Regularity estimates of solutions to complex Monge–Ampère equations on Hermitian manifolds

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Abstract

In this paper, we obtain the Bedford–Taylor interior $C^2$ estimate and local Calabi $C^3$ estimate for the solutions to complex Monge–Ampère equations on Hermitian manifolds.

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1. Introduction

The complex Monge–Ampère equation is one of the most important partial differential equations in complex geometry. The proof of the Calabi conjecture given by S.T. Yau [18] in 1976 yields significant applications of the Monge–Ampère equation in Kähler geometry. After that, many important geometric results, especially in Kähler geometry, were obtained by studying this equation. It is natural and also interesting to study the complex Monge–Ampère equations in a more general form and in different geometric settings.

There are many modifications and generalizations in the existing literature. In [17], Tosatti, Weinkove and Yau gave a partial affirmative answer to a conjecture of Donaldson in symplectic geometry by solving (under additional curvature assumption) the complex Monge–Ampère...
equation in an almost Kähler geometric setting. By studying a more general form of the Monge–Ampère equation on non-Kähler manifolds, Fu and Yau [8] gave a solution to the Strominger system which is motivated by superstring theory. Another direction worth studying is the corresponding equation on Hermitian manifolds. In such a case the equation is not so geometric, since Hermitian metrics do not represent positive cohomology classes. On the other hand the estimates for Hermitian manifolds are more complicated than the Kähler case because of the non-vanishing torsion.

In the eighties and nineties, some results regarding the Monge–Ampère equation in the Hermitian setting were obtained by Cherrier [3,4] and Hanani [10]. For the next few years there was no activity on the subject until very recently, when the results were rediscovered and generalized by Guan and Li [9]. Under additional conditions they generalized the a priori estimates due to Yau [18] from the Kähler case and got some existence results for the solution of the complex Monge–Ampère equation. At the same time, Zhang [19] independently proved similar a priori estimates in the Hermitian setting and he also considered a general form of the complex Hessian equation. Later, Tosatti and Weinkove [15,16] gave a more delicate a priori $C^2$-estimate and removed the conditions in [9]. Moreover, Dinh and Kolodziej [6] also studied the equation in the weak sense and obtained the $L^\infty$ estimates via suitably constructed pluripotential theory. In this paper, we want to study some other regularity properties of the complex Monge–Ampère equation on Hermitian manifolds: the Bedford–Taylor interior $C^2$-estimate and Calabi $C^3$-estimate.

The interior estimate for the second order derivatives is an important and difficult topic in the study of complex Monge–Ampère equation. It has many fundamental applications in complex geometric problems. In the cornerstone work of Bedford and Taylor [1], by using the transitivity of the automorphism group of the unit ball $B \subset \mathbb{C}^n$, they obtained the interior $C^2$-estimate for the following Dirichlet problem:

$$
\begin{align*}
\det(u_{ij}) &= f & \text{in } B, \\
u &= \phi & \text{on } \partial B,
\end{align*}
$$

where $\phi \in C^{1,1}(\partial B)$ and $0 \leq f \leq C^{1,1}(B)$.

Unfortunately for generic domains $\Omega \subset \mathbb{C}^n$, due to the non-transitivity of the automorphism group of $\Omega$, Bedford and Taylor’s method is not applicable and the analogous estimate is still open. Here, we exploit the method of Bedford and Taylor to study the interior estimate for the Dirichlet problem of the complex Monge–Ampère equation in the unit ball in the Hermitian setting (notice that for local arguments the shape of the domain is immaterial and hence it suffices to consider the balls). We consider the following equation

$$
\begin{align*}
(\omega + \sqrt{-1}\partial \bar{\partial} u)^n &= f \omega^n & \text{in } B, \\
u &= \phi & \text{on } \partial B,
\end{align*}
$$

where $0 \leq f \leq C^{1,1}(B)$ and $\omega$ is a smooth positive $(1, 1)$-form (not necessarily closed) defined on $\tilde{B}$. We denote by $PSH(\omega, \Omega)$ the set of all integrable, upper semicontinuous functions satisfying $(\omega + \sqrt{-1}\partial \bar{\partial} u) \geq 0$ in the current sense on the domain $\Omega$. Since $\omega$ is not necessarily Kähler, there are no local potentials for $\omega$, and thus Bedford–Taylor’s method cannot be applied directly in our case.

**Theorem 1.** Let $B$ be the unit ball on $\mathbb{C}^n$ and $\omega$ be a smooth positive $(1, 1)$-form (not necessary closed) on $\tilde{B}$. Let $u \in C(\tilde{B}) \cap PSH(\omega, \tilde{B}) \cap C^2(\tilde{B})$ solve the Dirichlet problem (1) with
\( \phi \in C^{1,1}(\partial B) \). Then, for arbitrary compact subset \( B' \subset B \), there exists a constant \( C \) dependent only on \( \omega \) and \( \text{dist}(B', \partial B) \) such that

\[
\| u \|_{C^2(B')} \leq C \| \phi \|_{C^{1,1}(\partial B)} + C \| f \|_{C^{1,1}(\partial B)}.
\]

**Remark 1.** Observe that this estimate is scale and translation invariant i.e. the same constant will work if we consider the Dirichlet problem in any ball with arbitrary small radius (and suitably rescaled set \( B' \)).

As we have already mentioned, another goal of this paper is to get a local version of the \( C^3 \)-estimate of the complex Monge–Ampère equation on Hermitian manifolds. Calabi’s \( C^3 \)-estimate for the real Monge–Ampère equation was first proved by Calabi himself in [2]. After that many mathematicians paid a lot of attention to this estimate. In Yau’s celebrated work [18] about the Calabi conjecture, he gave a detailed proof of the \( C^3 \)-estimate for the complex Monge–Ampère equation on Kähler manifolds, which was generalized to the Hermitian case by Cherrier [3].

All these \( C^3 \)-estimates are global. However, in some situations, a local \( C^3 \)-estimate is needed. For example Riebesehl and Schulz [14] gave a local version of Calabi’s estimate in order to study the Liouville property of Monge–Ampère equations on \( \mathbb{C}^n \). In a recent work by Dinew and the authors [7], aimed to study the \( C^{2,\alpha} \) regularity of solutions to complex Monge–Ampère equation, the local result in [14] also played an important role to get the optimal value of \( \alpha \). Thus, it is also natural to generalize this local estimate to Hermitian manifolds and find some interesting applications.

Let \((M, g)\) be a Hermitian manifold. We consider the following complex Monge–Ampère equation

\[
(\omega + \sqrt{-1} \partial \bar{\partial} \phi)^n = e^f \omega^n,
\]

where \( f(z) \in C^\infty(M) \) and \( \omega \) is the Hermitian form associated with the metric \( g \).

**Theorem 2.** Let \( \phi(z) \in PSH(\omega, M) \cap C^4(M) \) be a solution of the Monge–Ampère equation (2), satisfying

\[
\| \partial \bar{\partial} \phi \|_\omega \leq K.
\]

Let \( \Omega' \subset \Omega \subset M \). Then the third derivatives of \( \phi(z) \) of mixed type can be estimated in the form

\[
|\nabla_\omega \partial \bar{\partial} \phi|_\omega \leq C \quad \text{for } z \in \Omega',
\]

where \( C \) is a constant depending on \( K \), \( \| d\omega \|_\omega \), \( \| R \|_\omega \), \( \| \nabla R \|_\omega \), \( \| T \|_\omega \), \( \| \nabla T \|_\omega \), \( \text{dist}(\Omega', \partial \Omega) \) and \( \| \nabla^s f \|_\omega \), \( s = 0, 1, 2, 3 \). Here \( \nabla \) is the Chern connection with respect to the Hermitian metric \( \omega \), \( T \) and \( R \) are the torsion tensor and curvature form of \( \nabla \).

From the detailed proof in Yau’s paper [18] (see also [13]), in the Kähler case, we know that the quantity considered by Calabi

\[
S = \tilde{g}^{ij} \tilde{g}^{kl} \tilde{g}^{m\bar{n}} \phi_{jkm} \phi_{l\bar{n}i}
\]
satisfies the following elliptic inequality:

\[ \tilde{\Delta} S \geq -C_1 S - C_2. \] (4)

Here \( \phi \) is a smooth solution of Eq. (2), \( \tilde{g} \) denotes the Hermitian metric with respect to the form \( \omega_\phi = \omega + \sqrt{-1} \partial \bar{\partial} \phi \), \( \phi_{i\bar{j}} \) denotes the covariant derivative with respect to the Chern connection \( \nabla \). Riebesehl and Schulz [14] used the above elliptic inequality to get the \( L^p \) estimate for \( S \). Then, a standard theorem for linear elliptic equations gave the \( L^\infty \) estimate. For the Hermitian case, due to the non-vanishing torsion term, the estimates are more complicated. In [3], Cherrier proved the elliptic inequality corresponding to (4) on Hermitian manifolds:

\[ \tilde{\Delta} S \geq -C_1 S^3 - C_2, \] (5)

where \( \tilde{\Delta} \) is the canonical Laplacian with respect to the Hermitian metric \( \tilde{g} \) (i.e. \( \tilde{\Delta} f = 2 \tilde{g}^{ij} f_{ij} \)), positive constants \( C_1 \) and \( C_2 \) depend on \( K \), \( \| R \|_\omega \), \( \| \nabla R \|_\omega \), \( \| T \|_\omega \), \( \| \nabla T \|_\omega \), and \( \| \nabla^s f \|_\omega \), \( s = 0, 1, 2, 3 \).

By a similar method to that in [14], we obtain the \( L^p \) estimate for \( S \), and then use Moser iteration to get the \( L^\infty \) estimate.

The estimates obtained in this paper should be useful for the study of problems on Hermitian manifolds. As a simple application, following the lines of [7], one has the following corollary:

**Corollary 1.** Let \( \Omega \) be a domain in \( \mathbb{C}^n \) and \( \omega \) be a Hermitian form defined on \( \Omega \). Let \( \phi(z) \in PSH(\omega, \Omega) \cap C^2(\Omega) \) be a solution of the Monge–Ampère equation

\[ (\omega + \sqrt{-1} \partial \bar{\partial} \phi)^n = e^f \omega^n. \]

Suppose that \( f \in C^\alpha(\Omega) \) for some \( 0 < \alpha < 1 \). Then \( \phi \in C^{2,\alpha}(\Omega) \).

**Remark 2.** In the proof of Corollary 1, we don’t apply the local Calabi’s \( C^3 \) estimate to the original function \( \phi \in C^{1,1}(\Omega) \) directly. Instead of that, for any point \( x_0 \in \Omega' \subset \Omega \), we consider an approximation solution

\[
\begin{align*}
(\omega + \sqrt{-1} \partial \bar{\partial} u_k)^n &= e^{f(x_0)} \omega^n \quad \text{in } B(x_0, d\rho^k), \\
 u_k &= \phi \quad \text{on } \partial B(x_0, d\rho^k),
\end{align*}
\]

where \( \rho = \frac{1}{2} \) and \( d = \frac{1}{2} \text{dist}(\Omega', \partial \Omega'') \). Since \( \phi \) is only \( C^{1,1} \), we first consider the above Dirichlet problem with smooth boundary condition, i.e. instead of \( \phi \) by its mollification \( \phi^{(\epsilon)} \) for \( \epsilon \) small enough and \( \| \phi^{(\epsilon)} \|_{1,1} \to \| \phi \|_{1,1} \) as \( \epsilon \to 0 \). By the main theorem in [9] the solutions \( u_k \) (we suppress the indice \( \epsilon \) for the sake of readability) with the new boundary data coming from \( \phi^{(\epsilon)} \) are smooth. Now, by Bedford–Taylor’s interior \( C^2 \) estimate, one can get

\[ \| u_k \|_{C^2(B_{\rho^k})} \leq \tilde{c}_1 \left( \| \phi \|_{C^{1,1}(\Omega')} + \sup_{x \in \Omega'} e^{f(x)} \right), \]

where \( \tilde{c}_1 \) is a positive constant depending only on \( \omega \). This allows one to apply the complex version of Calabi estimate to the above Dirichlet problem. Thus, for any \( \gamma \in (0, 1) \), we have
where $\tilde{c}_1$ is a positive constant depending only on $\omega$, $d$, $n$, $\|\phi\|_{C^1(\Omega')}$ and $\sup_{x \in \Omega'} e^{f(x)}$. Letting now $\epsilon \to 0^+$, we obtain that this estimate remains true for the original function $u_k$. Then, using the $C^\alpha$ condition on $f$ and following the lines in Ref. [7], we use the regularity of $u_k$ to approximate the original $\phi$ and obtain a $C^{2,\alpha}$ estimate of $\phi$.

2. Proof of the interior estimates

In the proof of interior $C^2$-estimates, the comparison theorem will play the key role. Following the same idea as in [5], it’s easy to see that the comparison theorem is still true for the complex Monge–Ampère equation on Hermitian manifold $(M, \omega)$.

**Lemma 1.** Let $\Omega \subset M$ be a bounded set and $u, v \in C^0(\overline{\Omega}) \cap C^2(\Omega)$, with $\omega + \sqrt{-1} \partial \bar{\partial} u \geq 0$, $\omega + \sqrt{-1} \partial \bar{\partial} v > 0$ be such that

$$(\omega + \sqrt{-1} \partial \bar{\partial} v)^n \geq (\omega + \sqrt{-1} \partial \bar{\partial} u)^n$$

and

$$v \leq u \quad \text{on} \quad \partial \Omega,$$

then $v \leq u$ in $\overline{\Omega}$.

**Proof of Theorem 1.** As mentioned above, we will follow the idea of Bedford and Taylor from [1]. For $a \in B^n$, let $T_a \in \text{Aut}(B^n)$ be defined by

$$T_a(z) = \frac{\Gamma(a) z - a}{1 - a^\dagger z},$$

where $\Gamma(a) = \frac{a d^\dagger a}{1 - v(a)} - v(a) I$ and $v(a) = \sqrt{1 - |a|^2}$.

Note that $T_a(0) = 0, T_{-a} = T_a^{-1}$, and $T_a(z)$ is holomorphic in $z$, and a smooth function in $a \in B^n$. For any $a \in B(0, 1 - \eta) = \{ a \colon |a| < 1 - \eta \}$ set

$$L(a, h, z) = T_{a+h}^{-1} T_a(z)$$

and

$$U(a, h, z) = L_a^* u(z), \quad U(a, -h, z) = L_a^* u(z),$$

$$\Phi(a, h, z) = L_a^* \phi(z), \quad \Phi(a, -h, z) = L_a^* \phi(z), \quad \text{for} \quad z \in \partial B^n.$$
where $L_i^*$ means the pull-back of $L_i$ for $i = 1, 2$ and $L_1 = L(a, h, z)$, $L_2 = L(a, -h, z)$. Since $U(a, h, z) = \Phi(a, h, z)$ for $z \in \partial B^n$, it follows that $U \in C^{1,1}(B(0, 1 - \eta) \times B(0, \eta) \times \partial B^n)$. Consequently, for a suitable constant $K_1$, depending on $\eta > 0$, we have

$$
\frac{1}{2}(U(a, h, z) + U(a, -h, z)) - K_1|h|^2 \leq U(a, 0, z) = \phi(z) \tag{6}
$$

for all $|a| \leq 1 - \eta$, $|h| \leq \frac{1}{2} \eta$, and $z \in \partial B^n$. If it can be shown that $v(a, h, z)$ satisfies

$$(\omega + \sqrt{-1}\partial\bar{\partial}v)^n \geq f(z)\omega^n, \tag{7}$$

where

$$v(a, h, z) = \frac{1}{2}[U(a, h, z) + U(a, -h, z)] - K_1|h|^2 + K_2(|z|^2 - 1)|h|^2. \tag{8}$$

then it follows from the comparison theorem in the Hermitian case that $v(a, h, z) \leq u(z)$. Thus, if we set $a = z$, we conclude that

$$\frac{1}{2}[u(z + h) + u(z - h)] \leq u(z) + (K_1 + K_2)|h|^2$$

which would prove the theorem.

Let now

$$F(\omega + \sqrt{-1}\partial\bar{\partial}v) = \left(\frac{(\omega + \sqrt{-1}\partial\bar{\partial}v)^n}{(\sqrt{-1})^n dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n}\right)^{\frac{1}{n}} = (\det(g_{i\bar{j}} + v_{i\bar{j}}))^{rac{1}{n}}, \tag{9}$$

where $g_{i\bar{j}}$ is the local expression of $\omega$ under the standard coordinate $\{z_i\}_{i=1}^n$ in $\mathbb{C}^n$.

By the concavity of $F$, we have

$$F(\omega + \sqrt{-1}\partial\bar{\partial}v) = F\left(\omega + \frac{\sqrt{-1}}{2}(\partial\bar{\partial}L_1^*u + \partial\bar{\partial}L_2^*u + 2K_2|h|^2\partial\bar{\partial}|z|^2)\right)$$

$$= F\left(\frac{1}{2}(\omega - L_1^*\omega) + \frac{1}{2}(\omega - L_2^*\omega) + K_2|h|^2\sqrt{-1}\partial\bar{\partial}|z|^2 + \frac{1}{2}(L_1^*\omega + \sqrt{-1}\partial\bar{\partial}L_1^*u) + \frac{1}{2}(L_2^*\omega + \sqrt{-1}\partial\bar{\partial}L_2^*u)\right)$$

$$\geq \frac{1}{2} F(L_1^*\omega + \sqrt{-1}\partial\bar{\partial}L_1^*u) + \frac{1}{2} F(L_2^*\omega + \sqrt{-1}\partial\bar{\partial}L_2^*u)$$

$$+ \frac{1}{2} F((\omega - L_1^*\omega) + (\omega - L_2^*\omega) + 2K_2|h|^2\sqrt{-1}\partial\bar{\partial}|z|^2). \tag{10}$$

Since the Hermitian metric $\omega$ is smooth, one can find $K_2$ large enough, such that

$$(\omega - L_1^*\omega) + (\omega - L_2^*\omega) + 2K_2|h|^2\sqrt{-1}\partial\bar{\partial}|z|^2 \geq 0. \tag{11}$$
On the other hand, since $L(a, h, z)$ is holomorphic in $z$, it follows from Eq. (1) that

$$F(L^*_1 \omega + \sqrt{-1} \partial \bar{\partial} L^*_1 u) = F(L^*_1(\omega + \sqrt{-1} \partial \bar{\partial} u))$$

$$= \left( \frac{L^*_1(\omega + \sqrt{-1} \partial \bar{\partial} u)^n}{(\sqrt{-1})^n dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n} \right)^{\frac{1}{n}}$$

$$= \left( \frac{L^*_1(f(z)\omega^n)}{(\sqrt{-1})^n dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n} \right)^{\frac{1}{n}}$$

$$= F(L^*_1(f^\frac{1}{n} \omega)) = L^*_1(f^\frac{1}{n}) F(L^*_1(\omega)). \quad (12)$$

Similarly, we can get

$$F(L^*_2 \omega + \sqrt{-1} \partial \bar{\partial} L^*_2 u) = F(L^*_2(f^\frac{1}{n} \omega)) = L^*_2(f^\frac{1}{n}) F(L^*_2(\omega)).$$

Thus,

$$F(\omega + \sqrt{-1} \partial \bar{\partial} v) \geq \frac{1}{2} (F(L^*_1(f^\frac{1}{n} \omega)) + F(L^*_2(f^\frac{1}{n} \omega))) + \frac{1}{2} F(K_2|h|^2 \sqrt{-1} \partial \bar{\partial}|z|^2)$$

$$= F(f^\frac{1}{n} \omega) + \frac{1}{2} (F(L^*_1(f^\frac{1}{n} \omega)) + F(L^*_2(f^\frac{1}{n} \omega)) - 2F(f^\frac{1}{n} \omega))$$

$$+ \frac{1}{2} F(K_2|h|^2 \sqrt{-1} \partial \bar{\partial}|z|^2). \quad (13)$$

Again, since $\omega$ is smooth and $f^{1/n} \in C^{1,1}$, choosing $K_2$ large enough, we have

$$F(L^*_1(f^\frac{1}{n} \omega)) + F(L^*_2(f^\frac{1}{n} \omega)) - 2F(f^\frac{1}{n} \omega) \leq F(K_2|h|^2 \sqrt{-1} \partial \bar{\partial}|z|^2). \quad (14)$$

Finally, we obtain

$$F(\omega + \sqrt{-1} \partial \bar{\partial} v) \geq F(f^\frac{1}{n} \omega), \quad (15)$$

and thus, the inequality (7) follows.

3. Proof of the Calabi estimate

Let $(M, J, \omega)$ be a Hermitian manifold and $\nabla$ denote the Chern connection with respect to the metric $\omega$. Let locally $\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$, then the local formula for the connection 1-form reads $\theta = \partial g \cdot g^{-1}$. We also denote

$$\theta_\alpha = \partial_\alpha g \cdot g^{-1}, \quad \theta_{\alpha\beta}^\gamma = \frac{\partial g_{\beta\delta}}{\partial z^\alpha} g^{\gamma\delta}.$$ 

The torsion tensor of $\nabla$ is defined by
\[
T\left(\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^\beta}\right) = \nabla \frac{\partial}{\partial z^\alpha} \frac{\partial}{\partial z^\beta} - \nabla \frac{\partial}{\partial z^\beta} \frac{\partial}{\partial z^\alpha} - \left[\frac{\partial}{\partial z^\alpha}, \frac{\partial}{\partial z^\beta}\right]
\]
i.e., \( T_{\alpha\beta}^\gamma = \left(\frac{\partial g_{\beta\bar{\delta}}}{\partial z^\alpha} - \frac{\partial g_{\alpha\bar{\delta}}}{\partial z^\beta}\right) g_{\gamma\bar{\delta}}. \)

Notice that \( T = 0 \iff \omega \) is Kähler (and \( \nabla \) is the Levi-Civita connection on \( M \)).

The curvature form of \( \nabla \) is defined by \( R = \tilde{\partial} \partial = d \theta - \theta \wedge \theta = \tilde{\partial}(g \cdot g^{-1}) \). In local coordinates, we have

\[
R^i_{\alpha\bar{\beta}} = -\tilde{\partial} \left(\frac{\partial}{\partial z^\alpha} \right) g_{\gamma\bar{\delta}} \frac{\partial g_{\beta\bar{\delta}}}{\partial z^\gamma} = \frac{\partial^2 g_{\gamma\bar{\delta}}}{\partial z^\alpha \partial z^\beta} + \frac{\partial g_{\gamma\bar{\delta}}}{\partial z^\alpha} \frac{\partial g_{\beta\bar{\delta}}}{\partial z^\gamma} g_{\gamma\bar{\delta}}.
\]

\[
R^i_{\alpha\bar{\beta}} = g_{\gamma\bar{\delta}} R^i_{\gamma\alpha\bar{\beta}}.
\]

Note that \( R^{(2,0)} = R^{(0,2)} = 0 \) and \( T^{(1,1)} = 0 \), since the almost complex structure \( J \) is integrable and \( \nabla \) is the Chern connection.

**Proof of Theorem 2.** By the assumption (3) for the solution of Eq. (2), we know that

\[
1 = \frac{\lambda}{\lambda} g \leq g_\phi \leq \lambda g
\]
for some constant \( \lambda \),

where \( \lambda \) depends only on \( K \) and \( \|f\|_{C^0} \), and \( g_\phi \) denotes the Hermitian metric with respect to the form \( \omega_\phi = \omega + \sqrt{-1} \partial \bar{\partial} \phi \). Thus,

\[
S = (g_\phi)^{ij\bar{k}} (g_\phi)^{\bar{k}\bar{s}} (g_\phi)^{m\bar{l}} \phi \bar{\phi}_{jkm} \phi_{rsl} \lambda \leq \lambda (g_\phi)^{ij\bar{k}} (g_\phi)^{\bar{k}\bar{s}} (g_\phi)^{m\bar{l}} \phi \bar{\phi}_{jkm} \phi_{rsl}.
\]

On the other hand, we have

\[
g_{\phi}^{jk} g^{m\bar{l}} \phi_{jkm} = (g_{\phi}^{jk})^{m\bar{l}} \phi_{jkm} - (g_{\phi}^{jk})^{m\bar{l}} \phi_{jkm}
\]

\[
= g^{m\bar{l}} f_{m\bar{l}} + g_{\phi}^{jk} \phi_{j\bar{s}t} g_{\phi}^{m\bar{l}} \phi_{jkm},
\]

where we used Eq. (2) in the last equality above. Thus

\[
S \leq \lambda \left[ g_{\phi}^{jk} g^{m\bar{l}} \phi_{jkm} - \Delta f \right].
\]

Notice that \( g_{\phi}^{jk} g^{m\bar{l}} \phi_{jkm} = \Lambda_{g_\phi} (g^{m\bar{l}} \nabla_j \nabla_m (\sqrt{-1} \partial \bar{\partial} \phi)) \) is a globally defined quantity, where \( \Lambda_{g_\phi} \) is the contraction with \( \omega_\phi \), i.e. \( \Lambda_{g_\phi} \theta = \frac{\theta}{n} \nabla \theta \). In local coordinates, we have \( \Lambda_{g_\phi} \theta = \phi_{j\bar{s}t} g_{jkm} \). Therefore we can estimate for every sufficiently large exponents \( \rho, \sigma \), and every nonnegative test function \( \eta(z) \in C^1_0(\Omega) \):

\[
\int_{\Omega} S^\sigma \eta^{p+1} \frac{\alpha^n}{n!} \leq \lambda \int_{\Omega} S_{\phi}^{\sigma-1} \eta^{p+1} \left[ g_{\phi}^{jk} g^{m\bar{l}} \phi_{jkm} - \Delta f \right] \frac{\alpha^n}{n!}.
\]
Now, use the following identity:

\[
\phi_{j km} = \phi_{j km} + \phi_{sk} R^s_{jml} - \phi_{ji} R^i_{kml} \\
= \phi_{j mk} + \phi_{sk} R^s_{jml} - \phi_{ji} R^i_{lmk} - \phi_{jk} R^j_{kml},
\]

where \( C_1 \) is a constant depending on \( K \) and \(|R|_ω\). Therefore, we have

\[
\int_Ω S^σ\eta^{p+1} \frac{ω^n}{n!} \leq λ \left( \int_Ω S^{σ-1}\eta^{p+1} g^j φ g^m φ^j φ^m \frac{ω^n}{n!} \right. \\
+ \left. \int_Ω S^{σ-1}\eta^{p+1} (C_1 - Δf) \frac{ω^n}{n!} \right) \\
\leq λ \int_Ω S^{σ-1}\eta^{p+1} g^j(Δφ) j φ^j φ^m \frac{ω^n}{n!} + C_2 \int_Ω S^{σ-1}\eta^{p+1} \frac{ω^n}{n!},
\]

(19)

where \( C_2 \) is a constant depending on \( C_1 \) and \( Δf \).

Now, using integration by parts, it is easy to see that

\[
\int_Ω S^{σ-1}\eta^{p+1} g^j(Δφ) j φ^j φ^m \frac{ω^n}{n!} = \int_Ω e^{-f} S^{σ-1}\eta^{p+1} g^j(Δφ) j φ^j φ^m \frac{ω^n}{n!} \\
= \int_Ω e^{-f} S^{σ-1}\eta^{p+1} √-1 \tilde{d}φ^m φ^m \frac{ω^n}{n!} \\
= \int_Ω √-1 d(e^{-f} S^{σ-1}\eta^{p+1} \tilde{d}φ) \frac{ω^n}{n!} \\
- \int_Ω √-1 d(e^{-f} S^{σ-1}\eta^{p+1} \tilde{d}φ) \frac{ω^n}{n!},
\]

(20)

Next, we will estimate \(|I|\) and \(|II|\). First,

\[
I = \int_Ω √-1 d(e^{-f} S^{σ-1}\eta^{p+1} \tilde{d}φ) \frac{ω^n}{n!} \\
= - \int_Ω √-1 e^{-f} S^{σ-1}\eta^{p+1} \tilde{d}φ \frac{ω^n}{n!}.
\]

By the equivalence of two forms \( ω \) and \( ω_φ \) (i.e., the assumption (3) on \( φ \)), we know
\[
\left| \tilde{\partial} (\Delta \phi) \wedge d\omega_\phi \wedge \frac{\omega^{n-2}_\phi}{(n-2)!} \right| = \left| \tilde{\partial} (\Delta \phi) \wedge d\omega \wedge \frac{\omega^{n-2}_\phi}{(n-2)!} \right| \\
\leq C_3 |\tilde{\partial} (\Delta \phi)|_{d\omega_\phi} \frac{|d\omega_\phi|}{n!} \omega^n \\
\leq C_4 S^2 \frac{\omega^n}{n!},
\]

where \( C_4 \) is a constant depending on \(|d\omega_\phi|_g\), \( \| f \|_{C^0} \) and \( K \) (for the justification of the last inequality we refer to the formula of \( S \) given in Appendix A). This estimate yields

\[
|I| \leq C_5 \int_\Omega S^{\sigma - \frac{1}{2}} \eta^{p+1} \frac{\omega^n}{n!}
\]

for some constant \( C_5 \) dependent on \( \omega \), \( \| f \|_{C^0} \) and \( K \).

Let us now estimate the second term:

\[
II = \int_\Omega \sqrt{-1} d(e^{-f} S^{\sigma - 1} \eta^{p+1} \wedge \tilde{\partial} (\Delta \phi) \wedge \frac{\omega^{n-1}_\phi}{(n-1)!}) \\
+ (\sigma - 1) \int_\Omega \sqrt{-1} e^{-f} S^{\sigma - 2} \eta^{p+1} dS \wedge \tilde{\partial} (\Delta \phi) \wedge \frac{\omega^{n-1}_\phi}{(n-1)!} \\
+ (p + 1) \int_\Omega \sqrt{-1} e^{-f} S^{\sigma - 1} \eta^{p} d\eta \wedge \tilde{\partial} (\Delta \phi) \wedge \frac{\omega^{n-1}_\phi}{(n-1)!}
\]

Thus,

\[
|II| \leq C_6 \left( \int_\Omega S^{\sigma - \frac{1}{2}} \eta^{p+1} \frac{\omega^n}{n!} + (\sigma - 1) \int_\Omega S^{\sigma - \frac{3}{2}} |\nabla S| \eta^{p+1} \frac{\omega^n}{n!} \\
+ (p + 1) \int_\Omega S^{\sigma - \frac{1}{2}} \eta^{p} |\nabla \eta| \frac{\omega^n}{n!} \right),
\]

where \( C_6 \) is a constant depending on \( \| f \|_{C^1(\omega)} \) and \( K \).

By the estimates (22), (23) and using Cauchy’s inequality

\[
(\sigma - 1) \eta^{p+1} S^{\sigma - \frac{1}{2}} |\nabla S| \leq \frac{(\sigma - 1)^2}{4\epsilon} \eta^{p+1} S^{\sigma - 3} |\nabla S|^2 + \epsilon \eta^{p+1} S^\sigma
\]

we have, for \( \epsilon > 0 \) small enough,
\[
\int_\Omega S^{\sigma} \eta^{p+1} \frac{\omega_n}{n!} \leq C_7 \left( (\sigma - 1)^2 \int_\Omega S^{\sigma-3} |\nabla S|^2 \eta^{p+1} \frac{\omega_n}{n!} + \int_\Omega S^{\sigma-1} \eta^{p+1} \frac{\omega_n}{n!} \right. \\
+ \left. (p + 1) \int_\Omega S^{\sigma-1} \eta^p |\nabla \eta| \frac{\omega_n}{n!} + \int_\Omega S^{\sigma-1} \eta^{p+1} \frac{\omega_n}{n!} \right),
\]

where \(C_7\) is a constant depending on \(|d\omega|_{\omega} \), \(|R|_{\omega} , K, \|f\|_{C^1(\omega)}\) and \(\Delta f\).

Now we are in the place to use the elliptic inequality (5) in the introduction. Recall that

\[
\Delta \phi S \geq -CS^3 - C_0.
\]

Multiplying by \(S^{\sigma-2} \eta^{p+1}\) on both sides of the above inequality and integrating over \(\Omega\), we have

\[
-\int_\Omega S^{\sigma-2} \eta^{p+1} \frac{\omega_n}{n!} - C_0 \int_\Omega S^{\sigma-2} \eta^{p+1} \frac{\omega_n}{n!} \leq \int_\Omega S^{\sigma-2} \eta^{p+1} \Delta \phi S \frac{\omega_n}{n!} = \int_\Omega \frac{\omega_n}{n!}.
\]

The right-hand side of above inequality can be estimated as follows

\[
\int_\Omega S^{\sigma-2} \eta^{p+1} \Delta \phi S \frac{\omega_n}{n!}
\]

\[
= \int_\Omega e^{-f} S^{\sigma-2} \eta^{p+1} \sqrt{-1} \partial \bar{\partial} S \wedge \frac{\omega_{\phi}^{n-1}}{n-1}!
\]

\[
= \int_\Omega \sqrt{-1} d(e^{-f} S^{\sigma-2} \eta^{p+1} \bar{\partial} S) \wedge \frac{\omega_{\phi}^{n-1}}{n-1}!
\]

\[
- \int_\Omega \sqrt{-1} d(e^{-f} S^{\sigma-2} \eta^{p+1} \bar{\partial} S) \wedge \omega_{\phi}^{n-1}
\]

\[
= - \int_\Omega \sqrt{-1} e^{-f} S^{\sigma-2} \eta^{p+1} \bar{\partial} S \wedge d\omega \wedge \frac{\omega_{\phi}^{n-2}}{n-2}!
\]

\[
- \int_\Omega d(e^{-f}) S^{\sigma-2} \eta^{p+1} \wedge \bar{\partial} S \wedge \frac{\omega_{\phi}^{n-1}}{n-1}!
\]

\[
- (\sigma - 2) \int_\Omega \sqrt{-1} e^{-f} S^{\sigma-3} \eta^{p+1} \bar{\partial} S \wedge \bar{\partial} S \wedge \frac{\omega_{\phi}^{n-1}}{n-1}!
\]

\[
- (p + 1) \int_\Omega \sqrt{-1} e^{-f} S^{\sigma-2} \eta^p \bar{\partial} \eta \wedge \bar{\partial} S \wedge \frac{\omega_{\phi}^{n-1}}{n-1}!
\]
\[ 0 \leq -C_8 (\sigma - 2) \int_{\Omega} S^{\sigma - 3} \eta^{p+1} |\nabla S|^2 \frac{\omega^n}{n!} + C_9 \int_{\Omega} S^{\sigma - 2} \eta^{p+1} |\nabla S|^2 \frac{\omega^n}{n!} \]

\[ + C_9 (p + 1) \int_{\Omega} S^{\sigma - 2} \eta^p |\nabla \eta||\nabla S| \frac{\omega^n}{n!}, \]

for \( C_8 \) a positive constant. From above inequality, we obtain

\[ (\sigma - 2) \int_{\Omega} S^{\sigma - 3} \eta^{p+1} |\nabla S|^2 \frac{\omega^n}{n!} \leq C_{10} \left( (p + 1) \int_{\Omega} S^{\sigma - 2} \eta^{p+1} |\nabla \eta||\nabla S| \frac{\omega^n}{n!} + \int_{\Omega} S^{\sigma - 2} \eta^{p+1} |\nabla S|^2 \frac{\omega^n}{n!} \right) \]

Now, by Cauchy's inequality again,

\[ S^{\sigma - 2} \eta^{p+1} |\nabla S| \leq \epsilon |\nabla S|^2 S^{\sigma - 3} \eta^{p+1} + \frac{1}{4\epsilon} \eta^{p+1} S^{\sigma - 1} \]

\[ (p + 1) S^{\sigma - 2} \eta^p |\nabla \eta||\nabla S| \leq \epsilon |\nabla S|^2 S^{\sigma - 3} \eta^{p+1} + \frac{(p + 1)^2}{4\epsilon} \eta^{p-1} S^{\sigma - 1} |\nabla \eta|^2. \]

These two inequalities, together with (27) and (24) yield

\[ \int_{\Omega} S^\sigma \eta^{p+1} \frac{\omega^n}{n!} \leq C_{11} \sigma^2 (p + 1)^2 \left( \int_{\Omega} S^{\sigma - \frac{1}{2}} \eta^{p+1} \frac{\omega^n}{n!} + \int_{\Omega} S^{\sigma - \frac{1}{2}} \eta^{p+1} \frac{\omega^n}{n!} \right) \]

\[ + \int_{\Omega} S^{\sigma - \frac{1}{2}} \eta^p |\nabla \eta| \frac{\omega^n}{n!} + \int_{\Omega} S^{\sigma - 2} \eta^{p+1} \frac{\omega^n}{n!} \int_{\Omega} S^{\sigma - 1} \eta^{p-1} |\nabla \eta|^2 \frac{\omega^n}{n!} \]  

\[ \text{(28)} \]

for \( p \geq 2, \sigma \geq 4. \)

Now, let \( B_{R_0}(z) \subset \Omega \) be a ball, and let \( 0 < R \leq r < t \leq R_0, R_0 - R \leq 1. \) By choosing an appropriate testing function \( \eta(z), \) with \( 0 \leq \eta \leq 1, \eta|_{B_t} = 1, \eta|_{M/B_t} = 0, |\nabla \eta| \leq \frac{\epsilon}{t-r}, \) and putting \( p = \sigma - 1, \) we conclude that

\[ \int_{B_t(z)} (S\eta)^\sigma \frac{\omega^n}{n!} \leq C_{12} \sigma^4 \int_{B_t(z)} \left\{ \frac{1}{(t-r)^2} (S\eta)^{\sigma - 2} S \right\} \]

\[ + \frac{1}{t-r} (S\eta)^{\sigma - 1} S^\frac{1}{2} + (S\eta)^{\sigma - \frac{1}{2}} \eta^\frac{1}{2} + (S\eta)^{\sigma - 1} \eta + (S\eta)^{\sigma - 2} \eta^2 \frac{\omega^n}{n!}. \]  

\[ \text{(29)} \]
By Young’s inequality

\[ ab \leq \epsilon \frac{a^\alpha}{\alpha} + \frac{1}{\epsilon^{\beta/\alpha}} \frac{b^\beta}{\beta}, \quad \text{for } \epsilon > 0, \quad \frac{1}{\alpha} + \frac{1}{\beta} = 1. \]

It follows that,

\[
\begin{align*}
\frac{1}{t-r} (S \eta)^{\sigma-1} S^2 &\leq \frac{e}{\sigma-1} (S \eta)^{\sigma-1} \left( \frac{1}{t-r} S^2 \right)^{\frac{\sigma}{\sigma-1}}; \quad \alpha = \frac{\sigma}{\sigma-1}, \quad \beta = \sigma,
\frac{1}{(t-r)^2} (S \eta)^{\sigma-2} S &\leq \frac{e}{\sigma-2} (S \eta)^{\sigma-2} \left( \frac{1}{(t-r)^2} S \right)^{\frac{\sigma}{\sigma-2}}; \quad \alpha = \frac{\sigma}{\sigma-2}, \quad \beta = \frac{\sigma}{2},
(S \eta)^{\sigma-2} &\leq \frac{e}{\sigma-4} (S \eta)^{\sigma-4} \left( \frac{1}{(S \eta)^2} \right)^{\frac{\sigma}{\sigma-4}}; \quad \alpha = \frac{\sigma}{\sigma-4}, \quad \beta = \frac{\sigma}{4},
(S \eta)^{\sigma-1} &\leq \frac{e}{\sigma-2} (S \eta)^{\sigma-2} \left( \frac{1}{(S \eta)^2} \right)^{\frac{\sigma}{2}}; \quad \alpha = \frac{\sigma}{\sigma-2}, \quad \beta = \frac{\sigma}{2},
(S \eta)^{\sigma-\frac{1}{2}} &\leq \frac{e}{\sigma-1} (S \eta)^{\sigma-1} \left( \frac{1}{(S \eta)^2} \right)^{\frac{\sigma}{2}}; \quad \alpha = \frac{\sigma}{\sigma-1}, \quad \beta = \sigma.
\end{align*}
\]

All the above inequalities combined with (29), lead to

\[
\int_{B_r(z)} S^\sigma \omega_n n! \leq C_{13} B(\epsilon)^\sigma \left( \frac{1}{(t-r)^\sigma} + \frac{1}{(t-r)^\frac{\sigma}{2}} + 1 \right) \int_{B_r(z)} S^\sigma \omega_n n!
\leq C_{13} \frac{B(\epsilon)^\sigma}{(t-r)^\sigma} \left( \int_{B_r(z)} S^\sigma \omega_n n! \right)^{\frac{1}{2}},
\]

(30)

where \( B(\epsilon) \) is a constant depending on \( \epsilon \) which comes from the coefficients in Young’s inequalities above.

Now we can apply Meyers’ lemma:

**Lemma 2.** (See [12].) If \( u = u(x) \) is a nonnegative, non-decreasing continuous function in the interval \( [0, d) \), which satisfies the functional inequality:

\[ u(s) \leq \frac{c}{r-s} (u(r))^{1-\alpha}, \quad \text{for any } 0 \leq s < r < d, \]

with \( \alpha \) and \( c \) being constants \( (0 < \alpha < 1) \), then

\[ u(0) \leq \left( \frac{2^{\alpha+1} c}{(2^\alpha - 1)d} \right)^{\frac{1}{\alpha}}. \]
Using (30) and applying Meyers’ lemma with \(d = R_0 - R\), \(s = r - R\) and \(\Phi(s) = (\int_{B_{R+s}(z)} S^{\sigma} \omega_n^\sigma n!)^{\frac{1}{\sigma}}\), one can obtain

\[
\Phi(0) \leq C^\frac{1}{\sigma} B(\epsilon) R_0^{\frac{1}{\sigma}} \frac{R_0}{(R_0 - R)^2},
\]

and thus

\[
\left( \int_{B_R(z)} S^{\sigma} \omega_n^\sigma \frac{n!}{n!} \right)^{\frac{1}{\sigma}} \leq \frac{(CR_0)^{\frac{1}{\sigma}}}{(R_0 - R)^2} B(\epsilon).
\] (31)

From this, we obtain the \(L^p\) estimate of \(S\) for arbitrary \(p\). However, by tracking the constant \(B(\epsilon)\), one can find that \(B(\epsilon) \sim \sigma^4\). Thus, we cannot get the estimate for \(\sup_\Omega S\) by letting \(\sigma \to \infty\). Instead of that, we will use the standard Moser iteration to finish the \(L^\infty\) estimate for \(S\).

Recall that by inequality (27) we have

\[
(\sigma - 2) \int_\Omega S^{\sigma-3} \eta^{p+1} |\nabla S|^2 \frac{\omega_n^n}{n!} \leq C_{10} \left( (p + 1) \int_\Omega S^{\sigma-2} \eta^p |\nabla \eta| |\nabla S| \frac{\omega_n^n}{n!} + \int_\Omega S^{\sigma-2} \eta^{p+1} |\nabla S|^2 \frac{\omega_n^n}{n!} + \int_\Omega S^{\sigma-2} \eta^{p+1} \frac{\omega_n^n}{n!} \right).
\]

Coupling this with Young inequalities

\[
S^{\sigma-2} \eta^{p+1} |\nabla S| \leq \epsilon |\nabla S|^2 S^{\sigma-3} \eta^{p+1} + \frac{1}{4\epsilon} \eta^{p+1} S^{\sigma-1},
\]

\[
(p + 1)S^{\sigma-2} \eta^p |\nabla \eta| |\nabla S| \leq \epsilon |\nabla S|^2 S^{\sigma-3} \eta^{p+1} + \frac{(p + 1)^2}{4\epsilon} \eta^{p-1} S^{\sigma-1} |\nabla \eta|^2
\]

we have

\[
(\sigma - 2) \int_\Omega S^{\sigma-3} \eta^{p+1} |\nabla S|^2 \frac{\omega_n^n}{n!} \leq C_{14} \int_\Omega \frac{(p + 1)^2}{\sigma - 2} \eta^{p-1} S^{\sigma-1} |\nabla \eta|^2 + \frac{1}{\sigma - 2} S^{\sigma-1} \eta^{p+1} + S^{\sigma-2} \eta^{p+1} + S^{\sigma-2} \eta^{p+1} \frac{\omega_n^n}{n!}.
\] (32)

Let now \(q = \sigma - 1 \geq 2\), and \(p = 1\), then one obtains
\[
\int_{\Omega} S^{q-2} \eta^2 |\nabla S|^2 \frac{\omega_n}{n!} \leq C_{15} \int_{\Omega} \frac{1}{(q-1)^2} S^q |\nabla \eta|^2 + \frac{1}{(q-1)^2} S^q \eta^2 + \frac{1}{q-1} S^{q+\frac{1}{2}} \eta^2 + \frac{1}{q-1} S^{q-1} \frac{\omega_n}{n!}. \quad (33)
\]

By the Sobolev inequality
\[
\left( \int_{\Omega} v^{\frac{2m}{m-1}} \frac{\omega_n}{n!} \right)^{\frac{m-1}{2m}} \leq C \left( \int_{\Omega} |\nabla v|^2 \frac{\omega_n}{n!} \right)^{\frac{1}{2}} + C \left( \int_{\Omega} v^2 \frac{\omega_n}{n!} \right)^{\frac{1}{2}}
\]
applied to \( v = \eta S^\frac{q}{2} \), we conclude that
\[
\left( \int_{\Omega} (\eta S^\frac{q}{2})^{\frac{2m}{m-1}} \frac{\omega_n}{n!} \right)^{\frac{m-1}{2m}} \leq C_{16} \left[ \left( \int_{\Omega} |\nabla (\eta S^\frac{q}{2})|^2 \frac{\omega_n}{n!} \right)^{\frac{1}{2}} + \left( \int_{\Omega} (\eta S^\frac{q}{2})^2 \frac{\omega_n}{n!} \right)^{\frac{1}{2}} \right]
\]
\[
\leq C_{17} \left[ \left( \int_{\Omega} S^q |\nabla \eta|^2 + \left( \frac{q}{2} \right)^2 S^{q-2} \eta^2 |\nabla S|^2 \frac{\omega_n}{n!} \right)^{\frac{1}{2}} + \left( \int_{\Omega} \eta S^q \omega_n \right)^{\frac{1}{2}} \right]. \quad (34)
\]

Using the inequality (33), we have
\[
\left( \int_{\Omega} (\eta^2 S^q)^{\frac{m}{m-1}} \frac{\omega_n}{n!} \right)^{\frac{m-1}{m}} \leq C_{18} \int_{\Omega} \left( |\nabla \eta|^2 S^q + \eta^2 S^q + \frac{q^2}{(q-1)^2} S^q |\nabla \eta|^2 + \frac{q^2}{(q-1)^2} S^q \eta^2 
\right.
\]
\[
+ \frac{q^2}{q-1} S^{q+\frac{1}{2}} \eta^2 + \frac{q^2}{q-1} S^{q-1} \frac{\omega_n}{n!} \right) \omega_n \frac{\omega_n}{n!}
\]
\[
\leq C_{18} \int_{\Omega} \left( |\nabla \eta|^2 S^q + \eta^2 S^q + \frac{q^2}{(q-1)^2} S^q |\nabla \eta|^2 + \frac{q^2}{(q-1)^2} S^q \eta^2 
\right.
\]
\[
+ \frac{q^2}{q-1} S^{q+\frac{1}{2}} \eta^2 + \frac{q^2}{q-1} S^{q-1} \frac{\omega_n}{n!} \right) \omega_n \frac{\omega_n}{n!}
\]
\[
\left. \quad (35) \right)
\]

for any \( q > 4 \).

Again, let \( B_{R_1}(z) \subset \Omega \) be a ball, and let \( 0 < R \leq r_1 < r_2 \leq R_0, R_0 - R \leq 1 \). By choosing an appropriate testing function \( \eta(z) \), with \( 0 \leq \eta \leq 1, \eta|_{B_{r_1}} = 1, \eta|_{M/B_{r_2}} = 0, |\nabla \eta| \leq \frac{C}{r_2-r_1} \), we conclude that
\[
\left( \int_{B_{r_1}(z)} S^q \frac{m}{m-1} \frac{\omega_n}{n!} \right)^{\frac{m-1}{m}} \leq C_{18} \int_{B_{r_1}(z)} \left( |\nabla \eta|^2 S^q + \eta^2 S^q + \frac{q^2}{(q-1)^2} S^q |\nabla \eta|^2 + \frac{q^2}{(q-1)^2} S^q \eta^2 
\right.
\]
\[
+ \frac{q^2}{q-1} S^{q+\frac{1}{2}} \eta^2 + \frac{q^2}{q-1} S^{q-1} \frac{\omega_n}{n!} \right) \omega_n \frac{\omega_n}{n!}
\]
\[
\left. \quad (35) \right)
\]
\[
C_{19} \int_{B_2(z)} \left( 1 + \frac{q^2}{(q - 1)^2} \right) \left( \frac{1}{(r_2 - r_1)^2} + 1 \right) S^q + \frac{q^2}{q - 1} S^{q + \frac{1}{2}} + \frac{q^2}{q - 1} S^{q - 1} \right) \frac{\omega^n}{n!} \\
\leq q C_{20} \left( \frac{1}{(r_2 - r_1)^2} + 1 \right) \int_{B_2(z)} \left( S^q + S^{q - 1} + S^{q + \frac{1}{2}} \right) \frac{\omega^n}{n!} \\
\leq q C_{21} \left( \frac{1}{(r_2 - r_1)^2} + 1 \right) \int_{B_2(z)} S^{q + \frac{1}{2}} \frac{\omega^n}{n!}.
\]

Thus,
\[
\|S\|_{L^{qm/(m-1)}(B_1(z))} \leq \left[ C q \left( \frac{1}{(r_2 - r_1)^2} + 1 \right) \right]^{\frac{1}{q}} \|S\|^{\frac{q + \frac{1}{2}}{q}}_{L^{q + \frac{1}{2}}(B_2(z))} \quad (36)
\]

for any \(0 < R \leq r_1 < r_2 \leq R_0\).

Let \(\frac{q_m}{m-1} = q_{k+1} + \frac{1}{2}\) and \(r_k = R + (R_0 - R)2^{-k}\). Then,
\[
q_k = \left( \frac{m}{m - 1} \right)^k + \frac{m - 1}{2}, \quad \text{and} \quad |r_k - r_{k-1}| = (R_0 - R)2^{-k}.
\]

By (37), we have
\[
\|S\|_{L^{q_k+\frac{1}{2}}(B_{r_{k+1}}(z))} \leq \left[ C q_k \left( 1 + \frac{1}{(r_{k+1} - r_k)^2} \right) \right]^{\frac{1}{q_k}} \|S\|_{L^{q_k+\frac{1}{2}}(B_{r_k}(z))}^{a_k} \quad (38)
\]

where \(a_k := \frac{q_{k+1}}{q_k}\). By iteration, it follows from (38) that
\[
\|S\|_{L^{q_k+\frac{1}{2}}(B_{r_{k+1}}(z))} \leq \left[ \prod_{i=1}^{k} q_i^{\frac{1}{q_i}} \left( C \left( 1 + \frac{1}{(R_{i+1} - R_i)^2} \right) \right)^{\frac{1}{q_i}} 2^{\frac{2k}{q_k}} \|S\|_{L^{q_k+\frac{1}{2}}(B_{r_k}(z))}^{\prod_{i=1}^{k} a_i} \right]^{\prod_{i=1}^{k} a_i} \quad (39)
\]

Notice that \(a_k = \frac{q_{k+1}}{q_k} = \frac{q_{k-1}^{m}}{m-1} q_k\), so
\[
\prod_{i=1}^{k} a_i = \left( \frac{m}{m - 1} \right)^k \frac{q_0}{q_1} \cdots \frac{q_k}{q_k} = \left( \frac{m}{m - 1} \right)^k \frac{q_0}{q_k}
\]
and thus
\[
\lim_{k \to \infty} \prod_{i=1}^{k} a_i = q_0 = \frac{m + 1}{2}.
\]

Moreover,
\[
\prod_{i=1}^{k} q_i^{-\frac{1}{q_i}} \left( C \left( 1 + \frac{1}{(R_0 - R)^2} \right) \right)^{-\frac{1}{q_i}} 2^{\frac{2}{q_i}} = \prod_{i=1}^{k} q_i^{-\frac{1}{q_i}} \left( C \left( 1 + \frac{1}{(R_0 - R)^2} \right) \right)^{\frac{1}{q_i}} 2^{\frac{2}{q_i}} \sum_{i=1}^{k} \frac{1}{q_i} \sum_{i=1}^{k} \frac{2}{q_i}.
\]

When \( k \to \infty \), it is easy to show that \( \sum_{i=1}^{\infty} \frac{1}{q_i} < \infty \) and \( \sum_{i=1}^{\infty} \frac{2}{q_i} < \infty \). Notice also that \( \log(\prod_{i=1}^{\infty} q_i^{-\frac{1}{q_i}}) < \infty \). Thus,
\[
\lim_{k \to \infty} \prod_{i=1}^{k} q_i^{-\frac{1}{q_i}} \left( C \left( 1 + \frac{1}{(R_0 - R)^2} \right) \right)^{-\frac{1}{q_i}} 2^{\frac{2}{q_i}} < \infty.
\]

It follows from (39), by letting \( k \to \infty \),
\[
\|S\|_{L^\infty} \leq C \|S\|_{L^{\frac{m+1}{2}}(B_{R_0}(z))}.
\] (40)

Choosing now \( \sigma = q_1 + \frac{1}{2} = \frac{m}{m-1} + \frac{m}{2} \) in (31), we finally obtain
\[
\|S\|_{L^\infty} \leq C,
\] (41)

where \( C \) is a positive constant depending on \( k \), \(|d\omega|_{\omega}|, |R|_{\omega}|, |\nabla R|_{\omega}|, |T|_{\omega}|, |\nabla T|_{\omega}|,\) \( \operatorname{dist(\Omega', \partial \Omega)} \) and \( |\nabla^s f|_{\omega}, s = 0, 1, 2, 3 \).

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Appendix A

As mentioned in the introduction, using the idea from [13], we give a new proof for the elliptic inequality (5) in this section.

Proof of the elliptic inequality (5). Let \( \nabla \) and \( \tilde{\nabla} \) denote the Chern connections corresponding to the Hermitian metrics \( \omega \) and \( \omega + \sqrt{-1} \partial \bar{\partial} \phi \) respectively. Define
\[
h = \tilde{g} \cdot g^{-1}
\] (42)

and
\[
h^i_j = g^{ik} \tilde{g}_{jk}, \quad (h^{-1})^j_i = g_{ik} \tilde{g}^{jk}.
\]
In fact, $h$ can be thought to be an endomorphism $h: T^{1,0}(M) \to T^{1,0}(M)$, such that $\tilde{g}(X, Y) = g(h(X), Y)$.

Set

$$S = \tilde{g}^{i\bar{i}}\tilde{g}^{j\bar{k}}\tilde{g}^{m\bar{l}}\phi_{j\bar{k}m}\phi_{\bar{l}s\bar{i}},$$

where $\phi_{j\bar{k}m} = \nabla_m \nabla_{\bar{k}} \nabla_j \phi$.

By (42), we have

$$\tilde{\theta} = \partial \tilde{g} \cdot \tilde{g}^{-1} = \partial (h \cdot g) \cdot g^{-1} h^{-1}$$

$$= \partial h \cdot g \cdot g^{-1} \cdot h^{-1} + h \cdot \partial g \cdot g^{-1} \cdot h^{-1}$$

$$= \partial h \cdot h^{-1} + h \cdot \theta \cdot h^{-1}$$

$$= \partial h \cdot h^{-1} + h \cdot \theta \cdot h^{-1} - \theta \cdot h \cdot h^{-1} + \theta$$

$$= \theta + (\nabla^{1,0} h) \cdot h^{-1}.$$  \hspace{1cm} (44)

$$\tilde{R} = \tilde{\partial} \tilde{\theta} = \tilde{\partial} (\theta + (\nabla^{1,0} h) \cdot h^{-1})$$

$$= R + \tilde{\partial} (\nabla^{1,0} h) \cdot h^{-1}.$$  \hspace{1cm} (45)

By similar computation, we can get

$$\theta = \partial g \cdot g^{-1} = \tilde{\theta} - h^{-1} (\tilde{\nabla}^{1,0} h),$$  \hspace{1cm} (46)

$$R = \tilde{R} - \tilde{\partial} (h^{-1} \cdot (\tilde{\nabla}^{1,0} h)).$$  \hspace{1cm} (47)

Now, using the definitions, one can see that

$$\phi_{j\bar{k}m} = (\nabla_m \tilde{g})(\partial_j, \tilde{\partial}_k) = \tilde{g}_{j\bar{k},m}.$$  

Thus,

$$S = \tilde{g}^{i\bar{i}}\tilde{g}^{j\bar{k}}\tilde{g}^{m\bar{l}}\phi_{j\bar{k}m}\phi_{\bar{l}s\bar{i}} = |\nabla^{1,0} \tilde{g}|^2_{\tilde{g}}.$$  \hspace{1cm} (48)

On the other hand,

$$\nabla_m \tilde{g} = \nabla_m (h \cdot g) = \nabla_m h \cdot g = \left( \frac{\partial}{\partial z^m} h + h \cdot \theta_m - \theta_m \cdot h \right) \cdot g,$$

so

$$\tilde{\nabla}_m h = \frac{\partial}{\partial z^m} h + h \cdot \tilde{\theta}_m - \tilde{\theta}_m \cdot h$$

$$= \frac{\partial}{\partial z^m} h + h \cdot \theta_m - \theta_m \cdot h + h \cdot (\nabla_m h) \cdot h^{-1} - \nabla_m h$$

$$= h \cdot (\nabla_m h) \cdot h^{-1}.$$
Thus,
\[ \nabla_m g = \nabla_m h \cdot g = h^{-1} \cdot (\tilde{\nabla}_m h) \cdot h \cdot g = h^{-1} \cdot (\tilde{\nabla}_m h) \cdot \tilde{g}. \]

Finally we end up with the formula
\[ S = |\nabla^{1,0} \tilde{g}|_{\tilde{g}}^2 = |h^{-1} \cdot (\tilde{\nabla}^{1,0} h)|_{\tilde{g}}^2 = |\tilde{\theta} - \theta|_{\tilde{g}}^2 \] (49)
i.e. \( S \) can be thought as the \( \tilde{g} \)-norm of the difference between the two connection 1-forms.

Now, we can deduce the elliptic inequality:
\[
\tilde{\Delta} S = \tilde{\Delta} |h^{-1} \cdot (\tilde{\nabla}^{1,0} h)|_{\tilde{g}}^2 = \tilde{g}^{ij} \partial_i \partial_j |h^{-1} \cdot (\tilde{\nabla}^{1,0} h)|_{\tilde{g}}^2 = \tilde{g}^{ij} \partial_i \partial_j \tilde{g}^{ij} \tilde{\nabla}_j (h^{-1} \cdot (\tilde{\nabla}^{1,0} h)) \tilde{h}^{-1} \cdot (\tilde{\nabla}^{1,0} h) \tilde{g} + |(h^{-1} \cdot (\tilde{\nabla}^{1,0} h)) \tilde{\nabla}_j \tilde{h}^{-1} \cdot (\tilde{\nabla}^{1,0} h) |_{\tilde{g}}^2 + \tilde{g}^{ij} \tilde{\nabla}_j (h^{-1} \cdot (\tilde{\nabla}^{1,0} h) \tilde{g}) \tilde{h}^{-1} \cdot (\tilde{\nabla}^{1,0} h) \tilde{g} + |\tilde{\nabla}^{1,0} (h^{-1} \cdot (\tilde{\nabla}^{1,0} h) \tilde{g})|_{\tilde{g}}^2 + |\tilde{\nabla}^{0,1} (h^{-1} \cdot (\tilde{\nabla}^{1,0} h) \tilde{g})|_{\tilde{g}}^2. \] (50)

Using the relation \( R = \tilde{R} - \tilde{\theta} (h^{-1} \cdot (\tilde{\nabla}^{1,0} h)) \), we have
\[
\tilde{g}^{ij} \tilde{\nabla}_i \tilde{\nabla}_j (h^{-1} \cdot (\tilde{\nabla}^{1,0} h) \tilde{g}) \tilde{h}^{ij} = \tilde{g}^{ij} \tilde{\nabla}_i (\tilde{R}^{ij}_{m} - R^{ij}_{m}). \] (51)

Recall the Bianchi identities of curvature forms which can be found in [11] (p. 135):
\[
\sum (R(X, Y) Z) = \sum T(T(X, Y), Z) + (\nabla_X T)(Y, Z); \quad (52)
\]
\[
\sum \{\nabla_X R(Y, Z) + R(T(X, Y), Z) \} = 0, \quad (53)
\]
where \( X, Y, Z \in T M \) and \( T \) is the torsion of the connection \( \nabla \) (recall that \( \nabla \) is not necessarily the Levi-Civita connection), while \( \sum \) denotes the cyclic sum with respect to \( X, Y, Z \).

By the first Bianchi identity (52), one obtains
\[
\tilde{R}(\partial_i, \partial_j) \partial_m + \tilde{R}(\partial_j, \partial_m) \partial_i + \tilde{R}(\partial_m, \partial_i) \partial_j = \tilde{T}(\tilde{\nabla}_i \tilde{T}(\partial_i, \partial_j) \partial_m + \tilde{T}(\tilde{\nabla}_j \tilde{T}(\partial_j, \partial_m) \partial_i + \tilde{T}(\tilde{\nabla}_m \tilde{T}(\partial_m, \partial_i) \partial_j
\]
\[ + (\tilde{\nabla}_i \tilde{T})(\partial_j, \partial_m) + (\tilde{\nabla}_j \tilde{T})(\partial_m, \partial_i) + (\tilde{\nabla}_m \tilde{T})(\partial_i, \partial_j). \]

Recall the fact that \( \tilde{R}^{2,0} = \tilde{R}^{0,2} = 0, \tilde{T}^{1,1} = 0 \) (since \( \tilde{\nabla} \) is the Chern connection) and \( \tilde{T}(\partial_m, \partial_i) \in T^{1,0}(M) \). Also
\[ \tilde{T}(\partial_i, \partial_j) = \tilde{T}(\partial_j, \partial_m) = (\tilde{\nabla}_i \tilde{T})(\partial_j, \partial_m) = (\tilde{\nabla}_m \tilde{T})(\partial_i, \partial_j) = 0, \]
\[ \tilde{R}(\partial_m, \partial_i) \partial_j = 0. \]

Thus,
\[ \tilde{R}(\partial_i, \partial_j) \partial_m = \tilde{R}(\partial_j, \partial_m) \partial_i = (\tilde{\nabla}_j \tilde{T})(\partial_m, \partial_i). \]

By definition \( \tilde{R}(\partial_i, \partial_j) \partial_m = \tilde{R}^l_{mi,j} \partial_l \) and \( \tilde{R}^l_{mi,j} = -\tilde{R}^l_{mj,i} \), so we get
\[ \tilde{R}^l_{mi,j} = \tilde{R}^l_{im,j} + \tilde{T}^l_{mi,j}. \] (54)

Similarly, one can also obtain
\[ \tilde{R}^l_{kj,i} = \tilde{R}^l_{jik} + \tilde{T}^l_{jik}. \] (55)

Moreover, by the second Bianchi identity (53) and following the same step as above we have
\[ \tilde{R}^l_{mt,j,i} + \tilde{R}^l_{mji,t} + \tilde{R}^l_{mit,j} = -\tilde{R}(\tilde{T}(\partial_i, \partial_t), \partial_j) - \tilde{R}(\tilde{T}(\partial_j, \partial_t), \partial_i) - \tilde{R}(\tilde{T}(\partial_j, \partial_i), \partial_t) \]
and \( \tilde{R}^l_{mit,j} = 0, \tilde{T}(\partial_i, \partial_j) = \tilde{T}(\partial_j, \partial_i) = 0. \) Thus,
\[ \tilde{R}^l_{mi,j,t} = \tilde{R}^l_{mj,i} + \tilde{T}^l_{it} \tilde{R}^l_{ms,j}. \] (56)

Now, using the identities (54), (55) and (56), we obtain
\[ g^{ij} \tilde{\nabla}_i \tilde{R}^l_{mt,j} = g^{ij} \tilde{R}^l_{mi,j,t} = g^{ij} \tilde{R}^l_{mi,j} - g^{ij} \tilde{T}^s_{it} \tilde{R}^l_{ms,j} \]

\[ = g^{ij} \tilde{R}_{mki,t} g^{lk} - g^{ij} \tilde{T}^s_{it} \tilde{R}^l_{ms,j} \]

\[ = g^{ij} \tilde{R}_{kmj,t} + \tilde{T}^s_{mi,jt} g^{lk} - g^{ij} \tilde{T}^s_{it} \tilde{R}^l_{ms,j} \]

\[ = -g^{ij} \tilde{R}_{kim,t} g^{lk} + g^{ij} \tilde{T}^s_{mi,jt} - g^{ij} \tilde{T}^s_{it} \tilde{R}^l_{ms,j} \]

\[ = -g^{ij} \tilde{R}_{kim,t} g^{lk} - g^{ij} \tilde{T}^l_{jk,mi} g^{il} g^{lk} + g^{ij} \tilde{T}^l_{mi,jt} - g^{ij} \tilde{T}^s_{it} \tilde{R}^l_{ms,j} \]

\[ = \tilde{R}^l_{imk,t} g^{lk} - g^{ij} \tilde{T}^l_{jk,mi} g^{il} g^{lk} + g^{ij} \tilde{T}^l_{mi,jt} - g^{ij} \tilde{T}^s_{it} \tilde{R}^l_{ms,j}. \] (57)

From the Monge–Ampère equation (2), it follows that
\[ \tilde{R}^l_{imk,t} = \tilde{\nabla}_t R^l_{imk} - \tilde{\nabla}_t f_{mk}. \] (58)
In the following, we denote $\epsilon = O(S^\alpha)$ if there is a constant $C$ depending only on $K$, $|d\omega|_o$, $|R|_o$, $|\nabla R|_o$, $|T|_o$, $|\nabla T|_o$ and $|\nabla f|_o$, $s = 0, 1, 2, 3$, such that $\epsilon \leq C S^\alpha$. Note that $\tilde{V}$ is $O(S^{1/2})$, so

$$\tilde{R}_{i m k, i t}^{l} \tilde{g}_{i j}^{l k} = O(S^{1/2}) + O(1).$$

(59)

For the second term in (57)

$$\tilde{T}_{j k, m t}^{l} = ((\partial_{j} g_{n k} - \partial_{k} g_{n j}) \tilde{g}^{n i})_{m t} = (T_{j k n} \tilde{g}^{n i})_{m t} = \tilde{V}_{i} \tilde{V}_{m} T_{j k n} \tilde{g}^{n i} = \tilde{V}_{i} (\nabla_{m} T_{j k n} - (\tilde{\theta}_{m} - \tilde{\theta}_{m})_{n} T_{j k l}) \tilde{g}^{n i} = (\nabla_{i} (\nabla_{m} T_{j k n}) - (\tilde{\theta}_{i} - \tilde{\theta}_{i})_{m} \nabla_{j} T_{j k n} - \tilde{V}_{i} ((\tilde{\theta}_{m} - \tilde{\theta}_{m})_{n} T_{j k l}) - (\tilde{\theta}_{i} - \tilde{\theta}_{i})_{m} \nabla_{i} T_{j k l} - (\tilde{\theta}_{m} - \tilde{\theta}_{m})_{n} (\nabla_{i} T_{j k l} - (\tilde{\theta}_{i} - \tilde{\theta}_{i})_{l} T_{j k i})) \tilde{g}^{n i}. \tag{60}$$

Again, by the fact that $\tilde{V}$ is $O(S^{1/2})$ and $|h^{-1} \cdot (\tilde{V}^{1,0} h)|_g$ is also $O(S^{1/2})$, we have

$$\left| \tilde{g}_{i j}^{l} \tilde{T}_{j k, m t}^{l} \tilde{g}_{i j}^{l} \tilde{g}_{i j}^{l} \right| \leq O(S^{1/2}) + O(S) + C(\tilde{V}^{0,1} (h^{-1} \cdot (\tilde{V}^{1,0} h))) + O(1). \tag{61}$$

Similarly, we can get the estimate for the last two terms in (57)

$$\left| \tilde{g}_{i j}^{l} \tilde{T}_{m i, j t}^{l} \tilde{g}_{i j}^{l} \tilde{g}_{i j}^{l} \right| \leq O(S^{1/2}) + O(S) + C(\tilde{V}^{0,1} (h^{-1} \cdot (\tilde{V}^{1,0} h))) + O(1), \tag{62}$$

$$\left| \tilde{g}_{i j}^{l} \tilde{T}_{m i, j t}^{l} \tilde{g}_{i j}^{l} \tilde{g}_{i j}^{l} \right| \leq C(\tilde{V}^{0,1} (h^{-1} \cdot (\tilde{V}^{1,0} h))) + O(1). \tag{63}$$

Putting the above estimates (57)–(63) into (51), we can conclude that

$$\left| \tilde{g}_{i j}^{l} \tilde{V}_{i} \tilde{V}_{j} (h^{-1} \cdot (\tilde{V}^{1,0} h))_{m} \right| \leq O(S^{1/2}) + O(S) + C(\tilde{V}^{0,1} (h^{-1} \cdot (\tilde{V}^{1,0} h))) + C(\tilde{V}^{0,1} (h^{-1} \cdot (\tilde{V}^{1,0} h))), \tag{64}$$

One the other hand,

$$\tilde{g}_{i j}^{l} \tilde{V}_{i} \tilde{V}_{j} (h^{-1} \cdot (\tilde{V}^{1,0} h)) = \tilde{g}_{i j}^{l} \tilde{V}_{i} \tilde{V}_{j} (h^{-1} \cdot (\tilde{V}^{1,0} h)) - (\tilde{g}_{i j}^{l} \tilde{R}_{m i j}) \# (h^{-1} \cdot (\tilde{V}^{1,0} h))$$

where

$$\tilde{g}_{i j}^{l} \tilde{R}_{m i j} \# (h^{-1} \cdot (\tilde{V}^{1,0} h)) = \tilde{g}_{i j}^{l} \{ h^{-1} \cdot (\tilde{V}^{1,0} h)^{s}_{m} \tilde{R}_{s i j} - h^{-1} \cdot (\tilde{V}^{1,0} h)^{l}_{m} \tilde{R}_{i i j} - h^{-1} \cdot (\tilde{V}^{1,0} h)^{l}_{m} \tilde{R}_{i i j} \} \times d z' \otimes d z'' \otimes \frac{\partial}{\partial z'} \otimes \frac{\partial}{\partial z''}.$$
and
\[
\tilde{g}^{ij} \tilde{R}_{mij} = \tilde{g}^{ij} \tilde{R}_{imj} + \tilde{g}^{ij} \tilde{T}_{mij} = \tilde{g}^{ij} \tilde{R}_{imj} g^{lk} + \tilde{g}^{ij} \tilde{T}_{jk,m} g_{kl} + \tilde{g}^{ij} \tilde{R}_{mi,j}.
\]

Thus
\[
|\tilde{g}^{ij} \tilde{R}_{mij}| \leq O(S^2) + O(1).
\]

Hence we conclude that
\[
\left| \tilde{g}^{ij} \tilde{\nabla}_i \tilde{\nabla}_j (h^{-1} \cdot (\tilde{\nabla}^{1,0} h)) \right| \\
\leq |\tilde{g}^{ij} \tilde{\nabla}_j (h^{-1} \cdot (\tilde{\nabla}^{1,0} h))| + |(\tilde{g}^{ij} \tilde{R}_{mij})\#(h^{-1} \cdot (\tilde{\nabla}^{1,0} h))| \\
\leq O(S^{1/2}) + O(S) + C |\tilde{\nabla}^{1,0} (h^{-1} \cdot (\tilde{\nabla}^{1,0} h))| + C |\tilde{\nabla}^{0,1} (h^{-1} \cdot (\tilde{\nabla}^{1,0} h))|.
\] (65)

Finally, by (50) and (64), (65), we obtain the elliptic inequality:
\[
\tilde{\Delta} S \geq -C_1 S^{3/2} - C_2
\] (66)

where \(C_1, C_2\) are positive constants depending only on \(K, |d\omega|_\omega, |R|_\omega, |\nabla R|_\omega, |T|_\omega, |\nabla T|_\omega\) and \(|\nabla^s f|_\omega\), \(s = 0, 1, 2, 3\). □

References