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Journal of Combinatorial Theory, Series B 96 (2006) 164–181
Journal of Combinatorial Theory Series B
www.elsevier.com/locate/jctb

Tetravalent edge-transitive Cayley graphs with odd number of vertices

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Received 28 June 2004
Available online 1 September 2005

Abstract

A characterisation is given of edge-transitive Cayley graphs of valency 4 on odd number of vertices. The characterisation is then applied to solve several problems in the area of edge-transitive graphs: answering a question proposed by Xu [Automorphism groups and isomorphisms of Cayley graphs, Discrete Math. 182 (1998) 309–319] regarding normal Cayley graphs; providing a method for constructing edge-transitive graphs of valency 4 with arbitrarily large vertex-stabiliser; constructing and characterising a new family of half-transitive graphs. Also this study leads to a construction of the first family of arc-transitive graphs of valency 4 which are non-Cayley graphs and have a 'nice' isomorphic 2-factorisation.

Keywords: Cayley graphs; Edge-transitive

1. Introduction

A graph Γ is a *Cayley graph* if there exist a group G and a subset $S \subset G$ with $1 \notin S = S^{-1} := \{g^{-1} \mid g \in S\}$ such that the vertices of Γ may be identified with the elements of G

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[1] Partially supported by an ARC Discovery Project Grant and a QEII Fellowship.
[2] Partially supported by the NNSF and TYYF of China.

doi:10.1016/j.jctb.2005.07.003

in such a way that x is adjacent to y if and only if $yx^{-1} \in S$. The Cayley graph Γ is denoted by $\mathsf{Cay}(G, S)$. Throughout this paper, denote by $\mathbf{1}$ the vertex of $\mathsf{Cay}(G, S)$ corresponding to the identity of G.

It is well-known that a graph Γ is a Cayley graph of a group G if and only if the automorphism group $\mathsf{Aut}\,\Gamma$ contains a subgroup which is isomorphic to G and acts regularly on vertices. In particular, a Cayley graph $\mathsf{Cay}(G, S)$ is vertex-transitive of order $|G|$. However, a Cayley graph is of course not necessarily edge-transitive. In this paper, we investigate Cayley graphs that are edge-transitive.

Small valency Cayley graphs have received attention in the literature. For instance, Cayley graphs of valency 3 or 4 of simple groups have been investigated in [6,7,32]; Cayley graphs of valency 4 of certain p-groups are investigated in [8,30]. Refer to [4,20,23,24] for more results regarding edge-transitive graphs of small valencies. A relation between regular maps and edge-transitive Cayley graphs of valency 4 is studied in [22]. In the main result (Theorem 1.1) of this paper, we characterise edge-transitive Cayley graphs of valency 4 and odd order. To state this result, we need more definitions.

Let Γ be a graph with vertex set $V\Gamma$ and edge set $E\Gamma$. If a subgroup $X \leqslant \mathsf{Aut}\,\Gamma$ is transitive on $V\Gamma$ or $E\Gamma$, then the graph Γ is said to be *X-vertex-transitive* or *X-edge-transitive*, respectively. A sequence $v_0, v_1, \ldots, v_s$ of vertices of Γ is called an *s-arc* if $v_{i-1} \neq v_{i+1}$ for $1 \leqslant i \leqslant s-1$, and $\{v_i, v_{i+1}\}$ is an edge for $0 \leqslant i \leqslant s-1$. The graph Γ is called *(X, s)-arc-transitive* if X is transitive on the s-arcs of Γ; if in addition X is not transitive on the $(s+1)$-arcs, then Γ is said to be *(X, s)-transitive*. In particular, a 1-arc is simply called an *arc*, and an $(X, 1)$-arc-transitive graph is called *X-arc-transitive*.

A typical method for studying vertex-transitive graphs is taking certain quotients. For an X-vertex-transitive graph Γ and a normal subgroup $N \lhd X$, the *normal quotient graph* Γ_N induced by N is the graph that has vertex set $V\Gamma_N = \{v^N \mid v \in V\Gamma\}$ such that v_1^N and v_2^N are adjacent if and only if v_1 is adjacent in Γ to some vertex in v_2^N. If further the valency of Γ_N equals the valency of Γ, then Γ is called a *normal cover* of Γ_N.

Theorem 1.1. *Let G be a finite group of odd order, and let $\Gamma = \mathsf{Cay}(G, S)$ be connected and of valency 4. Assume that Γ is X-edge-transitive, where $G \leqslant X \leqslant \mathsf{Aut}\,\Gamma$. Then one of the following holds:*

(1) *G is normal in X, $X_{\mathbf{1}} \leqslant \mathrm{D}_8$, and $S = \{a, a^{-1}, a^\tau, (a^\tau)^{-1}\}$, where $\tau \in \mathsf{Aut}(G)$ such that either $o(\tau) = 2$, or $o(\tau) = 4$ and $a^{\tau^2} = a^{-1}$;*
(2) *there is a subgroup $M < G$ such that $M \lhd X$, and Γ is a normal cover of Γ_M;*
(3) *X has a unique minimal normal subgroup $N \cong \mathbb{Z}_p^k$ with p odd prime and $k \geqslant 2$ such that*
  (i) *$G = N \rtimes R \cong \mathbb{Z}_p^k \rtimes \mathbb{Z}_m$, where $m > 1$ is odd;*
  (ii) *$X = N \rtimes ((H \rtimes R).O) \cong \mathbb{Z}_p^k \rtimes ((\mathbb{Z}_2^l \rtimes \mathbb{Z}_m).\mathbb{Z}_t)$, and $X_{\mathbf{1}} = H.O$, where $H \cong \mathbb{Z}_2^l$ with $2 \leqslant l \leqslant k$, and $O \cong \mathbb{Z}_t$ with $t = 1$ or 2, satisfying the following statements:*
    (a) *there exist $x_1, \ldots, x_k \in N$ and $\tau_1, \ldots, \tau_k \in H$ such that $N = \langle x_1, \ldots, x_k \rangle$, $\langle x_i, \tau_i \rangle \cong \mathrm{D}_{2p}$ and $H = \langle \tau_i \rangle \times \mathbf{C}_H(x_i)$ for $1 \leqslant i \leqslant k$;*
    (b) *R does not centralise H;*
    (c) *$X/(NH) \cong \mathbb{Z}_m$ or D_{2m}, and Γ is X-arc-transitive if and only if $X/(NH) \cong \mathrm{D}_{2m}$;*

(4) *Γ is (X, s)-transitive, and X, X_1, s and G are as in the following table*:

X	X_1	s	G
A_5, S_5	A_4, S_4	2	$\mathbb{Z}_5$
PGL(2,7)	D_{16}	1	$\mathbb{Z}_7 \rtimes \mathbb{Z}_3$
PSL(2, 11), PGL(2, 11)	A_4, S_4	2	$\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$
PSL(2, 23)	S_4	2	$\mathbb{Z}_{23} \rtimes \mathbb{Z}_{11}$

Remarks on Theorem 1.1.

(a) The Cayley graph Γ in part (1), called *normal edge-transitive graph*, is studied in [25]. If further $X = \mathrm{Aut}\,\Gamma$, then Γ is called *a normal Cayley graph*, introduced in [31]. For this type of Cayley graph, the action of X on the graph Γ is well-understood.
(b) Part (2) is a reduction from the Cayley graph Γ to a smaller graph Γ_M, which is also an edge-transitive Cayley graph of valency 4. An edge-transitive Cayley graph is called *basic* if it is not a normal cover of a smaller edge-transitive Cayley graph. Theorem 1.1 shows that if Γ is not a normal Cayley graph then Γ is a cover of a well-characterised graph, that is a basic Cayley graph satisfying part (3) or part (4).
(c) Construction 3.2 shows that for every group X satisfying part (3) with $O = 1$ indeed acts edge-transitively on some Cayley graphs of valency 4.
(d) Part (4) tells us that there are only three 2-arc-transitive basic Cayley graphs of valency 4 and odd order. The graph in row 1 of the table is the complete graph K_5; the graph in row 2 of the table is the line graph of the Heawood graph.

The following corollary of Theorem 1.1 gives a solution to Problem 4 of [31], in particular, answering the question stated there in the negative.

Corollary 1.2. *There are infinitely many connected basic Cayley graphs of valency* 4 *and odd order which are not normal Cayley graphs*.

The proof of Corollary 1.2 follows from Lemma 3.3.

It is well-known that the vertex-stabiliser for an s-arc-transitive graph of valency 4 with $s \geqslant 2$ has order dividing $2^4 3^6$, see Lemma 2.5. However, in [2,26], 'non-trivial' arc-transitive graphs of valency 4 which have arbitrarily large vertex-stabiliser are constructed. Part (3) of Theorem 1.1 characterises edge-transitive Cayley graphs of valency 4 and odd order with this property.

Corollary 1.3. *Let Γ be a connected Cayley graph of valency* 4 *and odd order. Assume that Γ is X-edge-transitive for $X \leqslant \mathrm{Aut}\,\Gamma$. Then $|X_1| > 24$ if and only if Γ is a cover of a graph satisfying part* (3) *of Theorem* 1.1 *with* $l \geqslant 5$.

This characterisation provides a potential method for constructing edge-transitive graphs of valency 4 with arbitrarily large vertex-stabiliser, see Construction 3.2.

A graph Γ is called *half-transitive* if $\mathrm{Aut}\,\Gamma$ is transitive on the vertices and the edges but not transitive on the arcs of Γ. Constructing and characterising half-transitive graphs was initiated by Tutte (1965), and is a currently active topic in algebraic graph theory, see

[18,21,22] for references. Theorem 1.1 provides a method for characterising some classes of half-transitive graphs of valency 4. The following theorem is such an example:

Theorem 1.4. *Let* $G = N \rtimes \langle g \rangle = \mathbb{Z}_p^k \rtimes \mathbb{Z}_m < \mathrm{AGL}(1, p^k)$, *where* $k > 1$ *is odd*, p *is an odd prime and* m *is the largest odd divisor of* $p^k - 1$. *Assume that* Γ *is a connected edge-transitive Cayley graph of* G *of valency* 4. *Then* $\mathsf{Aut}\,\Gamma = G \rtimes \mathbb{Z}_2$, Γ *is half-transitive, and* $\Gamma \cong \Gamma_i = \mathsf{Cay}(G, S_i)$, *where* $1 \leqslant i \leqslant \frac{m-1}{2}$, $(m, i) = 1$, *and*

$$S_i = \{ag^i, a^{-1}g^i, (ag^i)^{-1}, (a^{-1}g^i)^{-1}\} \quad \text{where } a \in N \setminus \{1\}.$$

Moreover, $\Gamma_i \cong \Gamma_j$ *if and only if* $p^r i \equiv j$ or $-j \pmod m$ *for some* $r \geqslant 0$.

The following result is a by-product of analysing PGL(2, 7)-arc-transitive graphs of valency 4. (For two graphs Γ and Σ which have the same vertex set V and disjoint edge sets E_1 and E_2, respectively, denote by $\Gamma + \Sigma$ the graph with vertex set V and edge set $E_1 \cup E_2$. For a positive integer n and a cycle $\mathbf{C}_m$ of size m, denote by $n\mathbf{C}_m$ the vertex disjoint union of n copies of $\mathbf{C}_m$.)

Proposition 1.5. *Let* p *be a prime such that* $p \equiv -1 \pmod 8$, *and let* $T = \mathrm{PSL}(2, p)$ and $X = \mathrm{PGL}(2, p)$. *Then there exists an* X*-arc-transitive graph* Γ *of valency* 4 *such that the following hold*:

(i) $\Gamma = \Delta_1 + \Delta_2$, $\Delta_1 \cong \Delta_2 \cong \frac{p(p^2-1)}{48}\mathbf{C}_3$, $T \leqslant \mathsf{Aut}\,\Delta_1 \cap \mathsf{Aut}\,\Delta_2$, *and both* Δ_1 *and* Δ_2 *are* T*-arc-transitive*; *in particular*, Γ *is not* T*-edge-transitive*;
(ii) Γ *is a Cayley graph if and only if* $p = 7$.

Part (i) of this proposition is proved by Lemma 4.3, and part (ii) follows from Theorem 1.1.

Remark on Proposition 1.5. The factorisation $\Gamma = \Delta_1 + \Delta_2$ is an isomorphic 2-factorisa tion of Γ. The group X is transitive on $\{\Delta_1, \Delta_2\}$ with T being the kernel. Such isomorphic factorisations are called *homogeneous factorisations*, introduced and studied in [9,19]. The factorisation given in Proposition 1.5 are the first known example of non-Cayley graphs which have a homogeneous 2-factorisation, refer to [9, Lemma 2.7] for a characterisation of homogeneous 1-factorisations.

This paper is organized as follows. Section 2 collects some preliminary results which will be used later. Section 3 gives some examples of graphs appeared in Theorem 1.1. Then Section 4 constructs the graphs stated in Proposition 1.5. Finally, in Sections 5 and 6, Theorems 1.1 and 1.4 are proved, respectively.

2. Preliminary results

For a core-free subgroup H of X and an element $a \in X \setminus H$, let $[X: H] = \{Hx \mid x \in X\}$, and define the *coset graph* $\Gamma := \mathsf{Cos}(X, H, H\{a, a^{-1}\}H)$ to be the graph with vertex set

$[X : H]$ such that $\{Hx, Hy\}$ is an edge of Γ if and only if $yx^{-1} \in H\{a, a^{-1}\}H$. The properties stated in the following lemma are well-known.

Lemma 2.1. *For a coset graph $\Gamma = \mathsf{Cos}(X, H, H\{a, a^{-1}\}H)$, we have*

(i) *Γ is X-edge-transitive*;

(ii) *Γ is X-arc-transitive if and only if $HaH = Ha^{-1}H$, or equivalently, $HaH = HbH$ for some $b \in X \setminus H$ such that $b^2 \in H \cap H^b$*;

(iii) *Γ is connected if and only if $\langle H, a\rangle = X$*;

(iv) *the valency of Γ equals*

$$\mathsf{val}(\Gamma) = \begin{cases} |H\colon H \cap H^a| & \text{if } HaH = Ha^{-1}H, \\ 2|H\colon H \cap H^a| & \text{otherwise.} \end{cases}$$

Lemma 2.2. *Let Γ be a connected X-vertex-transitive graph where $X \leqslant \mathsf{Aut}\,\Gamma$, and let $N \lhd X$ be intransitive on $V\Gamma$. Assume that Γ is a cover of Γ_N. Then N is semiregular on $V\Gamma$, and the kernel of X acting on $V\Gamma_N$ equals N.*

Proof. Let K be the kernel of X acting on $V\Gamma_N$. Then $N \lhd K \lhd X$. Suppose that $K_v \neq 1$, where $v \in V\Gamma$. Then since Γ is connected and $K \lhd X$, it follows that $K_v^{\Gamma(v)} \neq 1$. Thus the number of K_v-orbits in $\Gamma(v)$ is less than $|\Gamma(v)|$, and so the valency of Γ_N is less than the valency of Γ, which is a contradiction. Hence $K_v = 1$, and it follows that $N = K$ is semiregular on $V\Gamma$. □

For a Cayley graph $\Gamma = \mathsf{Cay}(G, S)$, let $\mathsf{Aut}(G, S) = \{\alpha \in \mathsf{Aut}(G) \mid S^\alpha = S\}$. For the normal edge-transitive case, we have a simple lemma.

Lemma 2.3. *Let $\Gamma = \mathsf{Cay}(G, S)$ be connected of valency 4. Assume that $\mathsf{Aut}\,\Gamma$ has a subgroup X such that Γ is X-edge-transitive and $G \lhd X$. Then $X \leqslant \mathbf{N}_{\mathsf{Aut}\,\Gamma}(G) = G \rtimes \mathsf{Aut}(G, S)$, and either $X_{\mathbf{1}} \leqslant \mathrm{D}_8$, or Γ is $(X, 2)$-transitive and $|G|$ is even.*

Proof. Since Γ is connected, $\langle S\rangle = G$, and so $\mathsf{Aut}(G, S)$ acts faithfully on S. Hence $\mathsf{Aut}(G, S) \leqslant \mathrm{S}_4$. By [11, Lemma 2.1], we have that $X \leqslant \mathbf{N}_{\mathsf{Aut}\,\Gamma}(G) = G \rtimes \mathsf{Aut}(G, S)$. Thus $X_{\mathbf{1}} \leqslant \mathsf{Aut}(G, S) \leqslant \mathrm{S}_4$. Suppose that 3 divides $|X_{\mathbf{1}}|$. Then $X_{\mathbf{1}}$ is 2-transitive on S. Hence Γ is $(X, 2)$-transitive, and all elements in S are involutions, see for example [17]. In particular, $|G|$ is even. On the other hand, if 3 does not divide $|X_{\mathbf{1}}|$, then $X_{\mathbf{1}}$ is a 2-group, and hence $X_{\mathbf{1}} \leqslant \mathrm{D}_8$. □

Lemma 2.4. *Let G be a finite group of odd order, and let $\Gamma = \mathsf{Cay}(G, S)$ be connected and of valency 4. Assume that $N \lhd X \leqslant \mathsf{Aut}\,\Gamma$ such that $G \leqslant X$ and Γ is X-edge-transitive. Then one of the following statements holds*:

(i) *N has odd order and $N \leqslant G$*;

(ii) *N has even order, and either N is transitive on $V\Gamma$, or GN is transitive on $E\Gamma$.*

Proof. Let $Y = GN$. Then Y is transitive on $V\Gamma$. Suppose that $N \nleqslant G$. Then Y is not regular on $V\Gamma$. It follows that $Y_{\mathbf{1}}$ is a nontrivial $\{2, 3\}$-group. If $Y_{\mathbf{1}}$ has an orbit of size 3 on $\Gamma(\mathbf{1}) = S$, then Y has an orbit on $E\Gamma$ which is a 1-factor of Γ, which is not possible since $|V\Gamma| = |G|$ is odd. It follows that either $Y_{\mathbf{1}}$ is transitive on S, or $Y_{\mathbf{1}}$ has an orbit of size 2 on S. In particular, $|Y_{\mathbf{1}}|$ is even, so $|N|$ is even. Therefore, either N has odd order and $N \leqslant G$, as in part(i), or N has even order.

Assume now that $|N|$ is even. If $Y_{\mathbf{1}}$ is transitive on S, then Γ is Y-arc-transitive and hence Y-edge-transitive, so part (ii) holds. Thus assume that $Y_{\mathbf{1}}$ has an orbit of size 2 on S. Noting that $N \lhd X$, $N_{\mathbf{1}} \neq 1$ and Γ is connected and X-vertex-transitive, it is easily shown that $N_{\mathbf{1}}$ is non-trivial on S. Since $N_{\mathbf{1}} \leqslant Y_{\mathbf{1}}$, $N_{\mathbf{1}}$ has an orbit $\{x, y\}$ of size 2 on S. Suppose that N is intransitive on $V\Gamma$. Let $H = \mathbf{1}^N$ be the N-orbit containing $\mathbf{1}$. Then $H \cap S = \emptyset$ as Γ is X-edge-transitive. Further, $x^N = (\mathbf{1}^x)^N = \mathbf{1}^{(xNx^{-1})x} = (\mathbf{1}^N)^x = Hx$ and $y^N = (\mathbf{1}^y)^N = \mathbf{1}^{(yNy^{-1})y} = (\mathbf{1}^N)^y = Hy$, and so $Hx = x^N = y^N = Hy$. It is easily shown that H forms a subgroup of G. In particular, $xy^{-1} \in H$. If $y = x^{-1}$, then $x^2 = xy^{-1} \in H$, and $x \in H$ as $|H|$ is odd, a contradiction. Thus $S = \{x, y, x^{-1}, y^{-1}\}$. Clearly, $\{x, y\}$ is an orbit of $Y_{\mathbf{1}}$ on S. It follows that Y is transitive on $E\Gamma$, as in part (ii). □

By the result of [15], there is no 4-arc-transitive graph of valency at least 3 on odd number of vertices. Then by the known results about 2-arc-transitive graphs (see for example [29] or [16, Section 3.1]), the following result holds.

Lemma 2.5. *Let Γ be a connected (X, s)-transitive graph of valency 4. Then either $s \leqslant 4$ or $s = 7$, and further, s and the stabliser X_v are listed as following*

s	X_v
1	2-*group*
2	$A_4 \leqslant X_v \leqslant S_4$
3	$A_4 \times \mathbb{Z}_3 \leqslant X_v \leqslant S_4 \times S_3$
4	$\mathbb{Z}_3^2.SL(2,3) \leqslant X_v \leqslant \mathbb{Z}_3^2.GL(2,3)$
7	$[3^5].SL(2,3) \leqslant X_v \leqslant [3^5].GL(2,3)$

Moreover, if $|V\Gamma|$ is odd, then $s \leqslant 3$.

Finally, we quote a result about simple groups, which will be used later.

Lemma 2.6 (*Kazarin [13]*). *Let T be a non-abelian simple group which has a $2'$-Hall subgroup. Then $T = PSL(2, p)$, where $p = 2^e - 1$ is a prime. Further, $T = GH$, where $G = \mathbb{Z}_p \rtimes \mathbb{Z}_{\frac{p-1}{2}}$ and $H = D_{p+1} = D_{2^e}$.*

3. Existence of graphs satisfying Theorem 1.1

In this section, we construct examples of graphs satisfying Theorem 1.1.

First consider part (1) of Theorem 1.1. We observe that if Γ is a connected normal edge-transitive Cayley graph of a group G of valency 4, then $G = \langle a, a^\tau \rangle$, where $\tau \in \mathsf{Aut}(G)$ such that $a^{\tau^2} = a$ or a^{-1}. Conversely, if G is a group that has a presentation $G = \langle a, a^\tau \rangle$, where $\tau \in \mathsf{Aut}(G)$ such that $a^{\tau^2} = a$ or a^{-1}, then G has a connected normal edge-transitive

Cayley graph of valency 4, that is, $\mathsf{Cay}(G, S)$ where $S = \{a, a^{-1}, a^{\tau}, (a^{\tau})^{-1}\}$. Thus we have the following conclusion:

Lemma 3.1. *Let G be a group of odd order. Then G has a connected normal edge-transitive Cayley graph of valency* 4 *if and only if* $G = \langle a, a^{\tau}\rangle$, *where* $\tau \in \mathsf{Aut}(G)$ *such that* $a^{\tau^2} = a$ *or* a^{-1}.

See Construction 6.1 for an example of such graphs.

The following construction produces edge-transitive graphs admitting a group *X* satisfying part (3) of Theorem 1.1 with $O = 1$.

Construction 3.2. Let $X = N \rtimes (H \rtimes R) \cong \mathbb{Z}_p^k \rtimes (\mathbb{Z}_2^l \rtimes \mathbb{Z}_m)$, where p is an odd prime, m is odd and $2 \leqslant l \leqslant k$, such that $N \cong \mathbb{Z}_p^k$, $H \cong \mathbb{Z}_2^l$ and $R \cong \mathbb{Z}_m$ satisfy

(a) *N* is the unique minimal normal subgroup of *X*;
(b) there exist $x \in N \setminus \{1\}$ and $\tau \in H$ such that $x^{\tau} = x^{-1}$ and $H = \langle \tau \rangle \times \mathbf{C}_H(x)$;
(c) *R* does not centralise *H*.

Let $R = \langle \sigma \rangle \cong \mathbb{Z}_m$, and let $y = x\sigma$. Set

$$\Gamma(p, k, l, m) = \mathsf{Cos}(X, H, H\{y, y^{-1}\}H).$$

The next lemma shows that the graphs constructed here are as required.

Lemma 3.3. *Let* $\Gamma = \Gamma(p, k, l, m)$ *be a graph constructed in Construction* 3.2, *and let* $G = N \rtimes R \cong \mathbb{Z}_p^k \rtimes \mathbb{Z}_m$. *Then Γ is a connected X-edge-transitive Cayley graph of G of valency* 4, *and G is not normal in X.*

Proof. By the definition, *H* is core-free in *X*, and hence $X \leqslant \mathsf{Aut}\,\Gamma$. Now $X = GH$ and $G \cap H = 1$, and thus *G* acts regularly on the vertex set $[X{:}\ H]$. So Γ is a Cayley graph of *G*, which has odd order $p^k m$. Obviously, *G* is not normal in *X*.

For x and σ defined in Construction 3.2, set $x_i = x^{\sigma^{i-1}}$ for $i = 1, 2, \ldots, m$, and let $\alpha = (\sigma^{-1})^{\tau}\sigma$. Then, as $y = x\sigma$, $x_2 = \sigma^{-1}x\sigma$ and $\tau \in H$, we have

$$\alpha x_2^2 = ((\sigma^{-1})^{\tau}\sigma)(\sigma^{-1}x\sigma)^2 = (\sigma^{-1})^{\tau}x^2\sigma = (x^{-1}\sigma^{\tau})^{-1}(x\sigma) = (y^{\tau})^{-1}y \in \langle H, y\rangle.$$

As $\tau \in H$ and σ normalises H, we have $\alpha = (\sigma^{-1})^{\tau}\sigma = \tau(\tau^{\sigma}) \in H$. Thus, $x_2^2 = \alpha^{-1}(\alpha x_2^2) \in \langle H, y\rangle$, and as x_2 has odd order, $x_2 \in \langle H, y\rangle$. Then $x_3 = x_2^{\sigma} = x_2^{x_1\sigma} = x_2^{y} \in \langle H, y\rangle$. Similarly, we have that $x_i \in \langle H, y\rangle$ for $i = 2, 3, \ldots, m$. Then calculation shows that $y^m = x_1x_2\cdots x_m \in \langle H, y\rangle$. Thus $x = x_1 = y^m x_2^{-1}\cdots x_m^{-1} \in \langle H, y\rangle$, and so $\sigma = x^{-1}y \in \langle H, y\rangle$. Since *N* is a minimal normal subgroup of *X*, we conclude that $N = \langle x^{h\sigma^i} \mid h \in H, 0 \leqslant i \leqslant m-1\rangle$, and hence $N \leqslant \langle H, y\rangle$. So $\langle H, y\rangle \geqslant \langle N, H, \sigma\rangle = X$, and Γ is connected.

Finally, as σ normalises *H* and by condition (b) of Construction 3.2, we have that $H^x \cap H = \mathbf{C}_H(x)$ has index 2 in *H*. Thus $H^y \cap H = (H^x \cap H^{\sigma^{-1}})^{\sigma} = (H^x \cap H)^{\sigma} = \mathbf{C}_H(x)^{\sigma}$, which has index 2 in *H*. Since $X \leqslant \mathsf{Aut}\,\Gamma$, Γ is not a cycle. By Lemma 2.1, Γ is connected, *X*-edge-transitive and of valency 4. □

We end this section by presenting several groups satisfying (a), (b) and (c) of Construction 3.2, so we obtain examples of graphs satisfying Theorem 1.1(3).

Example 3.4. Let p be an odd prime, and m an odd integer.

(i) Let $X = (\langle x_1, \tau_1\rangle \times \langle x_2, \tau_2\rangle \times \cdots \times \langle x_m, \tau_m\rangle)\rtimes\langle\sigma\rangle \cong \mathrm{D}_{2p} \wr \mathbb{Z}_m = \mathrm{D}_{2p}^m \rtimes \mathbb{Z}_m$, where $\langle x_i, \tau_i\rangle \cong \mathrm{D}_{2p}$ and $(x_i, \tau_i)^\sigma = (x_{i+1}, \tau_{i+1})$ (reading the subscripts modulo m). Then $N = \langle x_1, x_2, \ldots, x_m\rangle \cong \mathbb{Z}_p^m$ is a minimal normal subgroup of X, and $H = \langle \tau_1, \tau_2, \ldots, \tau_m\rangle \cong \mathbb{Z}_2^m$ is such that $H = \langle\tau_i\rangle \times \mathbf{C}_H(x_i)$ for $1 \leqslant i \leqslant m$.

(ii) Let $Y < X$ with X as in part (i) such that $Y = \langle x_1, x_2, \ldots, x_m\rangle \rtimes \langle \tau_1\tau_2, \tau_2\tau_3, \ldots, \tau_{m-1}\tau_m\rangle \rtimes \langle\sigma\rangle \cong \mathbb{Z}_p^m \rtimes (\mathbb{Z}_2^{m-1} \rtimes \mathbb{Z}_m)$. Then $N = \langle x_1, x_2, \ldots, x_m\rangle$ is a minimal normal subgroup of Y, and $L := \langle \tau_1\tau_2, \tau_2\tau_3, \ldots, \tau_{m-1}\tau_m\rangle \cong \mathbb{Z}_2^{m-1}$ is such that $L = \langle\tau_i\tau_{i+1}\rangle \times \mathbf{C}_L(x_i)$ for $1 \leqslant i \leqslant m$.

Thus both X and Y satisfy the conditions of Construction 3.2.

Example 3.5. Let $N = \langle x_1, \ldots, x_k\rangle = \mathbb{Z}_p^k$, where p is an odd prime and $k \geqslant 3$. Let l be a proper divisor of k. Let $\sigma \in \mathsf{Aut}(N)$ be such that

$$x_i^\sigma = \begin{cases} x_{i+1} & \text{if } 1 \leqslant i \leqslant k-1, \\ x_1 x_{l+1} & \text{if } i = k. \end{cases}$$

Let $\tau \in \mathsf{Aut}(N)$ be such that

$$x_j^\tau = \begin{cases} x_j^{-1} & \text{if } l \mid j-1, \\ x_j & \text{otherwise.} \end{cases}$$

Let $o(\sigma) = m$, $H = \langle \tau^{\sigma^{i-1}} \mid 1 \leqslant i \leqslant m\rangle$ and $X = N \rtimes \langle\tau, \sigma\rangle$. Then N is a minimal normal subgroup of X and $H = \langle\tau\rangle \times \mathbf{C}_H(x_1) \cong \mathbb{Z}_2^l$. Thus, X satisfies the conditions of Construction 3.2.

For instance, taking $p = 3$, $k = 9$ and $l = 3$, so $m = 39$, and then applying Construction 3.2, we obtain an X-edge-transitive Cayley graph $\Gamma(3, 9, 3, 39)$ of valency 4 of the group $\mathbb{Z}_3^9 \rtimes \mathbb{Z}_{39}$, where $X = \mathbb{Z}_3^9 \rtimes (\mathbb{Z}_2^3 \rtimes \mathbb{Z}_{39})$.

4. A family of arc-transitive graphs of valency 4

Here, we construct a family of 4-arc-transitive cubic graphs and their line graphs. The smallest line graph is PGL(2, 7)-arc-transitive but not PSL(2, 7)-edge-transitive, which is one of the graphs stated in Theorem 1.1(4).

Construction 4.1. Let p be a prime such that $p \equiv -1 \pmod 8$, and let $T = \mathrm{PSL}(2, p)$ and $X = \mathrm{PGL}(2, p)$. Then T has exactly two conjugacy classes of maximal subgroups isomorphic to S_4 which are conjugate in X. Let $L, R < T$ be such that $L, R \cong \mathrm{S}_4$, $L \cap R \cong \mathrm{D}_8$, and L, R are not conjugate in T but $L^\tau = R$ for some involution $\tau \in X \setminus T$.

(1) Let $\Sigma = \mathsf{Cos}(T, L, R)$ be the *coset graph* defined as: the vertex set $V\Sigma = [T : L] \cup [T : R]$ such that Lx is adjacent to Ry if and only if $yx^{-1} \in LR$.

(2) Let Γ be the *line graph* of Σ, that is, the vertices of Γ are the edges of Σ and two vertices of Γ are adjacent if and only if the corresponding edges of Σ have exactly one common vertex.

Then it follows from the definition that Σ is bipartite with parts $[T:L]$ and $[T:R]$, and T acts by right multiplication transitively on the edge set $E\Sigma$. Further, we have the following properties:

Lemma 4.2. *The following statements hold for the graph Σ defined above*:

(i) *Σ is connected and of valency* 3;
(ii) *Σ may also be represented as the coset graph* $\mathsf{Cos}(X, L, L\tau L)$;
(iii) *Σ is $(X,4)$-arc-transitive*;
(iv) *Σ is T-vertex intransitive and locally $(T,4)$-arc-transitive.*

Proof. Since $\langle L, R\rangle = T$, part (i) follows from the definition, see [10, Lemma 2.7].

Part (ii) follows from the definitions of $\mathsf{Cos}(T, L, R)$ and $\mathsf{Cos}(X, L, L\tau L)$.

See [1] or [16, Example 3.5] for part (iii).

It follows from the definition that T is not transitive on the vertex set $V\Sigma$, and so part (iv) follows from part (iii). □

Next we study the line graph Γ in the following lemma.

Lemma 4.3. *Let Γ be the line graph of Σ defined as in Construction* 4.1. *Let v be the vertex of Γ corresponding to the edge $\{L, R\}$ of Σ. Then we have the following statements*:

(i) *Γ is connected, and has valency* 4 *and girth* 3;
(ii) *Γ is X-arc-transitive, and $X_v \cong \mathrm{D}_{16}$*;
(iii) *T is transitive on $V\Gamma$ and intransitive on $E\Gamma$, and $T_v \cong \mathrm{D}_8$*;
(iv) *T has exactly two orbits E_1, E_2 on $E\Gamma$, and letting $\Delta_1 = (V\Gamma, E_1)$ and $\Delta_2 = (V\Gamma, E_2)$, we have $\Delta_1 \cong \Delta_2 \cong \frac{p(p^2-1)}{48}\mathbf{C}_3$, and $\Gamma = \Delta_1 + \Delta_2$.*

Proof. We first look at the neighbors of the vertex v in Γ. Let $a \in L$ be of order 3, and let $b = a^\tau \in R$. Then the 3 neighbors of L in Σ are R, Ra and Ra^{-1}; and the 3 neighbors of R are L, Lb and Lb^{-1}. Write the corresponding vertices of Γ as: $u_1 = \{Lb, R\}$, $u_2 = \{Lb^{-1}, R\}$, $w_1 = \{L, Ra\}$ and $w_2 = \{L, Ra^{-1}\}$. Then the neighborhood $\Gamma(v) = \{u_1, u_2, w_1, w_2\}$.

Thus Γ is of valency 4. By the definition of a line graph, u_1 is adjacent to u_2, and w_1 is adjacent to w_2. Hence the girth of Γ is 3. Since Σ is connected, Γ is connected too, proving part (i).

Now $T_v = L \cap R \cong \mathrm{D}_8$ and $X_v = \langle L \cap R, \tau\rangle \cong \mathrm{D}_{16}$. Since T is transitive on $E\Sigma$ and is not transitive on the vertex set $V\Sigma$, there is no element of T maps the arc (L, R) to the arc (R, L). Since $T_v = L \cap R$, there exist $\sigma_1, \sigma_2 \in T_v$ such that $a^{\sigma_1} = a^{-1}$ and $b^{\sigma_2} = b^{-1}$. Thus $u_1^{\sigma_1} = u_2$ and $w_1^{\sigma_2} = w_2$. So T_v has exactly two orbits on $\Gamma(v)$, that is, $\{u_1, u_2\}$ and $\{w_1, w_2\}$. Further, $\langle b\rangle$ acts transitively on $\{v, u_1, u_2\}$. It follows that $E_1 := \{u_1, u_2\}^T$ is a self-paired orbital of T on $V\Gamma$. Therefore, Γ is not T-edge-transitive. Further, since τ interchanges L and R and also interchanges a and b, it follows that $\tau \in X_v$ and $\{u_1, u_2\}^\tau = \{w_1, w_2\}$. Thus Γ is X-arc-transitive.

Let $E_2 = \{w_1, w_2\}^T$, and let $\Delta_i = (V\Gamma, E_i)$ with $i = 1, 2$. Then $\Gamma = \Delta_1 + \Delta_2$, and Δ_i consists of cycles of size 3. Thus $|E_1| = |E_2| = |V\Gamma| = \frac{|X|}{|X_v|} = \frac{p(p^2-1)}{16}$, and Δ_i consists of $\frac{|E_i|}{3}$ cycles of size 3, that is, $\Delta_i \cong \frac{p(p^2-1)}{48}\mathbf{C}_3$. Finally, $E_1^\tau = E_2$ and so τ is an isomorphism between Δ_1 and Δ_2. □

5. Proof of Theorem 1.1

Let G be a finite group of odd order, and let $\Gamma = \mathsf{Cay}(G, S)$ be connected and of valency 4. Assume that Γ is X-edge-transitive, where $G \leqslant X \leqslant \mathsf{Aut}\,\Gamma$, and assume further that G is not normal in X.

We first treat the case where Γ has no non-trivial normal quotient of valency 4 in Sections 5.1 and 5.2.

Suppose that each non-trivial normal quotient of Γ is a cycle. Let N be a minimal normal subgroup of X. Then $N = T^k$ for some simple group T and some integer $k \geqslant 1$. Since $|V\Gamma| = |G|$ is odd, X has no nontrivial normal 2-subgroups. In particular, N is not a 2-group. Further we have the following simple lemma.

Lemma 5.1. *Either N is soluble, or $\mathbf{C}_X(N) = 1$.*

Proof. Suppose that N is insoluble and $C := \mathbf{C}_X(N) \neq 1$. Then $NC = N \times C$ and $C \lhd X$. Since $|N|$ is not semiregular on $V\Gamma$, C is intransitive. By the assumption that any non-trivial normal quotient of Γ is a cycle, Γ_C is a cycle. Let K be the kernel of X acting on $V\Gamma_C$. Then $X/K \leqslant \mathsf{Aut}\,\Gamma_C \cong \mathrm{D}_{2c}$, where $c = |V\Gamma_C|$. It follows that $N \leqslant K$. Let Δ be an arbitrary C-orbit on $V\Gamma$. Then Δ is N-invariant. Consider the action of NC on Δ, and let D be the kernel of NC acting on Δ. Then $NC/D = (ND/D) \times (CD/D)$. Since C is transitive on Δ, CD/D is also transitive on Δ. Then ND/D is semiregular on Δ. Noting that $|\Delta|$ is odd and $ND/D \cong N/(N \cap D) \cong T^{k'}$ for some $k' \geqslant 0$, it follows that ND/D is trivial on Δ, and hence $N \leqslant D$. Thus N is trivial on every C-orbit, and so N is trivial on $V\Gamma$, which is a contradiction. Therefore, either N is soluble, or $\mathbf{C}_X(N) = C = 1$. □

5.1. The case where N is transitive

Assume that N is transitive on the vertices of Γ. Our goal is to prove that $N = \mathrm{A}_5$, $\mathrm{PSL}(2, 7)$, $\mathrm{PSL}(2, 11)$ or $\mathrm{PSL}(2, 23)$ by a series of lemmas. The first shows that N is nonabelian simple.

Lemma 5.2. *The minimal normal subgroup N is a nonabelian simple group, X is almost simple, and $N = \mathsf{soc}(X)$.*

Proof. Suppose that N is abelian. Since N is transitive, N is regular, and hence $|N| = |G|$ is odd. By Lemma 2.3, we have that $N \leqslant G$, and so $G = N \lhd X$, which is a contradiction. Thus $N = T^k$ is nonabelian. Suppose that $k > 1$. Let L be a normal subgroup of N such that $L \cong T^{k-1}$. Since $N_{\mathbf{1}} \leqslant X_{\mathbf{1}}$ is a $\{2, 3\}$-group, it follows that L is intransitive on $V\Gamma$; further, since $|V\Gamma|$ is odd and $|T|$ is even, L is not semiregular. It follows from Lemma 2.2 that

Γ_L is a cycle. Then $\mathsf{Aut}\,\Gamma_L$ is a dihedral group. Thus N lies in the kernel of X acting on $V\Gamma_L$, and so N is intransitive on $V\Gamma$, which is a contradiction. Thus $k = 1$, and $N = T$ is nonabelian simple. By Lemma 5.1, $\mathbf{C}_X(N) = 1$, and hence N is the unique minimal normal subgroup of X. Thus X is almost simple, and $N = \mathsf{soc}(X)$. □

The 2-arc-transitive case is determined by the following lemma.

Lemma 5.3. *Assume Γ is $(X, 2)$-arc-transitive. Then one of the following holds*:

(i) $X = \mathrm{A}_5$ *or* S_5, *and* $X_{\mathbf{1}} = \mathrm{A}_4$ *or* S_4, *respectively*, *and* $G = \mathbb{Z}_5$;
(ii) $X = \mathrm{PSL}(2, 11)$ *or* $\mathrm{PGL}(2, 11)$, *and* $X_{\mathbf{1}} = \mathrm{A}_4$ *or* S_4, *respectively*, *and* $G = \mathbb{Z}_{11} \rtimes \mathbb{Z}_5$;
(iii) $X = \mathrm{PSL}(2, 23)$, $X_{\mathbf{1}} = \mathrm{S}_4$, *and* $G = \mathbb{Z}_{23} \rtimes \mathbb{Z}_{11}$.

Proof. Note that $X = GX_{\mathbf{1}}$ and $G \cap X_{\mathbf{1}} = 1$. By Lemma 2.5, $|X_{\mathbf{1}}|$ is a divisor of $2^4 3^2 = 144$, and hence a Sylow 2-subgroup of X is isomorphic to a subgroup of $\mathrm{D}_8 \times \mathbb{Z}_2$. Further, $|N : (G \cap N)| = |GN: G|$ divides $|X: G| = |X_{\mathbf{1}}|$. Let M be a maximal subgroup of N containing $G \cap N$. Then $[N: M]$ has size dividing 144, and N is a primitive permutation group on $[N: M]$. Inspecting the list of primitive permutation groups of small degree given in [3, Appendix B], we conclude that N is one of the following groups:

A_5, A_6, $\mathrm{PSL}(2, 7)$, $\mathrm{PSL}(2, 8)$, $\mathrm{PSL}(2, 11)$, M_{11}, $\mathrm{PSL}(2, 17)$, $\mathrm{PSL}(2, 23)$, $\mathrm{PSL}(2, 47)$, $\mathrm{PSL}(2, 71)$ and $\mathrm{PSL}(3, 3)$.

It is known that the groups M_{11}, $\mathrm{PSL}(2, 17)$, $\mathrm{PSL}(2, 47)$ and $\mathrm{PSL}(3, 3)$ have a Sylow 2-subgroup isomorphic to $\mathrm{Q}_8.\mathbb{Z}_2$, D_{16}, D_{16} and $\mathbb{Z}_2.\mathrm{Q}_8$, respectively. Thus N is none of these groups. Suppose that $N = \mathrm{A}_6$ or $\mathrm{PSL}(2, 8)$. Then $X = \mathrm{A}_6$, S_6, $\mathrm{PSL}(2, 8)$ or $\mathrm{PSL}(2, 8).\mathbb{Z}_3$. However, X has no factorisation $X = GX_{\mathbf{1}}$ such that $G \cap X_{\mathbf{1}} = 1$, and $X_{\mathbf{1}}$ is a $\{2, 3\}$-group, which is a contradiction. Suppose that $N = \mathrm{PSL}(2, 71)$. Then $X = \mathrm{PSL}(2, 71)$ or $\mathrm{PGL}(2, 71)$, and $X_{\mathbf{1}} = \mathrm{D}_{72}$ or D_{144}, respectively, and $G = \mathbb{Z}_{71} \rtimes \mathbb{Z}_{35}$. Thus $X_{\mathbf{1}}$ is a maximal subgroup of X, and X acts primitively on the vertex set $V\Gamma = [X: X_{\mathbf{1}}]$. This is not possible, see [28] or [18]. If $N = \mathrm{PSL}(2, 7)$, then $G = \mathbb{Z}_7$ and $N_{\mathbf{1}} = \mathrm{S}_4$. Then, however, N is 2-transitive on $V\Gamma = [N: N_{\mathbf{1}}]$, and so $\Gamma \cong \mathsf{K}_7$, which is a contradiction.

Therefore, $N = \mathrm{A}_5$, $\mathrm{PSL}(2, 11)$ or $\mathrm{PSL}(2, 23)$. Now either X is primitive on $V\Gamma$, or $X = N = \mathrm{PSL}(2, 11)$ and $G = \mathbb{Z}_{11} \rtimes \mathbb{Z}_5$. Then, by [27] and [12], we obtain the conclusion stated in the lemma. □

The next lemma determines X for the case where Γ is not $(X, 2)$-arc-transitive.

Lemma 5.4. *Suppose that Γ is not $(X, 2)$-arc-transitive. Then* $X = \mathrm{PGL}(2, 7)$, $X_{\mathbf{1}} = \mathrm{D}_{16}$ *and* $G = \mathbb{Z}_7 \rtimes \mathbb{Z}_3$.

Proof. Since Γ is not $(X, 2)$-arc-transitive, $X_{\mathbf{1}}$ is a 2-group. Since $X = GX_{\mathbf{1}}$ and $G \cap X_{\mathbf{1}} = 1$, G is a $2'$-Hall subgroup of X. Then $G \cap N$ is a $2'$-Hall subgroup of N. By Lemma 5.2, N is nonabelian simple. By Lemma 2.6, $N = \mathrm{PSL}(2, p)$, $G \cap N = \mathbb{Z}_p \rtimes \mathbb{Z}_{\frac{p-1}{2}}$, and $N_{\mathbf{1}} = \mathrm{D}_{p+1}$, where $p = 2^e - 1$ is a prime. If $e > 3$, then $N_{\mathbf{1}}$ is a maximal subgroup of N. Thus N is a primitive permutation group on $V\Gamma$ and has a self-paired suborbit of length 4, which is not possible, see [28] or [18]. Thus $e = 3$, $N = \mathrm{PSL}(2, 7)$, $G = \mathbb{Z}_7 \rtimes \mathbb{Z}_3$, and $N_{\mathbf{1}} = \mathrm{D}_8$. So $X = \mathrm{PSL}(2, 7)$ or $\mathrm{PGL}(2, 7)$.

Suppose that $X = \mathrm{PSL}(2,7)$. Now write Γ as coset graph $\mathsf{Cos}(X, H, H\{x, x^{-1}\}H)$, where $H = X_{\mathbf{1}} = \mathrm{D}_8$, and $x \in X$ is such that $\langle H, x\rangle = X$. Let $P = H \cap H^x$. Then $|H\colon P| = 2$ or 4.

Assume that $|H\colon P| = 4$. Then Γ is X-arc-transitive and $P = \mathbb{Z}_2$. By Lemma 2.1, we may assume that $x^2 \in P = H \cap H^x$ and x normalises P. If $P \lhd H$, then $P \lhd \langle H, x\rangle = X = \mathrm{PSL}(2,7)$, which is a contradiction. Thus P is not normal in H, and so $\mathbb{Z}_2^2 \cong \mathbf{N}_H(P) \lhd H$. Since $\mathbf{N}_X(P) \cong \mathrm{D}_8$, we have $\mathbf{N}_X(P) \neq H$. So $\mathbf{N}_H(P) \lhd \langle H, \mathbf{N}_X(P)\rangle = X$, which is a contradiction. Thus $|H : P| = 2$, and hence $P \lhd L := \langle H, H^x\rangle$. We conclude that $L \cong \mathrm{S}_4$. Then H and H^x are two Sylow 2-subgroups of L, and hence $H^x = H^y$ for some $y \in L$. Thus $H^{xy^{-1}} = H$, that is, $xy^{-1} \in \mathbf{N}_X(H) = H$, hence $x \in Hy \subseteq L$. Then $\langle x, H\rangle \leqslant L \neq X$, which is a contradiction. Thus $X \neq \mathrm{PSL}(2,7)$, and so $X = \mathrm{PGL}(2,7)$. □

5.2. *The case where N is intransitive*

Assume now that the minimal normal subgroup $N \lhd X$ is intransitive on $V\Gamma$. We are going to prove that part (3) of Theorem 1.1 occurs.

Lemma 5.5. *The minimal normal subgroup N is soluble, and $N < G$.*

Proof. Suppose that N is insoluble. Then $N = T^k$ and $N \nleqslant G$, where T is nonabelian simple and $k \geqslant 1$. Let $Y = NG$. Then by Lemma 2.4 Y is transitive on both of $V\Gamma$ and $E\Gamma$. Let $L \leqslant N$ be a non-trivial normal subgroup of Y. Then L is intransitive, and since $|V\Gamma|$ is odd, L is not semi-regular on $V\Gamma$. Thus the valency of the quotient graph Γ_L is less than 4. Since $|V\Gamma|$ is odd, Γ_L is a cycle of size $m \geqslant 3$. Let K be the kernel of Y acting on the L-orbits in $V\Gamma$. Then $Y/K \leqslant \mathsf{Aut}\,\Gamma_L \cong \mathrm{D}_{2m}$, where $m = |V\Gamma_L|$. Further, since $NK/K \cong N/(N \cap K) \cong T^l$ for some l, we conclude that $l = 0$ and $N \leqslant K$. Considering the action of N on an arbitrary L-orbit, we have that $L = N$. This particularly shows that N is a minimal normal subgroup of Y. As Γ_N is a cycle, Γ is not $(X,2)$-arc-transitive, and $X_{\mathbf{1}}$ is a nontrivial 2-group. In particular, $K_{\mathbf{1}}$ is a 2-group. Since $K = NK_{\mathbf{1}} \leqslant Y$ and $|Y : N|$ is odd, we know that $K = N$. Thus N itself is the kernel of X acting on $V\Gamma_N$. It follows that Y/N is the cyclic regular subgroup of $\mathsf{Aut}\,\Gamma_N$ acting on $V\Gamma_N$. Thus $Y = NG = N\langle a\rangle \cong N.\mathbb{Z}_m$ for some $a \in G \setminus N$.

Since $X_{\mathbf{1}}$ is a nontrivial 2-group, it is easily shown that $G \cap N$ is a $2'$-Hall subgroup of N, and $N = (G \cap N)N_{\mathbf{1}}$. Then $G \cap T = G \cap N \cap T$ is a $2'$-Hall subgroup of T. By Lemma 2.6, $T = \mathrm{PSL}(2,p)$ for a prime $p = 2^e - 1$. In particular, $\mathsf{Out}(T) \cong \mathbb{Z}_2$. By Lemma 5.1, $\mathbf{C}_X(N) = 1$, and hence $\mathbf{C}_Y(N) = 1$. Then N is the only minimal normal subgroup of Y and of X. So the element $a \in Y \leqslant X \leqslant \mathsf{Aut}(N) = \mathsf{Aut}(T) \wr \mathrm{S}_k$. Write $N = T_1 \times \cdots \times T_k$, where $T_i \cong T$. Then $\mathsf{Aut}(N) = (\mathsf{Aut}(T_1) \times \mathsf{Aut}(T_2) \times \cdots \times \mathsf{Aut}(T_k)) \rtimes \mathrm{S}_k$, and $a = b\pi$, where $b \in \mathsf{Aut}(T_1) \times \mathsf{Aut}(T_2) \times \cdots \times \mathsf{Aut}(T_k)$ and $\pi \in \mathrm{S}_k$.

Since N is a minimal normal subgroup of Y, we have that $\langle a\rangle$ acts by conjugation transitively on $\{T_1, T_2, \ldots, T_k\}$, and hence the permutation π is a k-cycle of S_k. Relabeling if necessary, we may assume $\pi = (12\ldots k) \in \mathrm{S}_k$. Then $T_k^a = T_1$ and $T_i^a = T_{i+1}$, where $i = 1, \ldots, k-1$. Further, $a^k = b^{\pi^k} \cdots b^{\pi} \in \mathsf{Aut}(T_1) \times \mathsf{Aut}(T_2) \times \cdots \times \mathsf{Aut}(T_k) = N \rtimes \mathbb{Z}_2^k$. Since a^k is of odd order, it follows that $a^k \in N$. Thus $Y/N \cong \mathbb{Z}_k$, and hence $m = k$. Set $a^k = t_1 t_2 \cdots t_k$, where $t_i \in T_i$. Since a centralises a^k, we have $t_1 t_2 \cdots t_k = a^k = (a^k)^a = t_1^a t_2^a \cdots t_k^a$. Since $t_k^a \in T_k^a = T_1$ and $t_i^a \in T_i^a = T_{i+1}$, it follows that $t_k^a = t_1$

and $t_i^a = t_{i+1}$, where $i = 1, \ldots, k-1$. Let $g = t_1^{-1}a$. Then $T_i = T_{i-1}^g = T_1^{g^{i-1}}$ and $g^i = a^i t_{i+1}^{-1} t_i^{-1} \ldots t_2^{-1}$ (reading the subscripts modular k), where $2 \leqslant i \leqslant k$. In particular, $g^k = a^k t_1^{-1} t_k^{-1} \ldots t_2^{-1} = 1$, and so the order of g is a divisor of k. Noting that $Y/N \cong \mathbb{Z}_k$ and $N\langle g\rangle = \langle N, g\rangle = \langle N, t_1^{-1}a\rangle = \langle N, a\rangle = Y$, it follows that $Y = N \rtimes \langle g\rangle$.

Let $H_1 = (T_1)_{\mathbf{1}}$ and $H_i := H_1^{g^{i-1}}$ for $1 \leqslant i \leqslant k$, and let $H = H_1 \times \cdots \times H_k$. Then $H_i \cong \mathrm{D}_{2^e}$ is a Sylow 2-subgroup of T_i, H is a Sylow 2-subgroup of N, and $H^g = H$. Since Γ_N is a k-cycle and $Y/N \cong \mathbb{Z}_k$, it follows that Γ is not Y-arc-transitive. Since Γ is Y-edge-transitive, we may write Γ as a coset graph $\Gamma = \mathsf{Cos}(Y, H, H\{g^j x, (g^j x)^{-1}\}H)$, where $1 \leqslant j < k$ and $x = x_1 \cdots x_k \in N$ for $x_i \in T_i$, such that $|H : (H \cap H^{g^j x})| = 2$ and $\langle H, g^j x\rangle = Y$. Now $H^{g^j x} = H^x = H_1^{x_1} \times H_2^{x_2} \times \cdots \times H_k^{x_k}$ and $H \cap H^{g^j x} = (H_1 \cap H_1^{x_1}) \times \cdots \times (H_k \cap H_k^{x_k})$. Thus we may assume that $|H_1 : (H_1 \cap H_1^{x_1})| = 2$ and $H_i \cap H_i^{x_i} = H_i$. Then $H_i^{x_i} = H_i$ for $i = 2, \ldots, k$. Since $\mathbf{N}_{T_i}(H_i) = H_i$, we know that $x_i \in H_i$ for $i \geqslant 2$. If $e > 3$, then H_1 is maximal in T_1, and hence $H_1 \cap H_1^{x_1} \lhd \langle H_1, H_1^{x_1}\rangle = T_1$, which is a contradiction. Thus $e = 3$, $T_1 \cong \mathrm{PSL}(2,7)$. Let $U_1 = \langle H_1, x_1\rangle$ and $U_i = U_1^{g^{i-1}}$ for $i = 2, 3, \ldots, k$. Then $\mathrm{S}_4 \cong U_i < T_i$. It follows that $\langle U_1, g\rangle = (U_1 \times \cdots \times U_k) \rtimes \langle g\rangle \cong (\mathrm{S}_4)^k \rtimes \mathbb{Z}_k$. Since Γ is connected, $Y = \langle H, g^j x\rangle \leqslant \langle H_1, x_1, g\rangle = \langle U_1, g\rangle \cong (\mathrm{S}_4)^k \rtimes \mathbb{Z}_k$, which is again a contradiction.

Thus N is soluble. Then by Lemma 2.4, we have $N < G$, completing the proof. □

We notice that, since N is intransitive on $V\Gamma$, the N-orbits in $V\Gamma$ form an X-invariant partition $V\Gamma_N$. The next lemma determines the structure of X.

Lemma 5.6. *Let K be the kernel of X acting on $V\Gamma_N$. Then the following statements hold*:

(i) *$X/K \cong \mathbb{Z}_m$ or D_{2m} for an odd integer $m > 1$, $K_{\mathbf{1}} \neq 1$, and Γ is X-arc-transitive if and only if $X/K \cong \mathrm{D}_{2m}$*;
(ii) *$G = N \rtimes R$, $X = N \rtimes ((K_{\mathbf{1}} \rtimes R).O)$ and R does not centralise $K_{\mathbf{1}}$, where $R \cong \mathbb{Z}_m$, and $O = 1$ or $\mathbb{Z}_2$*;
(iii) *$N \cong \mathbb{Z}_p^k$ for an odd prime p, and $K_{\mathbf{1}} \cong \mathbb{Z}_2^l$, where $2 \leqslant l \leqslant k$*;
(iv) *there exist $x_1, \ldots, x_k \in N$ and $\tau_1, \ldots, \tau_k \in K_{\mathbf{1}}$ such that $N = \langle x_1, \ldots, x_k\rangle$, $\langle x_i, \tau_i\rangle \cong \mathrm{D}_{2p}$ and $K_{\mathbf{1}} = \langle \tau_i\rangle \times \mathbf{C}_{K_{\mathbf{1}}}(x_i)$ for $1 \leqslant i \leqslant k$.*
(v) *N is the unique minimal normal subgroup of X*;

Proof. By Lemma 5.5, $N < G$ is soluble, hence $N \cong \mathbb{Z}_p^k$ for an odd prime p and an integer $k \geqslant 1$. In particular, N is semi-regular on $V\Gamma$. Since Γ_N is a cycle of size m say, $X/K \leqslant \mathsf{Aut}\,\Gamma_N = \mathrm{D}_{2m}$. Thus $K = N \rtimes K_{\mathbf{1}}$, $K_{\mathbf{1}}$ is a 2-group, and $X/K \cong \mathbb{Z}_m$ or D_{2m}. It follows that $G/N \cong GK/K \cong \mathbb{Z}_m$. If $K_{\mathbf{1}} = 1$, then $K = N$, and hence $G \lhd X$, which contradicts that G is not normal in X. Thus $K_{\mathbf{1}} \neq 1$. Further, Γ is X-arc-transitive if and only if $X/K \cong \mathrm{D}_{2m}$, so we have part (i).

Set $U = \mathbf{N}_X(K_{\mathbf{1}})$. Then $U \neq X$ since $K_{\mathbf{1}}$ is not normal in X. Noting that $(|N|, |K_{\mathbf{1}}|) = 1$, it follows that $\mathbf{N}_{X/N}(K/N) = \mathbf{N}_{X/N}(NK_{\mathbf{1}}/N) = \mathbf{N}_X(K_{\mathbf{1}})N/N = UN/N$. Since K/N is normal in X/N, it follows that $X = UN$. Since $N \lhd X$, $N \cap U \lhd U$. Further $N \cap U \lhd N$ as N is abelian. Then $N \cap U \lhd \langle U, N\rangle = UN = X$. If $N \leqslant U$, then $K = NK_{\mathbf{1}} = N \times K_{\mathbf{1}}$, and hence $K_{\mathbf{1}} \lhd X$, a contradiction. Thus $N \cap U < N$. Further, since N is a minimal normal subgroup of X, we know that $N \cap U = 1$, and hence $K \cap U = NK_{\mathbf{1}} \cap U = (N \cap U)K_{\mathbf{1}} = K_{\mathbf{1}}$. Now $X/K = UN/K = UK/K \cong U/(K \cap U) = U/K_{\mathbf{1}}$, and so $U = (K_{\mathbf{1}} \rtimes R).O$, where $R \cong \mathbb{Z}_m$ and $O = 1$ or $\mathbb{Z}_2$. Then $G = N \rtimes R$, and $X_{\mathbf{1}} = K_{\mathbf{1}}.O$. Further, since G is not normal in X, we conclude that R does not centralise $K_{\mathbf{1}}$, as in part (ii).

Let $Y = KR = N \rtimes (K_{\mathbf{1}} \rtimes R)$. Then Y has index at most 2 in X, and Γ is Y-edge-transitive by Lemma 2.4, but it is not Y-arc-transitive. Thus $\Gamma = \mathsf{Cos}(Y, K_{\mathbf{1}}, K_{\mathbf{1}}\{y, y^{-1}\}K_{\mathbf{1}})$, where $y \in Y$ is such that $\langle K_{\mathbf{1}}, y\rangle = Y$ and $K_{\mathbf{1}} \cap K_{\mathbf{1}}^y$ has index 2 in $K_{\mathbf{1}}$. We may choose $y \in N \rtimes R = G$ such that $R = \langle\sigma\rangle$ and $y = \sigma x$ where $x \in N$. Then $K_{\mathbf{1}} \cap K_{\mathbf{1}}^y = K_{\mathbf{1}} \cap K_{\mathbf{1}}^x$ has index 2 in $K_{\mathbf{1}}$.

We claim that $K_{\mathbf{1}} \cap K_{\mathbf{1}}^x = \mathbf{C}_{K_{\mathbf{1}}}(x)$. Let $\sigma \in K_{\mathbf{1}} \cap K_{\mathbf{1}}^x$. Then $\sigma^{x^{-1}} \in K_{\mathbf{1}}$, and so $\sigma^{-1}\sigma^{x^{-1}} \in K_{\mathbf{1}}$. Since $x \in N$ and $N \lhd NK_{\mathbf{1}}$, we have $\sigma^{-1}\sigma^{x^{-1}} = (\sigma^{-1}x\sigma)x^{-1} \in N$. Thus $\sigma^{-1}\sigma^{x^{-1}} \in N \cap K_{\mathbf{1}} = 1$, and so $\sigma^{x^{-1}} = \sigma$. Then σ centralises x. It follows that $K_{\mathbf{1}} \cap K_{\mathbf{1}}^x \leqslant \mathbf{C}_{K_{\mathbf{1}}}(x)$. Clearly, $\mathbf{C}_{K_{\mathbf{1}}}(x) \leqslant K_{\mathbf{1}} \cap K_{\mathbf{1}}^x$. Thus $\mathbf{C}_{K_{\mathbf{1}}}(x) = K_{\mathbf{1}} \cap K_{\mathbf{1}}^x$ as claimed.

Since N is a minimal normal subgroup of X and $X = NU$, we have that $N = \langle x\rangle \times \langle x^{\sigma_2}\rangle \times \cdots \times \langle x^{\sigma_k}\rangle$ where $\sigma_i \in U$. Then $\mathbf{C}_{K_{\mathbf{1}}}(x^{\sigma_i}) = \mathbf{C}_{K_{\mathbf{1}}}(x)^{\sigma_i} < K_{\mathbf{1}}^{\sigma_i} = K_{\mathbf{1}}$. The intersection $\cap_{i=1}^k \mathbf{C}_{K_{\mathbf{1}}}(x^{\sigma_i}) \leqslant \mathbf{C}_K(N) = N$, and hence $\cap_{i=1}^k \mathbf{C}_{K_{\mathbf{1}}}(x^{\sigma_i}) = 1$. Since each $\mathbf{C}_{K_{\mathbf{1}}}(x^{\sigma_i})$ is a maximal subgroup of $K_{\mathbf{1}}$, the Frattini subgroup $\Phi(K_{\mathbf{1}}) \leqslant \cap_{i=1}^k \mathbf{C}_{K_{\mathbf{1}}}(x^{\sigma_i}) = 1$. Hence $K_{\mathbf{1}}$ is an elementary abelian 2-group, say $K_{\mathbf{1}} \cong \mathbb{Z}_2^l$ for some $l \geqslant 1$. Noting that $\cap_{i=1}^k \mathbf{C}_{K_{\mathbf{1}}}(x^{\sigma_i}) = 1$, it follows that $l \leqslant k$. Suppose that $l = 1$. Then $K_{\mathbf{1}} \cong \mathbb{Z}_2$ and hence $|Y : G| = 2$. Then $G \lhd Y$, and hence $G \,\mathsf{char}\, Y \lhd X$. So $G \lhd X$, which contradicts the assumption that G is not normal in X. Thus $l > 1$, as in part (iii).

Since $|K_{\mathbf{1}} : \mathbf{C}_{K_{\mathbf{1}}}(x)| = 2$, there is $\tau_1 \in K_{\mathbf{1}}$ such that $K_{\mathbf{1}} = \langle\tau_1\rangle \times \mathbf{C}_{K_{\mathbf{1}}}(x)$. Let $x_1 = x^{-1}x^{\tau_1}$. Then $x_1 \neq 1$, $x_1^{\tau_1} = x_1^{-1}$ and $\mathbf{C}_{K_{\mathbf{1}}}(x) = \mathbf{C}_{K_{\mathbf{1}}}(x_1)$, and so $K_{\mathbf{1}} = \langle\tau_1\rangle \times \mathbf{C}_{K_{\mathbf{1}}}(x_1)$. Since N is a minimal normal subgroup of $X = NU$, there are $\mu_1 = 1, \mu_2, \ldots, \mu_k \in U$ such that $N = \langle x_1^{\mu_1}\rangle \times \cdots \times \langle x_1^{\mu_k}\rangle$. Let $x_i = x_1^{\mu_i}$ and $\tau_i = \tau_1^{\mu_i}$, where $i = 1, 2, \ldots, k$. Then $\mathbb{Z}_2^{l-1} \cong (\mathbf{C}_{K_{\mathbf{1}}}(x_1))^{\mu_i} = \mathbf{C}_{K_{\mathbf{1}}^{\mu_i}}(x_1^{\mu_i}) = \mathbf{C}_{K_{\mathbf{1}}}(x_i)$, and $K_{\mathbf{1}} = K_{\mathbf{1}}^{\mu_i} = \langle\tau_i\rangle \times \mathbf{C}_{K_{\mathbf{1}}}(x_i)$. Further, $x_i^{\tau_i} = x_1^{\tau_1\mu_i} = (x_1^{-1})^{\mu_i} = x_i^{-1}$, and hence $\langle x_i, \tau_i\rangle \cong \mathrm{D}_{2p}$, as in part (iv).

Now $N \cong \mathbb{Z}_p^k$ for an odd prime p and an integer $k > 1$. Suppose that X has a minimal normal subgroup $L \neq N$. Then $N \cap L = 1$, and $LK/K \lhd X/K \cong \mathbb{Z}_m$ or D_{2m}. It follows that either $L \leqslant K$, or L is cyclic and hence $|L|$ is an odd prime. If $L \leqslant K$, then L is a 2-group, it is not possible. Hence L is cyclic. It follows that L is intransitive and semiregular on $V\Gamma$. Then Γ_L is a cycle, and hence N is isomorphic a subgroup of $\mathsf{Aut}\,\Gamma_L$. It follows that N is cyclic, which is a contradiction. Thus N is the unique minimal normal subgroup of X, as in part (v). □

5.3. Proof of Theorem 1.1

If $G \lhd X$, then by Lemma 2.3, we have $X_{\mathbf{1}} \leqslant \mathrm{D}_8$. Thus by Lemma 3.1, $S = \{a, a^{-1}, a^\tau, (a^\tau)^{-1}\}$ for some involution $\tau \in \mathsf{Aut}(G)$, as in Theorem 1.1(1).

We assume that G is not normal in X in the following. Let $M \lhd X$ be maximal subject to that Γ is a normal cover of Γ_M. By Lemma 2.2, M is semiregular on $V\Gamma$ and equals the kernel of X acting on $V\Gamma_M$. Thus, setting $Y = X/M$ and $\Sigma = \Gamma_M$, Σ is Y-edge-transitive. Since $|M|$ is odd, by Lemma 2.3, we have $M \leqslant G$. Therefore, Σ is a Y-edge-transitive Cayley graph of G/M, as in Theorem 1.1(2).

We note that for the normal subgroup defined in the previous paragraph, we have that $G \lhd X$ if and only if $G/M \lhd X/M$. Thus, to complete the proof of Theorem 1.1, we only need to deal with the case where $M = 1$, that is, Γ has no non-trivial normal quotients of valency 4. Let N be a minimal normal subgroup of X. If N is intransitive on $V\Gamma$, then by Lemmas 5.5 and 5.6, part (3) of Theorem 1.1 occurs. If N is transitive on $V\Gamma$, then by Lemmas 5.2–5.3, Theorem 1.1(4) occurs. □

6. Proof of Theorem 1.4

Let p be an odd prime, and let $k > 1$ be an odd integer. Let m be the largest odd divisor of $p^k - 1$, and let

$$G = N \rtimes \langle g \rangle = \mathbb{Z}_p^k \rtimes \mathbb{Z}_m < \mathrm{AGL}(1, p^k).$$

It is easily shown that $\langle g \rangle$ acts by conjugation transitively on the set of subgroups of N of order p. We first construct a family of Cayley graphs of valency 4 of the group G.

Construction 6.1. Let i be such that $1 \leqslant i \leqslant m - 1$, and let $a \in N \setminus \{1\}$. Let

$$S_i = \{ag^i, a^{-1}g^i, (ag^i)^{-1}, (a^{-1}g^i)^{-1}\},$$
$$\Gamma_i = \mathsf{Cay}(G, S_i).$$

The following lemma gives some basic properties about G and Γ_i.

Lemma 6.2. *Let G be the group and let Γ_i be the graphs defined above. Then, we have the following statements*:

(i) $\mathsf{Aut}(G) = \mathrm{A\Gamma L}(1, p^k) \cong \mathbb{Z}_p^k \rtimes \mathrm{\Gamma L}(1, p^k)$;
(ii) *Γ_i is edge-transitive, and Γ_i is connected if and only if i is coprime to m*;
(iii) *$\Gamma_i \cong \Gamma_{m-i}$, and if $p^r i \equiv j \pmod m$, then $\Gamma_i \cong \Gamma_j$.*

Proof. See [5, Proposition 12.10] for part (i).

Since $\mathsf{Aut}(G) = \mathrm{A\Gamma L}(1, p^k)$ and $G < \mathrm{AGL}(1, p^k)$, there is an automorphism $\tau \in \mathsf{Aut}(G)$ such that $a^\tau = a^{-1}$ and $g^\tau = g$. Thus $S_i^\tau = S_i$ and $(ag^i)^\tau = a^{-1}g^i$ and $((ag^i)^{-1})^\tau = (a^{-1}g^i)^{-1}$. It follows that Γ_i is edge-transitive. It is easily shown that $\langle ag^i, a^{-1}g^i \rangle = G$ if and only if $(m, i) = 1$. Hence Γ_i is connected if and only if i is coprime to m.

Since g normalises N, there exists $a' \in N$ such that $(ag^i)^{-1} = a'g^{-i}$ and $(a^{-1}g^i)^{-1} = (a')^{-1}g^{-i}$. Thus $S_i = \{a'g^{-i}, (a')^{-1}g^{-i}, (a'g^{-i})^{-1}, ((a')^{-1}g^{-i})^{-1}\}$. Since $\mathrm{GL}(1, p^k)$ acts transitively on $N \setminus \{1\}$, there exists an element $\rho \in \mathsf{Aut}(G)$ such that $(a')^\rho = a$ and $g^\rho = g$. Thus $S_i^\rho = \{ag^{m-i}, a^{-1}g^{m-i}, (ag^{m-i})^{-1}, (a^{-1}g^{m-i})^{-1}\} = S_j$. So $\Gamma_i \cong \Gamma_{m-i}$.

Suppose that $p^r i \equiv j$ or $-j \pmod m$ for some $r \geqslant 0$. Noting that $\Gamma_{m-j} \cong \Gamma_j$, we may assume that $p^r i \equiv j \pmod m$. Since $g \in \mathrm{GL}(1, p^k) < \mathrm{\Gamma L}(1, p^k)$, there exists $\theta \in \mathrm{\Gamma L}(1, p^k)$ such that θ normalises N and $g^\theta = g^p$. Thus $S_i^{\theta^r} = \{a'g^{p^r i}, a'^{-1}g^{p^r i}, (a'g^{p^r i})^{-1}, (a'^{-1} g^{p^r i})^{-1}\}$, where $a' = a^{\theta^r} \in N$. Since $\mathrm{GL}(1, p^k)$ is transitive on $N \setminus \{1\}$ and fixes g, there exists $c \in \mathrm{GL}(1, p^k)$ such that $(S_i^{\theta^r})^c = S_j$, and so $\Gamma_i \cong \Gamma_j$. □

In the rest of this section, we aim to prove that every connected edge-transitive Cayley graph of G of valency 4 is isomorphic to some Γ_i, so completing the proof of Theorem 1.4.

Let $\Gamma = \mathsf{Cay}(G, S)$ be connected, edge-transitive and of valency 4. We will complete the proof of Theorem 1.4 by a series of steps, beginning with determining the automorphism group $\mathsf{Aut}\,\Gamma$.

Step 1: *G is normal in* $\mathsf{Aut}\,\Gamma$, *and* $\mathsf{Aut}\,\Gamma = G \rtimes \mathsf{Aut}(G, S)$.

Suppose that G is not normal in $\mathsf{Aut}\,\Gamma$. Since N is the unique minimal normal subgroup of G, it follows from Theorem 1.1 that either part (3) of Theorem 1.1 occurs with $X = \mathsf{Aut}\,\Gamma$, or Γ_N is a Cayley graph of G/N and isomorphic to one of the graphs in part (4) of Theorem 1.1. Assume that the latter case holds. Then $G/N \cong \mathbb{Z}_5$, $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$, $\mathbb{Z}_{11} \rtimes \mathbb{Z}_5$ or $\mathbb{Z}_{23} \rtimes \mathbb{Z}_{11}$. Therefore, as $G/N \cong \mathbb{Z}_m$, we have that $G/N \cong \mathbb{Z}_m \cong \mathbb{Z}_5$. By definition, $m = 5$ is the largest odd divisor of $p^k - 1$, which is not possible since p is an odd prime and $k > 1$ is odd. Thus the former case occurs, and $\mathsf{Aut}\,\Gamma = N \rtimes ((H \rtimes \langle g\rangle).O) \cong \mathbb{Z}_p^k \rtimes ((\mathbb{Z}_2^l \rtimes \mathbb{Z}_m).\mathbb{Z}_t)$, satisfying the properties in part (3) of Theorem 1.1. In particular, $2 \leqslant l \leqslant k$, and $\mathbf{C}_H(N) = 1$.

By Theorem 1.1(3), there exist $\tau_0 \in H \setminus \{1\}$ and $z_0 \in N$ such that $H = \langle \tau_0\rangle \times \mathbf{C}_H(z_0)$. It follows that for each $\sigma \in H$, we have $z_0^\sigma = z_0$ or z_0^{-1}. Since g normalises H and $\langle g\rangle$ acts transitively on the set of subgroups of N of order p, it follows that for each $x \in N$ and each $\sigma \in H$, we have $x^\sigma = x$ or x^{-1}. Suppose that there exist $x_1, x_2 \in N \setminus \{1\}$ such that $x_1^\sigma = x_1$ and $x_2^\sigma = x_2^{-1}$. Then $(x_1x_2)^\sigma = x_1x_2^{-1}$, which equals neither x_1x_2 nor $(x_1x_2)^{-1}$, a contradiction. Thus, as σ does not centralise N, we have $x^\sigma = x^{-1}$ for all $x \in N$. Since $H \cong \mathbb{Z}_2^l$ with $l \geqslant 2$, there exists $\tau \in H \setminus \langle\sigma\rangle$. Then similarly, τ inverts all elements of N, that is, $x^\tau = x^{-1}$ for all elements $x \in N$. However, now $x^{\sigma\tau} = x$ for all $x \in N$, and hence $\sigma\tau \in \mathbf{C}_H(N) = 1$, which is a contradiction.

Therefore, G is normal in $\mathsf{Aut}\,\Gamma$, and by Lemma 2.3, we have that $\mathsf{Aut}\,\Gamma = G \rtimes \mathsf{Aut}(G, S)$.

Step 2: $\mathsf{Aut}\,\Gamma = G \rtimes \langle\sigma\rangle = \mathbb{Z}_p^k \rtimes (\langle\sigma\rangle \times \langle f\rangle) \cong N \rtimes \mathbb{Z}_{2m} \cong G \rtimes \mathbb{Z}_2$, *and* $S = \{af^i, a^{-1}f^i, (af^i)^{-1}, (a^{-1}f^i)^{-1}\}$ *where $a \in N$ and $f \in G$ has order m such that $a^\sigma = a^{-1}$; in particular, Γ is not arc-transitive.*

By Lemma 6.2, we have $\mathsf{Aut}(G) \cong \mathrm{A\Gamma L}(1, p^k) \cong N \rtimes (\mathbb{Z}_{p^k-1} \rtimes \mathbb{Z}_k)$. Since k is odd, $\mathsf{Aut}(G)$ has a cyclic Sylow 2-subgroup, and thus all involutions of $\mathsf{Aut}(G)$ are conjugate. It is easily shown that every involution of $\mathsf{Aut}(G)$ inverts all elements of N. Since Γ is edge-transitive and $\mathsf{Aut}\,\Gamma = G \rtimes \mathsf{Aut}(G, S)$, $\mathsf{Aut}(G, S)$ has even order. On the other hand, since G is of odd order, by Lemma 2.3, we have that $\mathsf{Aut}(G, S)$ is isomorphic to a subgroup of D_8. Further, since a Sylow 2-subgroup of $\mathsf{Aut}(G)$ is cyclic, we have that $\mathsf{Aut}(G, S) = \langle\sigma\rangle \cong \mathbb{Z}_2$ or $\mathbb{Z}_4$. It follows that σ fixes an element of G of order m, say $f \in G$ such that $o(f) = m$ and $f^\sigma = f$. Then $G = N \rtimes \langle f\rangle$, and $X = \mathsf{Aut}\,\Gamma = G \rtimes \langle\sigma\rangle = N \rtimes \langle f, \sigma\rangle$.

Since Γ is connected, $\langle S\rangle = G$ and $\mathsf{Aut}(G, S)$ is faithful on S. Hence, we may write $S = \{x, y, x^{-1}, y^{-1}\}$ such that either $o(\sigma) = 2$ and $(x, y)^\sigma = (y, x)$, or $o(\sigma) = 4$ and $(x, y)^\sigma = (y, x^{-1})$, refer to Lemma 3.1. Now $x = af^i$, where $a \in N$ and i is an integer. Suppose that $o(\sigma) = 4$. Then $y = x^\sigma = (af^i)^\sigma = a^\sigma f^i$, and $a' f^{-i} = f^{-i}a^{-1} = (af^i)^{-1} = x^{-1} = x^{\sigma^2} = a^{\sigma^2} f^i = a^{-1} f^i$. It follows that $f^{2i} = 1$, and since f has odd order, $f^i = 1$. Thus $x = a$ and $y = x^\sigma = a^\sigma$, belonging to N, and so $\langle S\rangle \leqslant N < G$, which is a contradiction. Thus σ is an involution, and so $(x, y)^\sigma = (y, x)$, $x = af^i$, and $y = x^\sigma = a^\sigma f^i = a^{-1} f^i$. In particular, Γ is not arc-transitive, and $S = \{af^i, a^{-1}f^i, (af^i)^{-1}, (a^{-1}f^i)^{-1}\}$.

Step 3: *$\Gamma \cong \Gamma_j$ for some j such that $1 \leqslant j \leqslant \frac{m-1}{2}$ and $(j,m)=1$.*

By Step 2, we may assume that $\mathsf{Aut}\,\Gamma = N \rtimes \langle f, \sigma\rangle \leqslant \mathrm{AGL}(1, p^k)$. Since $g \in G$ has order m, it follows from Hall's theorem that there exists $b \in N$ such that $g^b \in \langle f, \sigma\rangle$. So $f^{b^{-1}} = g^r$ for some integer r. Let $\tau = \sigma^{b^{-1}}$. Then $\langle g, \tau\rangle \cong \langle f, \sigma\rangle \cong \mathbb{Z}_{2m}$, and $G = N \rtimes \langle g\rangle$ and $\mathsf{Aut}\,\Gamma = N \rtimes \langle g, \tau\rangle$. Further, $T := S^{b^{-1}} = \{ag^{ir}, a^{-1}g^{ir}, (ag^{ir})^{-1}, (a^{-1}g^{ir})^{-1}\}$. Let $j \equiv ir \pmod m$ and $1 \leqslant j \leqslant m-1$. Then $T = \{ag^j, a^{-1}g^j, (ag^j)^{-1}, (a^{-1}g^j)^{-1}\}$, and $(j,m)=1$ as $\Gamma \cong \mathsf{Cay}(G,T)$ is connected. By Lemma 6.2(iii), $\Gamma_j \cong \Gamma_{m-j}$, and so the statement in Step 3 is true.

Step 4: *Let Γ_i and Γ_j be as in Construction* 6.1 *with $(i,m)=(j,m)=1$. Then $\Gamma_i \cong \Gamma_j$ if and only if $p^r i \equiv j$ or $-j \pmod m$ for some $r \geqslant 0$.*

By Lemma 6.2, we only need to prove that if $\Gamma_i \cong \Gamma_j$ then $p^r i \equiv j$ or $-j \pmod m$ for some $r \geqslant 0$. Thus suppose that $\Gamma_i \cong \Gamma_j$. By Step 2, we have $\mathsf{Aut}\,\Gamma_i \cong \mathsf{Aut}\,\Gamma_j \cong G \rtimes \mathbb{Z}_2$. It follows that Γ_i and Γ_j are so-called CI-graphs, see [14, Theorem 6.1]. Thus $S_i^\gamma = S_j$ for some $\gamma \in \mathsf{Aut}(G)$. Since N is a characteristic subgroup of G, this γ induces an automorphism of $G/N = \langle \overline{g}\rangle$ such that $\overline{S}_i^\gamma = \overline{S}_j$, where $\overline{S}_i = \{\overline{g}^i, \overline{g}^{-i}\}$ and $\overline{S}_j = \{\overline{g}^j, \overline{g}^{-j}\}$ are the images of S_i and S_j under $G \to G/N$, respectively. Thus $(\overline{g}^i)^\gamma = \overline{g}^j$ or $\overline{g}^{-j}$. Since $\mathsf{Aut}(G) = \mathrm{A\Gamma L}(1, p^k)$, it follows that for each element $\rho \in \mathsf{Aut}(G)$, we have $g^\rho = cg^{p^r}$ for some $c \in N$ and some integer r with $0 \leqslant r \leqslant k-1$. Thus $(\overline{g}^i)^\gamma = \overline{g}^{p^r i}$, and hence $p^r i \equiv j$ or $-j \pmod m$.

This completes the proof of Theorem 1.4. □

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