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Asymptotic Equilibrium and Stability of Fuzzy Differential Equations

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Abstract—The local existence and uniqueness theorems and the global existence of solutions were investigated in [1–3], respectively, for the Cauchy problem of fuzzy-valued functions of a real variable whose values are in the fuzzy number space (E^n, D). In this paper, we first study the asymptotic equilibrium for fuzzy evolution equations. Then, the stability properties of the trivial fuzzy solution of the perturbed semilinear fuzzy evolution equations are investigated by extending the Lyapunov's direct method.

Keywords—Fuzzy number space E^n, Fuzzy differential equations, Asymptotic equilibrium, Lyapunov functions, Lyapunov stability.

1. INTRODUCTION

Using the H-differentiability due to Puri and Ralescu [4], fuzzy differential equations were studied by Kaleva [5,6] and Wu and Song [1–3], for the fuzzy-valued functions of a real variable whose values are normal, convex, upper semicontinuous, and are compactly supported fuzzy sets in R^n. The local existence and uniqueness theorems, for the Cauchy problem $x'(t) = f(t,x)$, $x(t_0) = x_0$ when the fuzzy valued function f satisfies the generalized Lipschitz condition, were given in [1]. The existence theorems under compactness-type conditions were studied in [2]. Based on these preceding works, the global existence of solutions of the Cauchy problem were investigated in [3]. This paper is devoted to the investigation of the asymptotic behavior and stability of fuzzy differential equations. In Section 2, some preliminaries concerning fuzzy number space, integrability, and differentiability [5] for fuzzy-valued functions are summarized and the comparison theorem for classical ordinary differential equations [7] are listed. The asymptotic equilibrium for fuzzy evolution equations are investigated in Section 3. Finally, in Section 4, we study the stability properties of perturbed semilinear fuzzy evolution equations by extending the Lyapunov's direct method.

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2. PRELIMINARIES

Let $P_k(R^n)$ denote the family of all nonempty compact convex subsets of R^n and define the addition and scalar multiplication in $P_k(R^n)$ as usual. Denote

$$E^n = \{u : R^n \to [0,1] \mid u \text{ satisfies (i)–(iv) below}\},$$

where

(i) u is normal, i.e., there exists an $x_0 \in R^n$, such that $u(x_0) = 1$,
(ii) u is fuzzy convex, i.e., $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in R^n$ and $0 \leq \lambda \leq 1$,
(iii) u is upper semicontinuous,
(iv) $[u]^0 = \mathrm{cl}\{x \in R^n \mid u(x) > 0\}$ is compact.

For $0 < \alpha \leq 1$, denote $[u]^\alpha = \{x \in R^n \mid u(x) \geq \alpha\}$. Then from (i)–(iv), it follows that the α-level set $[u]^\alpha \in P_k(R^n)$ for all $0 \leq \alpha < 1$.

According to Zadeh's extension principle, we have addition and scalar multiplication in fuzzy number space E^n as follows:

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha, \qquad [ku]^\alpha = k[u]^\alpha,$$

where $u, v \in E^n$, $k \in R$, and $0 \leq \alpha \leq 1$.

Define $D : E^n \times E^n \to [0, \infty)$ by the equation

$$D(u,v) = \sup_{0 \leq \alpha \leq 1} d([u]^\alpha, [v]^\alpha),$$

where d is the Hausdorff metric defined in $P_k(R^n)$. Then it is easy to see that D is a metric in E^n. Using the results in [8], we have

(1) (E^n, D) is a complete metric space;
(2) $D(u+w, v+w) = D(u,v)$ for all $u, v, w \in E^n$;
(3) $D(ku, kv) = |k| D(u,v)$ for all $u, v \in E^n$ and $k \in R$.

Let $T = [t_0, t_0 + p] \subset R$ $(p > 0)$ be a compact interval. The fuzzy-valued function $F : T \to E^n$ is called strongly measurable, if for every $\alpha \in [0,1]$ the set-valued function $F_\alpha : T \to P_k(R^n)$ defined by

$$F_\alpha(t) = [F(t)]^\alpha$$

is Lebesgue measurable, where $P_k(R^n)$ is endowed with the topology generated by the Hausdorff metric d.

A function $F : T \to E^n$ is called integrably bounded, if there exists an integrable function h, such that $\|x\| \leq h(t)$ for all $x \in F_0(t)$.

DEFINITION 2.1. *Let $F : T \to E^n$.The integral of F over T, denoted by $\int_T F(t)\,dt$, is defined levelwise by the equation*

$$\left[\int_T F(t)\,dt\right]^\alpha = \int_T F_\alpha(t)\,dt = \left\{ \int_T f(t)\,dt \,\middle|\, f : T \to R^n \text{ is a measurable selection for } F_\alpha \right\},$$

for all $0 < \alpha \leq 1$.

A strongly measurable and integrably bounded function $F : T \to E^n$ is said to be integrable over T if $\int_T F(t)\,dt \in E^n$. From [5], we know that if $F : T \to E^n$ is continuous, then it is integrable.

Let $x, y \in E^n$. If there exist a $z \in E^n$, such that $x = y + z$, then we call z the H-difference of x and y, denoted by $x - y$.

DEFINITION 2.2. *A function $F : T \to E^n$ is differentiable at $t_1 \in T$, if there exists an $F'(t_1) \in E^n$, such that the limits*

$$\lim_{h \to +0} \frac{F(t_1 + h) - F(t_1)}{h}$$

and

$$\lim_{h \to +0} \frac{F(t_1) - F(t_1 - h)}{h}$$

exist and are equal to $F'(t_1)$.

Here, the limits are taken in the metric space (E^n, D). At the endpoint of T, we consider only one-sided derivatives.

If $F : T \to E^n$ is differentiable at $t_1 \in T$, then we say that $F'(t_1)$ is the fuzzy derivative of $F(t)$ at the point t_1.

The integrability and H-differentiability properties of the fuzzy-valued functions can be referred to [5].

In addition, we denote a continuous fuzzy-valued function $f : T \times \Omega \to E^n$ by $f \in C[T \times \Omega, E^n]$, where $\Omega \subset E^n$ is an open set.

PROPOSITION 2.1. *Assume that $f \in C[T \times \Omega, E^n]$. A function $x : T \to \Omega$ is a solution to the problem $x' = f(t, x)$, $x(t_0) = x_0$ if and only if it is continuous and satisfies the integral equation*

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds,$$

for all $t \in T$ (see [5]).

PROPOSITION 2.2. *Let $G \subset R^2$ be an open set, $g \in C[G, R]$, $(t_0, u_0) \in G$. Suppose that the maximum solution to the initial value problem $u' = g(t, u)$, $u(t_0) = u_0$ is $r(t)$ and its largest interval of existence of right solution is $[t_0, t_0 + a)$. If $m(t) \in C[(t_0, t_0 + a), R]$ satisfies $(t, m(t)) \in G$ for all $t \in [t_0, t_0 + a)$; $m(t_0) \le u_0$, and*

$$D^+ m(t) \le g(t, m(t)), \qquad \forall\, t \in [t_0, t_0 + a) \setminus \Gamma,$$

where D^+ is one of the four Dini derivatives, Γ at most is a countable set on $[t_0, t_0 + a)$. Then we must have (see [7])

$$m(t) \le r(t), \qquad \forall\, t \in [t_0, t_0 + a).$$

PROPOSITION 2.3. ASCOLI-ARZELA THEOREM. *Let F be an equicontinuous family of fuzzy-valued functions from I into E^n. Let $x_n(t)$ be a sequence in F such that, for each $t \in I$, the set $\{x_n(t) : n \ge 1\}$ is relatively compact in E^n, i.e., the closure of the set $\{x_n(t) : n \ge 1\}$ is compact. Then there exists a subsequence $\{x_{n_k}(t)\}$ which converges uniformly on I to a continuous function $x(t)$ (see [9]).*

Kuratowski's measure of noncompactness [9] is summarized in the following.

DEFINITION 2.3. *Let S be an arbitrary bounded subset of the fuzzy number metric space (E^n, D), then Kuratowski's measure of noncompactness is defined as*

$$\alpha(S) = \inf \{\delta > 0 \mid S \text{ admits a finite covering of sets of diameter} \le \delta\}.$$

PROPOSITION 2.4. *The measure of noncompactness has the following properties (S, T denote two bounded subsets of E^n, k is a real number):*

(i) *$\alpha(S) = 0$ if and only if S is relatively compact;*
(ii) *$S \subset T$ implies $\alpha(S) \le \alpha(T)$;*
(iii) *$\alpha(\bar{S}) = \alpha(S)$;*

(iv) $\alpha(S \cup T) = \max\{\alpha(S), \alpha(T)\}$;
(v) $\alpha(kS) = |k|\alpha(S)$, *where* $kS = \{x = kz \mid z \in S\}$;
(vi) $\alpha(S + T) \leq \alpha(S) + \alpha(T)$, *where* $S + T = \{x = y + z \mid y \in S,\ z \in T\}$;
(vii) $\alpha(\bar{co} S) = \alpha(S)$;
(viii) $\alpha(S)$ *is uniform continuous on Hausdorff distance* d *defined as follows:*

$$d(S_1, S_2) = \max\left\{\sup_{x \in S_2} \rho(x, S_1), \sup_{x \in S_1} \rho(x, S_2)\right\},$$

where $\rho(x, S_1)$ *denotes the distance from the point* x *to the set* S_1, *i.e., for any given* $\varepsilon > 0$, *there exists an* $\delta > 0$, *such that for any two bounded subsets* S_1 *and* S_2 *with* $d(S_1, S_2) < \delta$ *implies* $|\alpha(S_1) - \alpha(S_2)| < \varepsilon$.

PROPOSITION 2.5. *Assume that*

(1) $f(t, x)$ *is locally Lipschitzian in* x *for* $(t, x) \in J \times E^n$;
(2) $D(f(t, x), \hat{0}) \leq g(t, D(x, \hat{0}))$, $\forall\, (t, x) \in J \times E^n$;
(3) $g \in C[J \times [0, \infty), [0, \infty)]$, $g(t, u)$ *is nondecreasing in* $u \geq 0$ *for each* $t \in J$, *and maximal solution* $r(t, t_0, u_0)$ *of the scalar initial value problem*

$$u' = g(t, u), \qquad u(t_0) = u_0$$

exists throughout J.

Then the largest interval of existence of any solution $x(t, t_0, x_0)$ *of (1) with* $D(x, \hat{0}) \leq u_0$ *is* J *(see [3]).*

3. ASYMPTOTIC EQUILIBRIUM OF FUZZY DIFFERENTIAL EQUATIONS

In this section, we shall continue to consider the following fuzzy differential equation:

$$x' = f(t, x), \qquad x(t_0) = x_0, \tag{1}$$

under the assumptions of Proposition 2.5, where we assume that $f \in C[J \times E^n, E^n]$, $J = [t_0, \infty)$, $x_0 \in E^n$.

DEFINITION 3.1. *We say that fuzzy equation (1) has asymptotic equilibrium if every solution of (1), such that* $(t_0, x_0) \in R_+ \times E^n$ *exists on* $[t_0, \infty)$ *and tends to a limit* $v \in E^n$ *as* $t \to \infty$, *and conversely, to every given vector* $v \in E^n$ *there exists a solution for (1) which tends to* v *as* $t \to \infty$. *In this paper, we denote* $R_+ = [0, \infty)$.

THEOREM 3.1. *Under the assumptions of Proposition 2.5 , given* $v \in E^n$, *there exists a* $T \in [t_0, \infty)$ *and a sequence* $\{x_n(t)\}_{n=1}^{\infty}$ *defined on* $[T, \infty)$, *such that*

(i) $\{x_n(t)\}_{n=1}^{\infty}$ *is equicontinuous on* $[T, \infty)$;
(ii) $\{x_n(t)\}_{n=1}^{\infty}$ *is uniformly bounded on* $[T, \infty)$, *that is, there exists* $M \in R_+$, *such that* $D(x_n(t), \hat{0}) \leq M$ *for all* $t \in [T, \infty)$ *and for all* n;
(iii) *for each* n, $x_n(t)$ *is a solution of* $x' = f(t, x)$, $x(T + n) = v$.

PROOF. Let $r(t, t_0, D(v, \hat{0}))$ be the maximal solution of $u' = g(t, u)$, $u(t_0) = D(v, \hat{0})$, which exists and is bounded on $[t_0, \infty)$. Since $g(t, u) \geq 0$, $r(t)$ is nondecreasing and consequently $\lim_{t \to \infty} r(t) = r(\infty) < \infty$. Also we have $D(v, \hat{0}) \leq r(t) \leq r(\infty)$ for all $t \in [t_0, \infty)$.

For $\lambda \in R_+$, let $u(t, t_0, \lambda)$ be the maximal solution of $u' = g(t, u)$, $u(t_0) = \lambda$. As before, we have $\lim_{t \to \infty} u(t) = u(\infty) < \infty$ and $\lambda = u(t_0) \leq u(t) \leq u(\infty)$ for all $t \in [t_0, \infty)$. However, $g(t, u)$ is nondecreasing in u for each t and we have

$$\int_{t_0}^{\infty} g(s, \lambda)\, ds \leq \int_{t_0}^{\infty} g(s, u(s))\, ds = u(\infty) - u(t_0) < \infty.$$

Thus, we have

$$\int_{t_0}^{\infty} g(s, \lambda)\, ds < \infty, \qquad \text{for any } \lambda \in R_+. \tag{2}$$

In particular,

$$\int_{t_0}^{\infty} g(s, 2r(\infty))\, ds < \infty,$$

and so there exists $T \in [t_0, \infty)$, such that

$$\int_{T}^{\infty} g(s, 2r(\infty))\, ds < r(\infty). \tag{3}$$

Applying Proposition 2.5 for each nonnegative integer n, we know there exists a solution $x_n(t)$ of $x' = f(t,x)$, $x(T+n) = v$, on $[T+n, \infty)$ and

$$D\left(x_n(t), \hat{0}\right) \le r_n\left(t, T+n, D\left(v, \hat{0}\right)\right), \qquad \text{on } [T+n, \infty), \tag{4}$$

where $r_n(t)$ is the maximal solution of $u' = g(t,u)$, $u(T+n) = D(v, \hat{0})$. But $r_n(T+n) = D(v, \hat{0}) = r(t_0) \le r(T+n)$, so by Proposition 2.2 and (4) we have

$$D\left(x_n(t), \hat{0}\right) \le r(t) \le r(\infty), \qquad \text{on } [T+n, \infty). \tag{5}$$

Let $u_n(t, T+n, D(v, \hat{0}))$ be a solution to the left for $u' = -g(t,u)$, $u(T+n) = D(v, \hat{0})$. Now the largest interval of existence will contain $[T, T+n]$ unless u_n becomes unbounded. Suppose $u_n(t)$ becomes unbounded and let $t_1, t_2 \in [T, T+n]$ with $t_1 < t_2$, $u_n(t_1) = 2r(\infty)$, $u_n(t_2) = r(\infty)$ and $2r(\infty) \ge u_n(t) \ge r(\infty)$ for all $t \in [t_1, t_2]$. Now

$$\begin{aligned} r(\infty) &= 2r(\infty) - r(\infty) \\ &= u_n(t_1) - u_n(t_2) \\ &= \left| \int_{t_1}^{t_2} -g(s, u_n(s))\, ds \right| \\ &= \int_{t_1}^{t_2} g(s, u_n(s))\, ds \\ &\le \int_{t_1}^{t_2} g(s, 2r(\infty))\, ds \\ &\le \int_{T}^{\infty} g(s, 2r(\infty))\, ds \\ &< r(\infty). \end{aligned}$$

Hence, for each n, we know $u_n(t)$ remains bounded by $2r(\infty)$ on $[T, T+n]$. Thus, for each n, the maximal solution of $u' = -g(t,u)$, $u(T+n) = D(v, \hat{0})$ to the left exists and is bounded by $2r(\infty)$ and, by Proposition 2.5, we know the interval of definition of $x_n(t)$ can be extended to $[t, \infty)$. Moreover, we have

$$D\left(x_n(t), \hat{0}\right) \le 2r(\infty), \qquad \text{for } t \in [T, \infty) \text{ and for each } n. \tag{6}$$

To complete the proof of this theorem, we need to show $\{x_n(t)\}_{n=1}^{\infty}$ is equicontinuous. Let $\varepsilon > 0$ and note that by (3), we have

$$\int_{T}^{\infty} g(s, 2r(\infty))\, ds < \infty,$$

and hence, there exists $S \in [T, \infty)$, such that

$$\int_{S}^{\infty} g(s, 2r(\infty))\, ds < \varepsilon.$$

On the set $[T, S+1] \times [0, 2r(\infty)]$ the continuous function $g(t, u)$ is bounded by a positive number L. Now if $|t_1 - t_2| < \min(1, \varepsilon/L)$, then

$$\begin{aligned} D\left(x_n(t_1), x_n(t_2)\right) &= D\left(\int_{t_1}^{t_2} f(s, x_n(s))\, ds, \hat{0}\right) \\ &\leq \left| \int_{t_1}^{t_2} g\left(s, D\left(x_n(s), \hat{0}\right)\right)\, ds \right| \\ &\leq \left| \int_{t_1}^{t_2} g(s, 2r(\infty))\, ds \right|. \end{aligned}$$

However, if $t_1, t_2 \in [S, \infty)$, we have

$$\left| \int_{t_1}^{t_2} g(s, r(\infty))\, ds \right| \leq \int_{S}^{\infty} g(s, r(\infty))\, ds < \varepsilon,$$

and if $t_1, t_2 \in [T, S+1]$, then

$$\left| \int_{t_1}^{t_2} g(s, 2r(\infty))\, ds \right| \leq L|t_2 - t_1| < \frac{L\varepsilon}{L} = \varepsilon.$$

Thus, $\{x_n(t)\}_{n=1}^{\infty}$ is equicontinuous. This completes the proof.

We need to complete the proof of asymptotic equilibrium. If we could conclude that for each $t \in [T, \infty)$ (or, in fact, for any infinite subinterval of $[T, \infty)$) that $\{x_n(t)\}_{n=1}^{\infty}$ is relatively compact, we could apply Ascoli-Arzela's theorem (Proposition 2.3). We need the following lemmas before we proceed.

LEMMA 3.1. *Suppose the hypotheses of Proposition 2.5 are satisfied. Let $v \in E^n$ and let $\{x_n(t)\}_{n=1}^{\infty}$ be the sequence of fuzzy-valued functions which exist by Theorem 3.1 of this paper. For each $t \in [T, \infty)$, let $m(t) = \alpha(\{x_n(t)\}_{n=1}^{\infty})$. Then $m(t)$ is uniformly continuous on $[T, \infty)$.*

PROOF. Notice that

$$\begin{aligned} m(t_1) - m(t_2) &= \alpha\left(\{x_n(t_1)\}_{n=1}^{\infty}\right) - \alpha\left(\{x_n(t_2)\}_{n=1}^{\infty}\right) \\ &\leq \alpha\left(\{x_n(t_1) - x_n(t_2)\}_{n=1}^{\infty}\right) \end{aligned}$$

by (vi) of Proposition 2.4. Using the equicontinuity of $\{x_n(t)\}_{n=1}^{\infty}$ and the fact $\alpha(A) \leq 2d$, where $\sup_{x \in A} D(x, \hat{0}) \leq d$, the proof is completed.

LEMMA 3.2. *Suppose the hypotheses of Proposition 2.5 are satisfied and f is uniformly continuous on bounded subsets of $[T, \infty) \times E^n$. Let $\{x_n(t)\}_{n=1}^{\infty}$ be the sequence of fuzzy-valued functions which exist by Theorem 3.1. For $s \in [T, \infty)$ and $h > 0$, we can then express*

$$x_n(s+h) = x_n(s) + hf(s, x_n(s)) + h\varepsilon_n(h)$$

and

$$x_n(s) = x_n(s-h) + hf(s, x_n(s)) + h\tilde{\varepsilon}_n(h),$$

where

$$\lim_{h \to +0} \alpha\left(\{\varepsilon_n(h)\}_{n=1}^{\infty}\right) = 0 \qquad \text{and} \qquad \lim_{h \to +0} \alpha\left(\{\tilde{\varepsilon}_n(h)\}_{n=1}^{\infty}\right) = 0.$$

PROOF. We shall prove the first expression only and the other proof is similar. Let $\varepsilon > 0$. Since f is uniformly continuous on $A = [s, s+1] \times \{x \in E^n \mid D(x, \hat{0}) < 1\}$, there exists $\delta > 0$, such that $|t_1 - t_2| + D(y_1, y_2) < \delta$ implies $D(f(t_1, y_1), f(t_2, y_2)) < \varepsilon$ for $(t_1, y_1), (t_2, y_2) \in A$.

Since $\{x_n(t)\}_{n=1}^{\infty}$ is equicontinuous, there exists $r > 0$, such that for $t \in (s-r, s+r)$ we have $D(x_n(t), x_n(s)) + |t-s| < \delta$ for all n. Thus, for $h < \delta$, we know

$$\begin{aligned} D\left(h\varepsilon_n(h), \hat{0}\right) &= D(x_n(s+h), x_n(s) + hf(s, x_n(s))) \\ &= D\left(\int_s^{s+h} f(\sigma, x_n(\sigma))\, d\sigma, \int_s^{s+h} f(s, x_n(s))\, d\sigma\right) \\ &\le \int_s^{s+h} D(f(\sigma, x_n(\sigma)), f(s, x_n(s)))\, d\sigma \\ &\le \int_s^{s+h} \varepsilon\, d\sigma \\ &= \varepsilon h. \end{aligned}$$

Thus, $D(\varepsilon_n(h), \hat{0}) \le \varepsilon$ for each n, which implies that $\alpha(\{\varepsilon_n(h)\}_{n=1}^{\infty}) \le 2\varepsilon$. Consequently, $\lim_{h\to 0} \alpha(\{\varepsilon_n(h)\}_{n=1}^{\infty}) = 0$.

THEOREM 3.2. *Suppose the hypotheses of Proposition 2.5 hold and also that*

(i) *f is uniformly continuous on bounded subsets of $[T, \infty) \times E^n$;*
(ii) *there exists $t^* \in [T, \infty)$, such that $m(t^*) = 0$;*
(iii) *$G \in C[[T, \infty) \times R_+, R_+]$, such that $G(t, 0) = 0$ and the only solution $u(t)$ of $u' = G(t, u)$, $u(s) = 0$ with*

$$\lim_{t\to s} \frac{u(t)}{t-s} = 0$$

is $u(t) \equiv 0$;
(iv) *for $h > 0$ and a bounded subset $A \subset E^n$, we have $\alpha(\{x + hf(t, x) \mid x \in A\}) - \alpha(A) \le hG(t, \alpha(A))$.*

Then $m(t) \equiv 0$ on $[t^*, \infty)$.

PROOF. Recall that $m(t) = \alpha(\{x_n(t)\}_{n=1}^{\infty})$. Now, using Lemma 3.2 and considering $h > 0$, we have

$$\begin{aligned} h^{-1}[m(t+h) - m(t)] &= h^{-1}\left[\alpha\left(\{x_n(t+h)\}_{n=1}^{\infty}\right) - \alpha\left(\{x_n(t)\}_{n=1}^{\infty}\right)\right] \\ &= h^{-1}\left[\alpha\left(\{x_n(t) + hf(t, x_n(t)) + h\varepsilon_n(h)\}_{n=1}^{\infty}\right) - \alpha\left(\{x_n(t)\}_{n=1}^{\infty}\right)\right] \\ &\le h^{-1}\left[\alpha\left(\{x_n(t) + hf(t, x_n(t))\}_{n=1}^{\infty}\right) - \alpha\left(\{x_n(t)\}_{n=1}^{\infty}\right) + \alpha(\{h\varepsilon_n(h)\}_{n=1}^{\infty})\right]. \end{aligned}$$

Now, using (iii) and (v) of Proposition 2.4, we have

$$\begin{aligned} h^{-1}[m(t+h) - m(t)] &\le h^{-1}\left[hG\left(t, \alpha\left(\{x_n(t)\}_{n=1}^{\infty}\right)\right) + h\alpha\left(\{\varepsilon_n(h)\}_{n=1}^{\infty}\right)\right] \\ &= G\left(t, \alpha\left(\{x_n(t)\}_{n=1}^{\infty}\right)\right) + \alpha\left(\{\varepsilon_n(h)\}_{n=1}^{\infty}\right). \end{aligned}$$

Hence, applying Lemma 3.2, we obtain $D_+m(t) \le G(t, m(t))$. Again, by Proposition 2.2, since $D_+m(t) \le G(t, m(t))$ on $[t, \infty)$ and $m(t^*) = 0$, thus, $m(t)$ is less than or equal to the maximal solution of $u' = G(t, u)$, $u(t^*) = 0$. Using Lemma 3.2, for $h > 0$, we have

$$\begin{aligned} h^{-1}[x_n(t^* + h)] &= h^{-1}\left[x_n(t^*) + hf(t^*, x_n(t^*)) + h\varepsilon_n(h)\right] \\ &= h^{-1}\left[x_n(t^*) + hf(t^*, x_n(t^*))\right] + \varepsilon_n(h). \end{aligned}$$

Thus,

$$\begin{aligned} 0 &\le \lim_{t\to t^*} \frac{m(t)}{t-t^*} = \lim_{t\to t^*} \frac{\alpha\left(\{x_n(t)\}_{n=1}^{\infty}\right)}{t-t^*} = \lim_{h\to +0} \frac{\alpha\left(\{x_n(t^*+h)\}_{n=1}^{\infty}\right)}{h} \\ &\le \lim_{h\to +0} h^{-1}\left[\alpha\left(\{x_n(t^*) + hf(t^*, x_n(t^*))\}_{n=1}^{\infty}\right)\right] + \lim_{h\to +0} \alpha\left(\{\varepsilon_n(h)\}_{n=1}^{\infty}\right) \\ &\le \lim_{h\to +0} h^{-1}\left[hG\left(t^*, \alpha\left(\{x_n(t^*)\}_{n=1}^{\infty}\right)\right)\right] + 0 = G\left(t^*, \alpha\left(\{x_n(t^*)\}_{n=1}^{\infty}\right)\right) \\ &= G(t^*, m(t^*)) = G(t^*, 0) = 0. \end{aligned}$$

Hence,

$$\lim_{t\to t^*}\frac{m(t)}{t-t^*}=0,$$

so by (iii), we have $m(t)=0$ for all $t\in[t^*,\infty)$. By the existence theorem under compactness-type conditions (see Theorem 4.1 in [2]), it is easy to see that the fuzzy differential system (3.1) has a local solution under the assumptions of Theorem 3.2. Thus, we have sufficient hypotheses to assure local existence, and so the assumption about local existence in Proposition 2.5 is superfluous whenever we assume that the conditions of Theorem 3.2 are satisfied.

In Theorem 3.2, we assumed the existence of $t^*\in[T,\infty)$, such that $m(t^*)=0$. Theorem 3.3 yields sufficient conditions to assure that such a t^* exists.

THEOREM 3.3. *Suppose the hypotheses of Proposition 2.5 hold and also*

(i) *f is uniformly continuous on bounded subsets of $[T,\infty)\times E^n$;*
(ii) *$F\in C[[T,\infty)\times R_+,R_+]$ and $u'=F(t,u)$, $u(s)=\omega$ has a solution to the left for each $(s,\omega)\in[T,\infty)\times R_+$ and for any solution $u(t)$ there exists $t_1\in(T,s]$, such that $u(t_1)=0$;*
(iii) *for $h<0$, a bounded subset of $A\subset E^n$ and $t\in[T,s]$, we have $\alpha(\{x+hf(t,x)\mid x\in A\})-\alpha(A)\le hF(t,\alpha(A))$.*

Then there exists a $t^*\in[T,\infty)$, such that $m(t^*)=0$.

PROOF. Recall that $m(t)=\alpha(\{x_n(t)\}_{n=1}^{\infty})$ and, from Lemma 3.1, $m(t)$ is continuous. Now, using Lemma 3.2 and considering $h>0$, we have

$$\begin{aligned}h^{-1}[m(t)-m(t-h)]&=h^{-1}\left[\alpha\left(\{x_n(t)\}_{n=1}^{\infty}\right)-\alpha\left(\{x_n(t-h)\}_{n=1}^{\infty}\right)\right]\\&=h^{-1}\left[\alpha\left(\{x_n(t)\}_{n=1}^{\infty}\right)-\alpha\left(\{x_n(t)-hf(t,x_n(t))-h\tilde{\varepsilon}_n(h)\}_{n=1}^{\infty}\right)\right].\end{aligned}$$

Now, by (vi) of Proposition 2.4, we have

$$\begin{gathered}h^{-1}[m(t)-m(t-h)]\ge h^{-1}\left[\alpha\left(\{x_n(t)\}_{n=1}^{\infty}\right)\right.\\\left.-\alpha\left(\{x_n(t)-hf(t,x_n(t))\}_{n=1}^{\infty}\right)+\alpha\left(\{-h\tilde{\varepsilon}_n(h)\}_{n=1}^{\infty}\right)\right].\end{gathered}$$

Next, using (vi) and (v) of Proposition 2.4, we have

$$\begin{aligned}h^{-1}[m(t)-m(t-h)]&\ge h^{-1}\left[hF\left(t,\alpha\left(\{x_n(t)\}_{n=1}^{\infty}\right)\right)+h\alpha\left(\{\tilde{\varepsilon}_n(h)\}_{n=1}^{\infty}\right)\right]\\&=F\left(t,\alpha\left(\{x_n(t)\}_{n=1}^{\infty}\right)\right)+\alpha\left(\{\tilde{\varepsilon}_n(h)\}_{n=1}^{\infty}\right).\end{aligned}$$

Thus,

$$\begin{aligned}D_-m(t)&=\lim_{h\to+0}h^{-1}[m(t)-m(t-h)]\\&\ge\lim_{h\to+0}F\left(t,\alpha\left(\{x_n(t)\}_{n=1}^{\infty}\right)\right)+\lim_{h\to+0}\alpha\left(\{\tilde{\varepsilon}_n(h)\}_{n=1}^{\infty}\right).\end{aligned}$$

Since $\lim_{h\to+0}\alpha(\{\tilde{\varepsilon}_n(h)\}_{n=1}^{\infty})=0$, by Lemma 3.2, then $D_-m(t)\ge F(t,m(t))$. From (3.6), we have $D(x_n(t),\hat{0})\le 2r(\infty)$ on $[T,\infty)$ for any n, and hence, it follows that $m(t)\le 4r(\infty)$. For $s\in(T,\infty)$, let $R(t,s,4r(\infty))$ be the maximal solution to the left for $u'=F(t,u)$, $u(\tilde{t})=4r(\infty)$. The existence of $R(t,s,4r(\infty))$ on $[T,s]$ follows from (ii). By Proposition 2.2, since $D_-m(t)\ge F(t,m(t))$, we have $m(t)\le R(t,s,4r(\infty))$ on $[T,s]$. However, by (ii), there exists $t_1\in[T,s]$ with $R(t_1)=0$, and so there exists $t^*\in[t_1,s]$ with $\alpha(\{x_n(t^*)\}_{n=1}^{\infty})=m(t^*)=0$, which means $\{x_n(t^*)\}_{n=1}^{\infty}$ is relatively compact.

THEOREM 3.4. *Suppose the hypotheses of Proposition 2.5, Theorems 3.2, and 3.3 are satisfied. Then equation (1) has asymptotic equilibrium.*

PROOF. The sequence $\{x_n(t)\}_{n=1}^{\infty}$ constructed in Theorem 3.1 has the property that $\{x_n(t)\}_{n=1}^{\infty}$ is relatively compact for each $t \in [t^*, \infty)$, by Theorems 3.2 and 3.3. Applying Proposition 2.4 (Ascoli-Arzela's theorem), we know there exists a subsequence which converges pointwise on $[t^*, \infty)$ and uniformly on compact subsets of $[t^*, \infty)$. To simplify the notation, we will use $\{x_n(t)\}_{n=1}^{\infty}$ to denote the subsequence. Let $x(t) = \lim_{n\to\infty} x_n(t)$ for each $t \in [t^*, \infty)$ and note $x(t)$ is continuous by the uniform convergence on compact subsets. For each $t \in [t^*, \infty)$, we have

$$x_n(t) = x_n(t^*) + \int_{t^*}^{t} f(\sigma, x_n(\sigma))\, d\sigma.$$

Since convergence is uniform on compact subsets, it then follows that

$$x(t) = x(t^*) + \int_{t^*}^{t} f(\sigma, x(\sigma))\, d\sigma.$$

Thus, $x(t)$ is a solution of the system $x' = f(t,x)$, $x(t^*) = x(t^*)$. It remains only to show that $\lim_{t\to\infty} x(t) = v$. Let $\varepsilon > 0$. Then by (2), there exists an integer $S > t^*$, such that $\int_S^{\infty} g(s, 2r(\infty))\, ds < \varepsilon/2$. If $s \in [S, \infty)$, there exists N, such that for all $n > N$ we know $D(x_n(s), x(s)) < \varepsilon/2$. Consider $N + S$ and recall that $x_{N+S}(T + N + S) = v$. Thus,

$$\begin{aligned}
D(x(s), v) &= D(x(s), x_{N+S}(T + N + S)) \\
&\le D(x(s), x_{N+S}(s)) + D(x_{N+S}(s), x_{N+S}(T + N + S)) \\
&\le \frac{\varepsilon}{2} + D\left(\int_s^{T+N+S} f(\sigma, x_{N+S}(\sigma))\, d\sigma, \hat{0}\right) \\
&\le \frac{\varepsilon}{2} + \left|\int_s^{T+N+S} g\left(t, D\left(x_{N+S}(\sigma), \hat{0}\right)\right)\, d\sigma\right| \\
&\le \frac{\varepsilon}{2} + \int_S^{\infty} g(\sigma, 2r(\infty))\, d\sigma < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

and so $\lim_{t\to\infty} x(t) = v$. This completes the proof.

4. LYAPUNOV ASYMPTOTIC STABILITY

Based on the metric D defined on the fuzzy number space (E^n, D), a variety of properties of Lyapunov stability in Banach space can be investigated. In the following, we shall first study the Lyapunov stability properties of the following fuzzy differential equations:

$$x' = A(t)x + f(t,x), \qquad x(t_0) = x_0, \tag{7}$$

where $f \in C[R_+ \times E^n, E^n]$, and for each $t \in R_+$, $A(t) : E^n \to E^n$ is a semilinear operator that has the following properties:

(i) $A(t)(x + y) = A(t)(x) + A(t)(y)$, $x, y \in E^n$;
(ii) $A(t)(\lambda x) = \lambda A(t)(x)$, $\lambda \in R_+$, $x \in E^n$.

Assume the existence of solutions for equation (7) and also assume that $A(t)$ is the contraction operator having the following property: there exists a scalar γ with $0 \le \gamma < 1$, such that

$$D(A(t)x, A(t)y) \le \gamma D(x,y),$$

for any $x, y \in E_n$. Then for each $t \in R_+$ and for all $h > 0$ sufficiently small, the operator

$$R[h, A(t)] \equiv I + hA(t) + h^2A^2(t) + \cdots + h^nA^n(t) + \cdots$$

exists as a bounded operator defined on E^n and for each $x \in E^n$

$$\lim_{h\to 0} R[h, A(t)]x = x. \tag{8}$$

It is easy to show that relation (8) can be satisfied. The following comparison theorem is basic in the discussion of stability criteria.

THEOREM 4.1. *Assume that*

(i) $V \in C[R_+ \times E^n, R_+]$ *and for* $(t, x_1), (t, x_2) \in R_+ \times E^n$,

$$|V(t, x_1) - V(t, x_2)| \leq L(t) D(x_1, x_2),$$

where $L(t) \geq 0$ *is continuous on* R_+;

(ii) $g \in C[R_+ \times R_+, R]$ *and for* $(t, x) \in R_+ \times E^n$,

$$D^+ V(t, x) \equiv \lim_{h \to +0} \sup \frac{1}{h} [V(t + h, R[h, A(t)]x + hf(t, x)) - V(t, x)] \leq g(t, V(t, x));$$

(iii) *the maximal solution* $r(t, t_0, u_0)$ *of the scalar differential equation*

$$u' = g(t, u), \qquad u(t_0) = u_0 \geq 0 \tag{9}$$

exists on $[t_0, \infty)$.

Then $V(t_0, x_0) \leq u_0$ *implies that*

$$V(t, x(t, t_0, x_0)) \leq r(t, t_0, x_0), \qquad t \geq t_0. \tag{10}$$

PROOF. Let $x(t) = x(t, t_0, x_0)$ be any solution of (1), such that $V(t_0, x_0) \leq u_0$. Define $m(t) = V(t, x(t, t_0, x_0))$. For small $h > 0$, by (i) we have

$$\begin{aligned} m(t+h) - m(t) \leq\ & L(t+h) D(x(t+h), R[h, A(t)]x(t) + hf(t, x(t))) \\ & + V(t+h, R[h, A(t)]x(t) + hf(t, x(t))) - V(t, x(t)). \end{aligned}$$

For every $x \in E^n$, from the sense of $R[h, A(t)]$, we know $R[h, A(t)] = I + hA(t)R[h, A(t)]$, so that

$$R[h, A(t)]x = x + h(R[h, A(t)]A(t))x.$$

It follows that

$$R[h, A(t)]x + hf(t, x) = x + hf(t, x)) + h(R[h, A(t)]A(t))x.$$

Thus, we have

$$\begin{aligned} m(t+h) - m(t) \leq\ & L(t+h) D(x(t+h), x(t) + h(A(t)x(t) + f(t, x(t)))) \\ & + L(t+h) h D(R[h, A(t)]A(t)x(t), A(t)x(t)) \\ & + V(t+h, R[h, A(t)]x(t) + hf(t, x(t))) - V(t, x(t)). \end{aligned}$$

Hence, by (7),(8) and (ii), we obtain

$$D^+ m(t) \leq g(t, m(t)).$$

By Proposition 2.2, we have the desired estimate (10). For the special case $V(t, x) = D(x, \hat{0})$, we have the following corollary.

COROLLARY 4.1. *Let* $g \in C[R_+ \times R_+, R]$, *and for* $(t, x) \in R_+ \times E^n$,

$$\lim_{h \to 0^+} \frac{1}{h} \left(D \left(R[h, A(t)]x + hf(t, x), \hat{0} \right) - D \left(x, \hat{0} \right) \right) \leq g \left(t, D \left(x, \hat{0} \right) \right). \tag{11}$$

Then $D(x_0, \hat{0}) \leq u_0$ *implies*

$$D \left(x(t, t_0, x_0), \hat{0} \right) \leq r(t, t_0, x_0), \qquad t \geq t_0, \tag{12}$$

where $r(t, t_0, x_0)$ *is the maximal solution of (9).*

Using the comparison Theorem 4.1, it is easy to prove various stability criteria corresponding to the results in Euclidean space. We state a typical result.

THEOREM 4.2. *In addition to Hypotheses (i) and (ii) of Theorem 4.1, assume that*

(iii) $f(t,\hat{0}) \equiv \hat{0}$, $g(t,0) \equiv 0$, *and* $V(t,\hat{0}) \equiv 0$;

(iv) $b(D(x,\hat{0})) \le V(t,x) \le a(D(x,\hat{0}))$, $(t,x) \in R_+ \times E^n$, $a,b \in K$, *where* $K = \{\phi \in C[R_+,R_+];$ $\phi(0) = 0$ *and* $\phi(u)$ *strictly increasing in* $u\}$.

Then any stability property of $u = 0$ *of the scalar differential equation (9) implies the corresponding stability properties of the trivial fuzzy solution of (7).*

EXAMPLE 4.1. Consider the fuzzy differential equation

$$x' = \frac{1}{t^p}x, \qquad x(2) = x_0, \qquad x, x_0 \in E^1, \qquad t \ge 2. \tag{13}$$

Then for $p \ge 2$, $A(t) = (1/t^p)I$ is a semilinear operator as mentioned above. So for sufficient small h satisfying $0 < h < 1$, we have

$$\begin{aligned} R[h, A(t)] &= I + hA(t) + h^2A^2(t) + \cdots \\ &= \left(1 + \frac{h}{t^p} + \frac{h^2}{t^{2p}} + \cdots\right) I \\ &= \frac{t^p}{t^p - h} I. \end{aligned}$$

Here I is an identity operator from E^1 to E^1. We take $g(t,u) = u/t^2$ and consider the following scalar differential equation:

$$u' = \frac{u}{t^2}, \qquad u(2) = u_0 \ge 0. \tag{14}$$

It is easy to show that the solution $u = u_0 e^{1/2-1/t}$ of (14) is asymptotically stable. Using the Lyapunov function $V(t,x) = D(x,\hat{0})$, the assumptions of Theorem 4.2 are satisfied, so the solution of fuzzy differential equation (13) is asymptotically stable.

In fact, from equation (13), we have the following scalar differential equations group:

$$x'_\alpha(t,\lambda) = \frac{1}{t^p}x_\alpha(t,\lambda), \qquad x_\alpha(2,\lambda) = x^0_\alpha(\lambda), \qquad \alpha \in \{0,1\}, \quad \lambda \in [0,1]. \tag{15}$$

It is clear that the solution of (15) is

$$x_\alpha(t,\lambda) = x^0_\alpha(\lambda) e^{(2^{(1-p)} - t^{(1-p)})/(p-1)}. \tag{16}$$

So if $D(x_0,\hat{0}) \to 0$, then by the definition of metric D on fuzzy number space (E^n, D), the solution sequence $\{x^0_\alpha(\lambda)\}$ converges to zero uniformly on parameter λ. Thus, the solution sequence $\{x_\alpha(t,\lambda)\}$ of (16) converges to zero uniformly on parameter λ as $t \to \infty$. This implies also the asymptotic stability of the fuzzy differential equation (13).

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