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Irreducible Odd Representations of $PSL(n, q)$

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Let $m_p(G)$ be the number of inequivalent, irreducible characters of group G whose degree is relatively prime to p. In [6] McKay tabulated $m_2(G)$, the number of odd degree characters, for certain simple groups and some infinite simple families of groups. From [4] several additions can be made to this list for the infinite families $PSL(3, q)$, $PSU(3, q^2)$ and $PSL(4, q)$ $d = 1$:

G	d	$q = p^t$, p a prime	$m_2(G)$
$PSL(3, q)$	3	even ($q = 2^t$ t even)	$\frac{1}{3}(q^2 + 8)$
	3	odd ($q \equiv 1 \pmod 6$)	$\frac{2}{3}(q - 1)$
	1	odd	$2(q - 1)$
	1	even ($q = 2^t$, t odd)	q^2
$PSU(3, q^2)$	3	even ($q = 2^t$, t odd)	$\frac{1}{3}(q^2 + 8)$
	3	odd ($q \equiv 5 \pmod 6$)	$\frac{2}{3}(q + 1)$
	1	odd	$2(q + 1)$
	1	even ($q = 2^t$ t even)	q^2
$PSL(4, q)$	1	even ($q = 2^t$)	q^3

$$(d = n, q + \delta) \qquad \delta = \begin{cases} -1 & \text{for} \quad G = PSL(n, q) \\ +1 & \text{for} \quad G = PSU(n, q^2) \end{cases}$$

It has been conjectured (Bannai–Enomoto) that if L is a complex, simple Lie algebra of rank l and G is the group defined by Chevalley and constructed from L over $GF(2^t)$ then $m_2(G) = 2^{tl}$. From [6] and the above table we note that $m_2(PSL(n, 2^t), d = 1) = q^{n-1} = 2^{t(n-1)}$ $n = 2, 3, 4$ which lends support to the conjecture.

Using the method described by McKay [6] we shall prove the following theorem.

THEOREM 1. *$m_p(PSL(n, q), d = 1) = q^{n-1}$ where $q = p^t$.*

An immediate corollary is the specific case of the Bannai–Enomoto conjecture noted above:

COROLLARY. *$m_2(PSL(n, 2^t), d = 1) = 2^{t(n-1)}$.*

Proof of Theorem 1. Let

$$\alpha = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & 1\ 1 \end{pmatrix} \in PSL(n, q)$$

when $d = (n, q - 1) = 1$, $q = p^t$. We can calculate the order of the centralizer of α in $GL(n, q)$ using formulas found in several papers. (The formula given in [1] for $U(n, q^2)$ can be used for $GL(n, q^2)$ with a couple very minor changes). We find that $|\, C(\alpha)|_{GL} = q^{n-1}(q - 1)$. Now since $d = 1$, $|\, C(\alpha)|_{SL} = |\, C(\alpha)|_{GL}/(q - 1) = q^{n-1}$ and $|\, C(\alpha)|_{PSL} = q^{n-1}$.

The order of α is a power of the characteristic of $GF(p^t)$ so by the repeated use of the congruence relation established by Frame [3], we get that $\chi_i(\alpha) \equiv \chi_i(1) \bmod p$ for all characters χ_i of $PSL(n, q)$.

If we can show that $\chi_i(\alpha) = \pm 1$ or 0 $\forall i$, we are done since $\chi_i(\alpha) \equiv \chi_i(1) \bmod p$ implies $\chi_i(\alpha) = \pm 1$ if and only if the degree of χ_i is relatively prime to p, and thus $m_p(PSL(n, q)) = (\chi_i(\alpha), \chi_i(\alpha)) = |\, C(\alpha)|_{PSL} = q^{n-1}$. To show that $\chi_i(\alpha) = \pm 1$ or 0 we look at the characters ψ_j of $GL(n, q)$. Since $d = 1$, we can obtain the irreducible characters of $PSL(n, q)$ from those of $GL(n, q)$ without splitting any character or conjugacy classes of GL. Thus if $\psi_j(\alpha) = \pm 1$ or 0 then $\chi_i(\alpha) = \pm 1$ or 0. The fact that we need only examine the characters of GL is advantageous because Green in [5] develops a method of constructing the character table of $GL(n, q)\, \forall n, q$ from certain 'primary' characters. We shall show that using Green's procedure to calculate the entries $\psi_j(\alpha)\, \forall j$ will always result in ± 1 or 0.

In order to conserve space, all necessary definitions and theorems from [5] will be referenced by page rather than restated here.

Denote by u_n the $n \times n$ matrix

$$\begin{pmatrix} u & & & & \\ u & u & & & \\ & & \ddots & & \\ & & & u & \\ & & & u & u \end{pmatrix} \quad u \in GF(q)$$

and let its conjugacy class be c.

Let $n = n_1 + n_2 + \cdots + n_k$ be a partition of n into positive integers n_i and let a_i be a character of $GL(n_i, q)$ for $i = 1, 2,..., k$. On page 403 an operation called the '$\circ$-product' is defined. This operation enables us to construct characters of $GL(n, q)$ from those of $GL(n_i, q)$ by an inducing process. Such a character of $GL(n, q)$ is called an '$\circ$-product' and is denoted by $a_1 \circ a_2 \circ \cdots \circ a_k$. The value of any $\circ$-product for $GL(n, q)$ is particularly simple on the class c. Using [Theorem 2 (p. 410)] we find that

$$a_1 \circ a_2 \circ \cdots \circ a_k(u_k) = a_1(u_{n_1}) \cdot a_2(u_{n_2}) \cdots a_k(u_{n_k})$$

i.e., it is an ordinary product of characters.

By [Theorem 14 (p. 443)] we know that every irreducible character of $GL(n, q)$ can be expressed as an $\circ$-product of certain 'primary characters'. Since these $\circ$-products for the class c are ordinary products of the primary characters, we need only show that the primary characters all have the value ± 1 or 0 on c.

Take a divisor d of n and let $v = n/d$. Consider the multiplicative group of $GF(q^d)$. It is abelian so all its characters are linear and thus form a group, X_d, under multiplication. Since $| X_d | = q^d - 1$, the map $\psi \to \psi^q$ of X_d into itself is a permutation of order d which divides X_d into orbits, the length of each orbit being a divisor of d.

Take any such orbit of length d. The set $\{\psi\} = \{\psi, \psi^q,..., \psi^{q^{d-1}}\}$ we call a 'd-simplex'.

For any d-simplex $\{\psi\}$ and any partition λ of v we can construct a 'primary' irreducible character, $J(\psi, \lambda)$, of $GL(n, q)$ and all such primary irreducible characters can be constructed for a suitable choice of d, $\{\psi\}$, and λ. (Notational remark: $J(\psi, \lambda)$ is denoted in [5, p. 439] by (g^λ) where g denotes the d-simplex $k, kq,..., kq^{d-1}$. The ψ and k are related by the fact that if θ is the generator of $X_{n!}$, then ψ is the restriction of θ^k to $GF(q^d)$.)

Each primary irreducible character $J(\psi, \lambda)$ is composed of independent functions called 'principle parts' and denoted by U_ρ. By [5, Theor. 12, p. 439] we see that $U_\rho = 0$ unless ρ is a partition of n of the form

$\rho = (d^{p_1}, (2d)^{p_2} \cdots)$ where $\pi = (1^{p_1}, 2^{p_2} \cdots)$ is a partition of v. For ρ of this form we get:

$$U_\rho(\xi^\rho) = (-1)^{(d-1)v} \chi_\pi{}^\lambda \prod_e \prod_{i=1}^{e} \prod_{j=0}^{d-1} \psi^{q^j}(N_{de;d}\xi_{de,i}) \tag{1}$$

A 'principle class of type ρ' is a conjugacy class whose characteristic polynomial $F(t)$ has r_d factors of degree d ($d = 1, 2, \ldots, n$), where ρ is the partition $\{1^{r_1}, 2^{r_2}, \cdots n^{r_n}\}$ of $n = r_1 + 2r_2 + \cdots + nr_n$ [5, p. 407]. ξ^ρ is the set of eigenvalues of a typical principle class of type ρ; e.g., $\xi_{de,i}$ is the root of this class, which happens to have degree de. ($\xi_{de,i}$ is a root of the irreducible polynomial $GF(q)[t]$ of degree de). We define $N_{de;d}(x)$ to be the product $x \cdot x^{q^d} \cdot x^{q^{2d}} \cdots x^{q^{(e-1)d}}$ which lies in $GF(q^d)$ for x being any nonzero element of $GF(q^{de})$.

$\chi_\pi{}^\lambda$ denotes a character of the symmetric group S_v in standard notation.

To calculate $J(\psi, \lambda)$ on the class c we must know how to combine the principle parts U_ρ. This is given by the 'degeneracy rule' [5, p. 423, (18)]. To use this rule we need the 'modes of substitution' [5, p. 422] of the ρ variables into u_n for each partition ρ of n. The fact that all the eigenvalues of u_n are the same makes this calculation much easier.

Write $f_u(t) = t - u$, a linear polynomial in t over $GF(q)$. Using the notation established in [4, p. 420] we get the following for a fixed ρ.

(i) There is exactly one substitution of the ρ variables χ^ρ into the class c of u_n; it takes each variable to f_u; or in terms of the ρ eigenvalues $\xi_{de,i}$, each eigenvalue is taken to u.

(ii) If m is the mode of this substitution, then $\rho(m, f_u) = \rho$.

(iii) $v_c(f_u) = \{n\}$, the partition whose only part is n.

(iv) $Q_\rho^{\{n\}}(q) = 1$ for all ρ (see [4, p. 455]).

Putting (i)–(iv) into (18) and using U_ρ given by (1) of this paper, we can calculate the value of the primary characters on the class c:

$$J(\psi, \lambda)(u_n) = (-1)^{(d-1)v} \Big(\sum_{\pi | v} \frac{1}{z_\pi} \chi_\pi{}^\lambda\Big) \cdot \psi^v(u)$$

$$= \begin{cases} (-1)^{(d-1)v}\psi^v(u) & \text{if } \lambda = \{v\} \\ 0 & \text{if } \lambda \neq \{v\} \end{cases}$$

Recalling that ψ is a character of the abelian multiplicative group X_d of $GF(q^d)$ so that $\psi(1) = 1$, we see that the substitution of 1 in for u above gives:

$$J(\psi, \lambda)(1_n) = \pm 1 \quad \text{or} \quad 0 \ \forall n.$$

Q.E.D.

In [2] and [4] it was conjectured that the character tables for $U(n, q^2)$, $SU(n, q^2)$, $PSU(n, q^2)$ can be obtained from the tables of $GL(n, q)$, $SL(n, q)$, $PSL(n, q)$ respectively by the simple means of replacing q everywhere by $-q$ and multiplying each character by -1 if necessary to keep the degree positive. For $n = 2, 3$ this conjecture was verified. If the above is true, then

$$m_q(PSU(n, q^2)) = q^{n-1} \qquad \text{if} \quad d = (n, q+1) = 1.$$

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