$$g(x) = f(x) \qquad \text{if} \quad x \notin \bigcup \{U: U \in \mathscr{U}'_n\},$$

$$g(x) = \varphi_U(x) \qquad \text{if} \quad x \in U, U \in \mathscr{U}'_n,$$

is easily seen to be continuous. Also, by (*), we have $\rho(g, f) \leqslant \varepsilon$. Let $\tilde{\mathscr{V}} = \{\tilde{V}_U: U \in \mathscr{U}_n\}$ be any family in $\mathbb{W}_{n,k}$ such that $\tilde{V}_U = W_U$ for every $U \in \mathscr{U}'_n$. Then we have $g \in A_n(\tilde{\mathscr{V}})$. Indeed, we claim that

$$\sup g(U) \geqslant \sup g(U \backslash \tilde{V}_U) + \varepsilon/2$$

for every $U \in \mathscr{U}_n$. This is obvious if $U \notin \mathscr{U}'_n$, for in this case $\sup g(U) = \sup f(U) = +\infty$. If $U \in \mathscr{U}'_n$, then the above claim follows from

$$\begin{aligned} \sup g(U) &\geqslant g(x_U) = f(x_U) + \varepsilon \\ &\geqslant \sup f(U) + \varepsilon/2 \geqslant \sup f(U \backslash \tilde{V}_U) + \varepsilon/2 \\ &\geqslant \sup g(U \backslash \tilde{V}_U) + \varepsilon/2. \end{aligned}$$

Finally, to complete the proof, we show that $M(X)$ contains a dense G_δ-set in $C(X)$, namely $\bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} B_{n,k}$.

Let $f \in \bigcap_{n=1}^{\infty} \bigcap_{k=1}^{\infty} B_{n,k}$. We have to show that if $\Omega \subset X$ is open and nonempty, then $\Omega \cap M(f) \neq \emptyset$. This is obvious if $\Omega \backslash X' \neq \emptyset$. So, let $\Omega \subset X'$, hence $\Omega \subset \operatorname{Int}(X')$, and, replacing Ω with a nonempty open subset if necessary, let the condition $\sup f(\Omega) < +\infty$ be satisfied. As $\mathscr{U}$ is a π-base for $\operatorname{Int}(X')$, there exist an integer n^* and a set $U^* \in \mathscr{U}_{n^*}$ such that $U^* \subset \Omega$. As the function f is in $\bigcap_{k=1}^{\infty} B_{n^*,k}$, then for every $k = 1, 2, \ldots,$ there is a nonempty open set $V_k \subset U^*$ such that $\delta_{U^*}(V_k) \leqslant 1/k$ and $\sup f(U^*) \geqslant \sup f(U^* \setminus V_k)$. The family $\{V_k : k = 1, 2, ...\}$ has the finite intersection property because, for every $k = 1, 2, ...$, the inequalities

$$\sup f(U^*) > \sup f(U^* \setminus V_i), \qquad i = 1, ..., k,$$

imply

$$\sup f(U^*) > \sup f\left(\bigcup_{i=1}^{k} (U^* \setminus V_i)\right) = \sup f\left(U^* \setminus \left(\bigcap_{i=1}^{k} V_i\right)\right),$$

hence $\bigcap_{i=1}^{k} V_i$ cannot be empty. Since (U^*, d_{U^*}) is a complete metric space, then, by a version of the Cantor theorem [2, p. 337, Theorem 4.3.10], $\bigcap_{k=1}^{\infty} (U^* \cap \overline{V_k})$ is nonempty. Let p be a point in $\bigcap_{k=1}^{\infty} (U^* \cap \overline{V_k})$. We will prove that $p \in M(f)$ and this will conclude the proof.

We first note that $f(p) = \sup f(U^*)$. This is proved by contradiction as follows. Assume that a point $q \in U^*$ exists such that $f(q) > f(p)$. and let W be an open neighborhood of p, $W \subset U^*$, such that $f(q) > f(y)$ for every $y \in W$. Since $\delta_{U^*}(V_k) \to 0$ as $k \to \infty$ and $p \in \overline{V_k}$, we have that, for k large enough, $V_k \subset W$, hence $\sup f(V_k) \leqslant f(q)$, whence the contradiction $\sup f(U^*) = \sup f(U^* \setminus V_k)$.

Next, we show that $f(p) > f(x)$ for every $x \in U^* \setminus \{p\}$. Indeed, if $x \neq p$, then $x \notin V_k$ if k is large enough, and so

$$f(p) = \sup f(U^*) > \sup(U^* \setminus V_k) \geqslant f(x). \quad \blacksquare$$

It is worth pointing out explicitly the following particular case of Theorem 4.

COROLLARY 2. *Let X be a completely metrizable space. Then $M(X) \cap m(X)$ is a residual set in $C(X)$ endowed with the topology of uniform convergence.*

Proof. Let $\mathscr{B}$ be a σ-discrete base for X and, if $\operatorname{Int}(X') \neq \emptyset$, let $\mathscr{U} = \{B \cap \operatorname{Int}(X') : B \in \mathscr{B}\}$. Then the assumptions of Theorem 4 are satisfied. $\blacksquare$

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