

Normal ordering and boundary conditions for fermionic string coordinates

Nelson R.F. Braga^{a,*}, Hector L. Carrion^b, Cresus F.L. Godinho^c

^a Instituto de Física, Universidade Federal do Rio de Janeiro, Caixa Postal 68528, 21941-972 Rio de Janeiro, RJ, Brazil

^b Instituto de Física, Universidade de São Paulo, Caixa Postal 66318, 05315-970, São Paulo, SP, Brazil

^c Centro Brasileiro de Pesquisas Físicas, Rua Dr Xavier Sigaud 150, 22290-180, Rio de Janeiro, RJ, Brazil

Received 23 February 2006; received in revised form 12 April 2006; accepted 10 May 2006

Available online 24 May 2006

Editor: M. Cvetič

Abstract

We build up normal ordered products for fermionic open string coordinates consistent with boundary conditions. The results are obtained considering the presence of antisymmetric tensor fields. We find a discontinuity of the normal ordered products at string endpoints even in the absence of the background. We discuss how the energy–momentum tensor also changes at the world-sheet boundary in such a way that the central charge keeps the standard value at string end points.

© 2006 Elsevier B.V. Open access under [CC BY license](https://creativecommons.org/licenses/by/4.0/).

PACS: 11.25.-w

Recent progress in string theory [1–3] indicates scenarios where our four-dimensional space–time with standard model fields corresponds to a D3-brane [4] embedded in a larger manifold. Since D-branes correspond, in type II string theories, to the space where the open string endpoints are attached, our space–time would be affected by string boundary conditions. One important consequence is the possible non-commutativity of space–time coordinates at very small length scales [5–7] since commuting coordinates are incompatible with open string boundary conditions in the presence of antisymmetric tensor backgrounds. This is one of the reasons for the increasing interest in many aspects of non-commutative quantum field theories as can be seen for example in [7,8]. Furthermore, this illustrates the fact that string boundary conditions may play a non-trivial role in our four-dimensional physics.

In quantum field theory, products of quantum fields at the same space–time points are in general singular objects. The same thing happens in string theory with position operators, that can be taken as conformal fields on the world-sheet. This situation is well known and one can remove the singular part of

the operator products by defining normal ordered non-singular objects [9]. The explicit characterization of these singularities is important when one investigates the realization, at quantum level, of the classical symmetries. The singular terms that show up in products of energy–momentum tensors are associated with the central charge and play a crucial role in fixing the critical dimension where conformal invariance holds.

Normal ordered products of operators are usually defined so as to satisfy the classical equations of motion at quantum level. For open strings the position operators are defined in a manifold with boundary (the world-sheet). So they also have to satisfy boundary conditions. For the bosonic sector of string coordinates, it was shown in [10] how to build up normal ordered products of operators consistent with boundary conditions and equations of motion. In that bosonic case the effect of the boundary was to add extra terms to the normal ordering. This causes no change in the calculation of the central charge.

Here we calculate the normal ordered products for fermionic open string coordinates in a flat embedding space with a constant antisymmetric tensor background, taking the boundary conditions into account. We will see that the products of fermionic position operators of the same kind get a factor of 1/2 at string endpoints. This factor has to be considered in the calculation of the critical dimension for string theory, since normal ordering is used to build up the algebra of conformal transfor-

* Corresponding author.

E-mail addresses: braga@if.ufrj.br (N.R.F. Braga), hlc@fma.if.usp.br (H.L. Carrion), godinho@cbpf.br (C.F.L. Godinho).

mations. However we will see that the boundary conditions also affect the energy–momentum tensor on the boundary in such a way that the central charge is unchanged.

A superstring in Minkowski space in the presence of a constant antisymmetric tensor field $B^{\mu\nu}$ can be described by the action [11–14]

$$S = \frac{-1}{4\pi\alpha'} \int_{\Sigma} d\tau d\rho \left[\partial_a X^\mu \partial^a X_\mu + \epsilon^{ab} B_{\mu\nu} \partial_a X^\mu \partial_b X^\nu + i\psi_{\mu(-)} E^{\nu\mu} \partial_+ \psi_{\nu(-)} + i\psi_{\mu(+)} E^{\nu\mu} \partial_- \psi_{\nu(+)} \right], \quad (1)$$

where $\partial_+ = \partial_\tau + \partial_\rho$, $\partial_- = \partial_\tau - \partial_\rho$ and $E^{\mu\nu} = \eta^{\mu\nu} + B^{\mu\nu}$.

In a flat embedding space, as is the case here, bosonic and fermionic string coordinates satisfy independent boundary conditions (for string boundary conditions in general spaces see [15,16]). It is interesting to remark that the coupling of the superstring to the antisymmetric tensor field background has to be defined in such a way that boundary conditions do not spoil supersymmetry. For the action (1) it was shown in [11,12] that boundary conditions are consistent with supersymmetry in the sense that the supersymmetry transformation of the fermionic boundary conditions leads to the bosonic boundary conditions.

From now on we will just consider the fermionic sector of the string action since the bosonic one was already studied in [10]. The volume terms coming from the minimum action condition imply the classical equations of motion:

$$\partial_+ \psi_{\nu(-)} = 0, \quad \partial_- \psi_{\nu(+)} = 0. \quad (2)$$

While the boundary term leads to the condition

$$(\psi_{(-)}^\mu E_{\nu\mu} \delta \psi_{(-)}^\nu - \psi_{(+)}^\mu E_{\nu\mu} \delta \psi_{(+)}^\nu) \Big|_0^\pi = 0, \quad (3)$$

that is satisfied imposing the constraints

$$E_{\nu\mu} \psi_{(+)}^\nu(0, \tau) = E_{\mu\nu} \psi_{(-)}^\nu(0, \tau), \quad (4)$$

$$E_{\nu\mu} \psi_{(+)}^\nu(\pi, \tau) = \lambda E_{\mu\nu} \psi_{(-)}^\nu(\pi, \tau), \quad (5)$$

at the endpoints $\rho = 0$ and $\rho = \pi$, where $\lambda = \pm 1$ corresponding to Ramond or Neveu–Schwarz boundary conditions.

Now in order to change to complex world-sheet coordinates we first take the Euclidean form of the fermionic action by means of the transformation $\rho = -i\sigma$. Then we introduce the complex variables: $z = \tau + i\sigma$, $\bar{z} = \tau - i\sigma$ and the action takes the form

$$S = \frac{-i}{4\pi\alpha'} \int_{\Sigma} dz d\bar{z} \left[\psi_{\mu(-)} E^{\nu\mu} \partial_{\bar{z}} \psi_{\nu(-)} + \psi_{\mu(+)} E^{\nu\mu} \partial_z \psi_{\nu(+)} \right]. \quad (6)$$

In this complex coordinates the string end-points corresponds to the region $z = \bar{z}$ and $z = \bar{z} + 2i\pi$.

As usual, one can study the properties of quantum operators by considering the corresponding expectation values, defined in terms of path integrals. Using the fact that the path integral of a total functional derivative vanishes and considering the insertion of one fermionic operator one finds

$$\int [d\psi] \left[\frac{\delta}{\delta \psi_{(a)}^\mu(z, \bar{z})} \left[e^{-S} \psi_{(b)}^\nu(z', \bar{z}') \right] \right] = 0, \quad (7)$$

where $a, b = +, -$. Considering first the case of $\psi_{(b)}^\nu(z', \bar{z}')$ inside the world-sheet and not at the boundary, this equation implies the following expectation values

$$\begin{aligned} \langle \partial_z \psi_{(+)}^\mu(z, \bar{z}) \psi_{(+)}^\nu(z', \bar{z}') \rangle &= 2\pi i \alpha' \langle \eta^{\mu\nu} \delta^2(z - z', \bar{z} - \bar{z}') \rangle, \\ \langle \partial_{\bar{z}} \psi_{(-)}^\mu(z, \bar{z}) \psi_{(-)}^\nu(z', \bar{z}') \rangle &= 2\pi i \alpha' \langle \eta^{\mu\nu} \delta^2(z - z', \bar{z} - \bar{z}') \rangle, \\ \langle \partial_z \psi_{(-)}^\mu(z, \bar{z}) \psi_{(+)}^\nu(z', \bar{z}') \rangle &= \langle \partial_{\bar{z}} \psi_{(+)}^\mu(z, \bar{z}) \psi_{(-)}^\nu(z', \bar{z}') \rangle = 0. \end{aligned} \quad (8)$$

From these results we find the appropriate way to define normal ordered products that satisfy the equations of motion for fermionic operators that are not at the world-sheet boundary

$$\begin{aligned} :\psi_{(+)}^\mu(z, \bar{z}) \psi_{(+)}^\nu(z', \bar{z}') : &= \psi_{(+)}^\mu(z, \bar{z}) \psi_{(+)}^\nu(z', \bar{z}') - \frac{i\alpha'}{\bar{z} - \bar{z}'} \eta^{\mu\nu}, \\ :\psi_{(-)}^\mu(z, \bar{z}) \psi_{(-)}^\nu(z', \bar{z}') : &= \psi_{(-)}^\mu(z, \bar{z}) \psi_{(-)}^\nu(z', \bar{z}') - \frac{i\alpha'}{z - z'} \eta^{\mu\nu}, \\ :\psi_{(+)}^\mu(z, \bar{z}) \psi_{(-)}^\nu(z', \bar{z}') : &= 0, \\ :\psi_{(-)}^\mu(z, \bar{z}) \psi_{(+)}^\nu(z', \bar{z}') : &= 0. \end{aligned} \quad (9)$$

These products satisfy the quantum equations of motion, that in complex coordinates read $\partial_z \psi_{(+)}^\mu = 0$; $\partial_{\bar{z}} \psi_{(-)}^\mu = 0$, as for example $\partial_{\bar{z}} :\psi_{(-)}^\mu(z, \bar{z}) \psi_{(-)}^\nu(z', \bar{z}') : = 0$ and are not subject to boundary conditions since they are defined for points inside the world-sheet.

Let us now consider the case of an insertion of a fermionic string coordinate $\psi_{(\pm)}^\nu(z', \bar{z}')$ located at the world-sheet boundary. Working out Eq. (7), but now subject to constraint (4) we find

$$\begin{aligned} \langle \partial_z \psi_{(+)}^\mu(z, \bar{z}) \psi_{(+)}^\nu(z', \bar{z}') \rangle &= \pi i \alpha' \langle \eta^{\mu\nu} \delta^2(z - z', \bar{z} - \bar{z}') \rangle, \\ \langle \partial_{\bar{z}} \psi_{(-)}^\mu(z, \bar{z}) \psi_{(-)}^\nu(z', \bar{z}') \rangle &= \pi i \alpha' \langle \eta^{\mu\nu} \delta^2(z - z', \bar{z} - \bar{z}') \rangle, \\ \langle \partial_z \psi_{(+)}^\mu(z, \bar{z}) \psi_{(-)}^\nu(z', \bar{z}') \rangle &= \pi i \alpha' \langle [(\eta + B)^{-1}(\eta - B)]^{\nu\mu} \delta^2(z - z', \bar{z} - \bar{z}') \rangle, \\ \langle \partial_{\bar{z}} \psi_{(-)}^\mu(z, \bar{z}) \psi_{(+)}^\nu(z', \bar{z}') \rangle &= \pi i \alpha' \langle [(\eta - B)^{-1}(\eta + B)]^{\nu\mu} \delta^2(z - z', \bar{z} - \bar{z}') \rangle. \end{aligned} \quad (10)$$

So the appropriate normal ordering for fermionic string coordinates at the boundary is

$$\begin{aligned} :\psi_{(+)}^\mu(z, \bar{z}) \psi_{(+)}^\nu(z', \bar{z}') : &= \psi_{(+)}^\mu(z, \bar{z}) \psi_{(+)}^\nu(z', \bar{z}') - \frac{i\alpha'}{2(\bar{z} - \bar{z}')} \eta^{\mu\nu}, \\ :\psi_{(-)}^\mu(z, \bar{z}) \psi_{(-)}^\nu(z', \bar{z}') : &= \psi_{(-)}^\mu(z, \bar{z}) \psi_{(-)}^\nu(z', \bar{z}') - \frac{i\alpha'}{2(z - z')} \eta^{\mu\nu}, \\ :\psi_{(+)}^\mu(z, \bar{z}) \psi_{(-)}^\nu(z', \bar{z}') : &= \psi_{(+)}^\mu(z, \bar{z}) \psi_{(-)}^\nu(z', \bar{z}') - \frac{i\alpha' [(\eta + B)^{-1}(\eta - B)]^{\nu\mu}}{2(\bar{z} - \bar{z}')}, \\ :\psi_{(-)}^\mu(z, \bar{z}) \psi_{(+)}^\nu(z', \bar{z}') : &= \psi_{(-)}^\mu(z, \bar{z}) \psi_{(+)}^\nu(z', \bar{z}') - \frac{i\alpha' [(\eta + B)^{-1}(\eta - B)]^{\nu\mu}}{2(z - z')}. \end{aligned} \quad (11)$$

These are new results. We found normal orderings that incorporate the effect of boundary conditions. We see that even

in the absence of the antisymmetric tensor background \mathcal{B} the normal ordering is discontinuous at the boundary. The products $:\psi_{(+)}^{\mu}\psi_{(+)}^{\nu}:$ and $:\psi_{(-)}^{\mu}\psi_{(-)}^{\nu}:$ are reduced by a factor 1/2 on the boundary. Normal ordered products are important to calculate the central charge that gives us the critical dimension. Let us see what happens on the boundary.

The energy–momentum tensor for the fermionic sector for points inside the world-sheet reads

$$\begin{aligned} T^{zz} &= -\frac{1}{2}\psi_{\mu(+)}E^{\nu\mu}\partial_{\bar{z}}\psi_{\nu(+)} \equiv \bar{T}, \\ T^{\bar{z}\bar{z}} &= -\frac{1}{2}\psi_{\mu(-)}E^{\nu\mu}\partial_z\psi_{\nu(-)} \equiv T, \end{aligned} \quad (12)$$

while for at the boundary, the conditions (4) relating $\psi_{\nu(-)}$ to $\psi_{\nu(+)}$ lead to

$$\bar{T} = -\frac{1}{2}\psi_{\mu(+)}E^{\nu\mu}(\partial_{\bar{z}} + \partial_z)\psi_{\nu(+)} = T. \quad (13)$$

The central charge can be calculated from the most singular term in the normal ordered product of energy–momentum tensors. This term involves two contractions of fermionic coordinate operator products and is proportional to

$$\begin{aligned} &\int dz' \dots \int dz'' \dots \left[\frac{-i\alpha'}{2a(z' - z'')} \frac{\delta}{\delta\psi_{\mu(-)}(z')} \frac{\delta}{\delta\psi_{\mu(-)}(z'')} \right] \\ &\cdot \left[\frac{-i\alpha'}{2a(z''' - z''')} \frac{\delta}{\delta\psi_{\nu(-)}(z''')} \frac{\delta}{\delta\psi_{\nu(-)}(z''')} \right] \\ &\times [T(z_1)T(z_2)]. \end{aligned} \quad (14)$$

The contributions involving the antisymmetric tensor $\mathcal{B}_{\mu\nu}$ cancel out. For points inside the world-sheet $a = 1$ and the energy–momentum tensor T is given by Eq. (12). As it is well known, the central charge is $D/2$ where D is the space–time dimension.

For points on the boundary $a = 2$ as a consequence of the normal ordering of Eq. (11) but the energy–momentum tensor takes the form (13). After taking the functional field derivatives in (14) and considering the points z_1 and z_2 to be fixed on the boundary, where $d\bar{z} = dz$, the derivatives will act as $\partial_{\bar{z}} = \partial_z$. So, each energy momentum tensor will contribute with a factor of 2 with respect to the case of points inside the world-sheet. As a result, the contribution to the central charge will have, as expected, the same value $D/2$ for world-sheet boundary points.

It is interesting to compare the present fermionic results with those for the bosonic string coordinates. In the bosonic case, the

normal ordering at the boundary reads [10]

$$\begin{aligned} &:\hat{X}^{\mu}(w, \bar{w})\hat{X}^{\nu}(w', \bar{w}') : \\ &= \hat{X}^{\mu}(w, \bar{w})\hat{X}^{\nu}(w', \bar{w}') + \frac{\alpha'}{2}\eta^{\mu\nu}\ln|w - w'|^2 \\ &+ \frac{\alpha'}{2}([\eta + \mathcal{B}]^{-1}[\eta - \mathcal{B}])^{\mu\nu}\ln(w - \bar{w}') \\ &+ \frac{\alpha'}{2}([\eta + \mathcal{B}][\eta - \mathcal{B}]^{-1})^{\mu\nu}\ln(\bar{w} - w') + \alpha'D^{\mu\nu}. \end{aligned} \quad (15)$$

The effect of the boundary conditions is the presence of the terms involving the \mathcal{B} field in this expression. They are additive factors that do not contribute to the central charge. In contrast to the fermionic case studied here, the bosonic normal ordering does not have a discontinuity on the boundary. So, this is a more trivial situation than the fermionic case where there are multiplicative factors coming from the normal ordering that cancel with others coming from the energy–momentum tensors.

Acknowledgements

The authors are partially supported by CNPq, FAPERJ, CLAF and FAPESP.

References

- [1] J. Maldacena, *Adv. Theor. Math. Phys.* 2 (1998) 231.
- [2] L. Randall, R. Sundrum, *Phys. Rev. Lett.* 83 (1999) 3370.
- [3] L. Randall, R. Sundrum, *Phys. Rev. Lett.* 83 (1999) 4690.
- [4] J. Polchinski, *Phys. Rev. Lett.* 75 (1995) 4724.
- [5] C.S. Chu, P.M. Ho, *Nucl. Phys. B* 550 (1999) 151.
- [6] A.Y. Alekseev, A. Recknagel, V. Schomerus, *JHEP* 9909 (1999) 023.
- [7] N. Seiberg, E. Witten, *JHEP* 9909 (1999) 032.
- [8] R.J. Szabo, *Phys. Rep.* 378 (2003) 207.
- [9] See for example J. Polchinski, *String Theory*, vol. 1, Cambridge Univ. Press, Cambridge, UK, 1998.
- [10] N.R.F. Braga, H.L. Carrion, C.F.L. Godinho, *J. Math. Phys.* 46 (2005) 062302.
- [11] P. Haggi-Mani, U. Lindstrom, M. Zabzine, *Phys. Lett. B* 483 (2000) 443.
- [12] U. Lindstrom, M. Rocek, P. van Nieuwenhuizen, *Nucl. Phys. B* 662 (2003) 147.
- [13] N.R.F. Braga, C.F.L. Godinho, *Phys. Lett. B* 570 (2003) 111.
- [14] B. Chakraborty, S. Gangopadhyay, A.G. Hazra, F.G. Scholtz, *Phys. Lett. B* 625 (2005) 302.
- [15] C. Albertsson, U. Lindstrom, M. Zabzine, *Commun. Math. Phys.* 233 (2003) 403.
- [16] C. Albertsson, U. Lindstrom, M. Zabzine, *Nucl. Phys. B* 678 (2004) 295.