Note

On Symmetric Designs with Parameters \((176, 50, 14)\)

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In [1] Graham Higman has discovered the unique symmetric design \(D\) with parameters \((176, 50, 14)\) whose full automorphism group is the Higman-Sims simple group \(G\) of order 44352000. It turns out that \(G\) operates double transitively on points and on blocks of \(D\). It is of interest to take some large subgroup \(H\) of \(G\) and then determine all symmetric designs with parameters \((176, 50, 14)\) on which operates \(H\) as an automorphism group. We hope that we obtain in this way some new symmetric designs with these parameters which seem to be exceptional in some way. The assumptions of the following theorem are satisfied by \(G\) Higman design \(D\).

**Theorem.** We assume that the group \(H = (\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4) \cdot F_{21}\) (a faithful extension of the abelian group \(\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4\) of order 64 by the Frobenius group \(F_{21}\) of order 21) operates on a symmetric design with parameters \((176, 50, 14)\) acting in two orbits of lengths 64 (with the stabilizer isomorphic to \(F_{21}\)) and 112 (with the stabilizer isomorphic to a cyclic group \(\mathbb{Z}_{12}\) of order 12) on points and blocks. Then there exist up to isomorphism exactly two such designs: the Graham Higman design \(D\) and the new such symmetric design \(D'\) whose full automorphism group is just the group \(H\) itself (which is a second-maximal subgroup of the Higman-Sims simple group). Both designs are self-dual and we find them here explicitly.

**Proof.** We assume that the above group \(H\) (which is a solvable subgroup of the Higman-Sims simple group \(G\)) acts as an automorphism group on a symmetric design with parameters \((176, 50, 14)\). In terms of generators and relations the group \(H\) is given by:

\[
H = \langle a, b, c, d, e \mid a^7 = 1, b^4 = 1, c^4 = 1, d^4 = 1, bc = cb, bd = db, cd = dc, a^{-1}ba = c, a^{-1}ca = d, a^{-1}da = bc^{-1}d^2, e^3 = 1, e^{-1}ae = a^2, eb = be \rangle.
\]
Here we set: \( \langle b, c, d \rangle = Z_4 \times Z_4 \times Z_4, \langle a, e \rangle = F_{21} \) and \( \langle e, b \rangle = Z_{12} \). The subgroups \( F_{21} \) and \( Z_{12} \) are stabilizers of \( H \)-orbits of lengths 64 and 112 respectively on points and blocks. The group \( H \) has order 1344 and is obviously also generated with \( a, b, e \).

The \( H \)-orbits of points have lengths \( m_1 = 64, m_2 = 112 \), and the \( H \)-orbits of blocks have lengths \( n_1 = 64 \) and \( n_2 = 112 \). The \( H \)-orbit matrix (in the sense of [2]) turns out to be unique:

\[
M = \begin{pmatrix} 22 & 28 \\ 16 & 34 \end{pmatrix}
\]

We denote the points of the \( H \)-orbit of length 64 (64-points orbit) with \( 1, 2, 3, ..., 64 \) and the points of the \( H \)-orbit of length 112 (112-points orbit) with \( 1', 2', 3', ..., 112' \). Then the generating permutations \( a, b, e \) of \( H \) on the above 176 points are given here explicitly:

\[
a = (1)(2\ 5\ 8\ 20\ 23\ 24\ 25)(3\ 6\ 9\ 35\ 36\ 37\ 38)
(4\ 7\ 10\ 40\ 48\ 49)(11\ 17\ 21\ 53\ 26\ 57\ 31)
(12\ 18\ 22\ 58\ 62\ 51\ 42)(13\ 19\ 30\ 14\ 27\ 34\ 54)
(15\ 59\ 41\ 29\ 28\ 63\ 43)(16\ 32\ 33\ 39\ 44\ 61\ 45)
(46\ 56\ 52\ 60\ 50\ 55\ 64)
(1'\ 2'\ 3'\ 4'\ 5'\ 6'\ 7')(8'\ 25'\ 43'\ 53'\ 94'\ 95'\ 17')
(9'\ 26'\ 98'\ 52'\ 59'\ 74'\ 83')(10'\ 27'\ 42'\ 57'\ 102'\ 79'\ 84')
(11'\ 31'\ 103'\ 104'\ 67'\ 58'\ 18')(12'\ 23'\ 38'\ 50'\ 70'\ 71'\ 85')
(13'\ 105'\ 41'\ 54'\ 69'\ 72'\ 86')(14'\ 32'\ 106'\ 46'\ 63'\ 78'\ 88')
(15'\ 22'\ 37'\ 49'\ 66'\ 81'\ 89')(16'\ 28'\ 34'\ 47'\ 64'\ 80'\ 90')
(19'\ 20'\ 107'\ 109'\ 55'\ 33'\ 73')(21'\ 30'\ 110'\ 44'\ 61'\ 76'\ 92')
(24'\ 39'\ 51'\ 68'\ 112'\ 108'\ 100')(29'\ 40'\ 56'\ 60'\ 75'\ 87'\ 96')
(35'\ 45'\ 62'\ 77'\ 91'\ 99'\ 101')(36'\ 48'\ 65'\ 82'\ 93'\ 97'\ 111')
\]

\[
b = (1\ 2\ 3\ 4)(5\ 11\ 12\ 13)(6\ 26\ 35\ 50)(7\ 22\ 34\ 46)
(8\ 14\ 15\ 16)(9\ 27\ 38\ 42)(10\ 28\ 39\ 51)(17\ 29\ 40\ 52)
(18\ 25\ 31\ 53)(19\ 30\ 41\ 24)(20\ 33\ 44\ 55)(21\ 23\ 45\ 56)
(32\ 37\ 63\ 36)(43\ 47\ 57\ 59)(48\ 61\ 62\ 58)(49\ 54\ 60\ 64)
(1')(2'\ 22'\ 23'\ 24')(3'\ 34'\ 35'\ 36')(4'\ 44'\ 45'\ 46')
(5'\ 33'\ 59'\ 60')(6'\ 58'\ 71'\ 72')(7'\ 17'\ 83'\ 84')(8')
(9')(10')(11')(12')(13')(14')(15')(16')
(18'\ 88'\ 89'\ 90')(19'\ 85'\ 87'\ 91')(20')(21')
\]
Denote with \( l_1 \) the \( F_{21} \)-invariant basic block of our design which corresponds to the first row of the orbit matrix \( M \). Hence the first 22 points on \( l_1 \) are from 64-points orbit and the last 28 points on \( l_1 \) are from 112-points orbit. Also denote with \( l_2 \) the \( Z_{12} \)-invariant basic block of our design which corresponds to the second row of \( M \). Hence the first 16 points on \( l_2 \) are from 64-points orbit and the last 34 points on \( l_2 \) are from 112-points orbit. Using the fact that any two blocks intersect in exactly 14 points we get exactly four possibilities (four designs) \( D_1, D_2, D_3, D_4 \) for the basic blocks \( l_1 \) and \( l_2 \).

However, it is easy to find the unique permutation \( f \) of degree 176 which centralizes \( F_{21} \) and acts invertingly on \( \langle b, c, d \rangle \). Hence \( f \) normalizes the group \( H \) and we see that \( f \) establishes an isomorphism between designs \( D_1 \) and \( D_2 \) and also between designs \( D_3 \) and \( D_4 \).
We give here explicitly the basic blocks $l_1$ and $l_2$.

For the design $D_1$ (Higman design) we have:

$$l_1 = \{ 1 \ 2 \ 5 \ 8 \ 20 \ 23 \ 24 \ 25 \ 3 \ 6 \ 9 \ 35 \ 36 \ 37 \ 38 \ 4 \ 7 \ 10 \ 47 \ 40 \ 48 \ 49 \ 1' \ 2' \ 3' \ 4' \ 5' \ 6' \ 7' \ 8' \ 25' \ 43' \ 53' \ 94' \ 95' \ 17' \ 11' \ 31' \ 103' \ 104' \ 67' \ 58' \ 18' \ 21' \ 30' \ 110' \ 44' \ 61' \ 76' \ 92' \}.$$  

$$l_2 = \{ 1 \ 2 \ 3 \ 4 \ 5 \ 11 \ 12 \ 13 \ 14 \ 15 \ 16 \ 21 \ 23 \ 45 \ 56 \ 1' \ 8' \ 11' \ 21' \ 9' \ 12' \ 99' \ 10' \ 13' \ 14' \ 27' \ 32' \ 107' \ 111' \ 37' \ 40' \ 41' \ 42' \ 63' \ 69' \ 64' \ 68' \ 51' \ 56' \ 104' \ 57' \ 75' \ 82' \ 95' \ 78' \ 86' \ 93' \ 108' \ 92' \}.  $$

For the design $D_3$ (new design) we have:

$$l_1 = \{ 1 \ 2 \ 5 \ 8 \ 20 \ 23 \ 24 \ 25 \ 3 \ 6 \ 9 \ 35 \ 36 \ 37 \ 38 \ 4 \ 7 \ 10 \ 47 \ 40 \ 48 \ 49 \ 1' \ 2' \ 3' \ 4' \ 5' \ 6' \ 7' \ 15' \ 22' \ 37' \ 49' \ 66' \ 81' \ 89' \ 16' \ 28' \ 34' \ 47' \ 64' \ 80' \ 90' \ 19' \ 20' \ 107' \ 109' \ 55' \ 33' \ 73' \},$$  

$$l_2 = \{ 1 \ 2 \ 3 \ 4 \ 18 \ 25 \ 31 \ 53 \ 19 \ 30 \ 41 \ 24 \ 20 \ 33 \ 44 \ 55 \ 1' \ 9' \ 12' \ 99' \ 15' \ 16' \ 20' \ 96' \ 100' \ 97' \ 27' \ 32' \ 107' \ 111' \ 37' \ 40' \ 41' \ 42' \ 63' \ 69' \ 64' \ 68' \ 51' \ 56' \ 104' \ 57' \ 75' \ 82' \ 95' \ 78' \ 86' \ 93' \ 108' \ 92' \}.  $$

It remains to examine the designs $D_1$ and $D_3$. In case of the design $D_1$ any three blocks intersect in exactly 3, 4, or 8 points and any type of intersection occurs. In case of the design $D_3$ any three blocks intersect in exactly 0, 1, 2, 3, 4, 5, 6, 7 or 8 points and any type of intersection occurs. Hence the designs $D_1$ and $D_3$ are non-isomorphic. Also the dual of $D_1$ has the same types of intersections of three blocks as $D_1$ itself. In the same way we see that the dual of $D_3$ has the same types of intersections as $D_3$ itself. This shows that the designs $D_1$ and $D_3$ are both self-dual.

We examine now which of the two designs could have possibly a doubly transitive automorphism group (on blocks and then also on points). In case of the design $D_1$ taking any two fixed blocks $k_1$ and $k_2$ we examine the
174 intersections \( k_1 \cap k_2 \cap k \), where \( k \) is any other block different from \( k_1 \) and \( k_2 \). We see that 72 such intersections contain exactly 3 points, 90 such intersections contain exactly 4 points and finally 12 such intersections contain exactly 8 points. This result does not depend on the choice of the blocks \( k_1 \) and \( k_2 \). Hence the design \( D_1 \) could have a doubly transitive automorphism group. In case of the design \( D_3 \) we examine on one hand all intersections \( l_1 \cap k_1 \cap k_2 \) of the basic block \( l_1 \) with any other unordered pair of blocks \( k_1 \) and \( k_2 \) different from \( l_1 \). On the other hand we examine all intersections \( l_2 \cap k_1 \cap k_2 \) of the basis block \( l_2 \) with any other unordered pair of blocks \( k_1 \) and \( k_2 \) different from \( l_2 \). The intersection numbers obtained in these two cases are different. This shows that the full automorphism group of the design \( D_3 \) does not act (even) transitively on blocks of \( D_3 \).

Hence the design \( D_1 \) must be isomorphic to the Graham Higman design \( D \) whose full automorphism group is the Higman-Sims simple group. On the other hand the design \( D_3 = D' \) must be new. The full automorphism group of \( D' \) (using a computer program for calculating an automorphism group of a symmetric block design) has order 1344 and so must be isomorphic to our group \( H \). This completes the proof of our theorem.

REFERENCES

2. Z. JANKO, Coset enumeration in groups and constructions of symmetric designs, Combinatorics 90 (1992), 275-277.