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V-Subdifferentials of Convex Operators

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For any convex operator f from a convex set C of a topological vector space E into another one F endowed with a convex cone F_+ a notion of V-subdifferential $\partial_V f(a)$ of f at $a \in C$ is introduced. Although it is equivalent to the notion of ε-subdifferential when $F = \mathbb{R}$, it enjoys many important properties which are not satisfied by the ε-subdifferential whenever int $F_+ = \emptyset$. The nonvacuity of $\partial_V f(a)$ is proved whenever V is a neighbourhood of zero in F and f belongs to a large class of mappings analogous to the class of lower semicontinuous real-valued convex functions. Other properties of V-subdifferentials are studied and applications to differentiability of f are made. © 1986 Academic Press, Inc.

INTRODUCTION

The notion of ε-subdifferential of a real-valued convex function has been introduced by Bronsted and Rockafellar [9]. Important properties of this notion has been studied by many authors. Bronsted [8] gave a characterisation of the subdifferential of the supremum of two lower semicontinuous real-valued convex functions in terms of ε-subdifferentials. Moreau [22, 23] and Asplund and Rockafellar [2] established some results on equicontinuity of ε-subdifferentials. But one of the major results on the ε-subdifferential (proved by Asplund and Rockafellar [2]) is the continuity of the ε-subdifferential multifunction (for a lower semicontinuous convex function defined on a Banach space and for $\varepsilon > 0$) with respect to the Hausdorff topology on subsets. This property which is one of the strongest properties which can be required on a multifunction is not satisfied by the subdifferential multifunction. Actually more can be said. Indeed Hiriart-Urruty [14] (for Banach spaces) and Nurminskii [25] (for $\mathbb{R}^n$) have recently proved that the ε-subdifferential multifunction of a lower semicontinuous real-valued convex function f is locally Lipschitz on the set of points of continuity of f.

Besides many studies of convex operators and their subdifferentials (see, e.g., [3–7, 16–19, 21, 26, 29, 33–37, 39) Borwein [6], Kutateladze [20],

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Thera [32] have considered the notion of ε-subdifferentials for convex mappings taking values in an ordered topological vector space and for ε in the positive cone of that ordered vector space. Essentially these authors have established an operational calculus for ε-subdifferentials. Their definition has the same formulation than the one for real-valued functions. More precisely, if F is a topological vector space ordered by a convex cone F_+, f is a convex mapping from a convex set C of a topological vector space E into F and ε is in F_+, the ε-subdifferential of f at $a \in C$, $\partial^{\varepsilon} f(a)$, is the set of all continuous linear mappings T from E into F such that

$$T(x-a) \leqslant f(x) - f(a) + \varepsilon \qquad \text{for all } x \in C.$$

Although this notion has some interesting properties, it does not enjoy the above locally Lipschitz behavior if int $F_+ = \varnothing$. This state of affairs leads us to define the notion of V-subdifferentials which, as will be proved in [36], enjoys the locally Lipschitz behavior whenever V is a neighbourhood of zero. Moreover taking $V = [-\varepsilon, \varepsilon]$ one sees this notion encompass that of ε-subdifferentials since $\partial_V f(a)$ is the set of all continuous linear mappings T from E into F such that

$$T(x-a) \in f(x) - f(a) + V - F_+ \qquad \text{for all } x \in C.$$

This paper is concerned with the study of some properties of V-subdifferentials. It is divided in five parts. In the first section we recall some preliminary definitions which will be used in the next parts. Section two is devoted to the introduction of a class of mappings which is important for the notion of V-subdifferentials, the class of mappings f which are at each point of their domain, the limit of nested families of continuous affine minorants for f. In section three we introduce and we define the notion of V-subdifferential. Non vacuity of V-subdifferentials of mappings in the above class is proved. In section four we establish some equicontinuity properties of V-subdifferentials as subsets of continuous linear mappings and in section five we study the relationship between the differentiability and the V-subdifferentials of convex operators following the way opened by Asplund and Rockafellar [2]. In particular we prove that a convex operator f is Fréchet-differentiable at a point on a neighourhood of which f is continuous and subdifferentiable if and only if the subdifferential multifunction of f is continuous at that point in the Hausdorff sense. So we complete a result of Borwein (see [3, Theorem 5.5]) which states that this property is implied by the Fréchet-differentiability of the convex operator f.

Before concluding this introduction let us indicate that a substantial survey on ε-subdifferentials of real-valued convex functions can be found in

Hiriart–Urruty [15] and that we refer the reader to the papers of Borwein [3] and [5] for many important results on conditions for subdifferentials of convex operators to be nonempty.

1. Preliminaries

Throughout this paper E and F will be two (real separated) topological vector spaces. We shall always assume that F_+ is a *convex cone* in F (i.e., $sF_+ + tF_+ \subset F_+$ for all real numbers $s, t \geq 0$ and $F_+ \cap (-F_+) = \{0\}$) and hence it induces an ordering in F by $y \leqslant y'$ if $y' - y \in F_+$. So F is an ordered topological vector space.

We shall denote by $L(E, F)$ the set of all continuous linear mappings from E into F and by $L_+(F, F)$ the set of all mappings $T \in L(F, F)$ satisfying $T(x) \geqslant 0$ for all $x \geqslant 0$. Such mappings are called positive continuous linear mappings and we write $S \leqslant T$ whenever $T(x) - S(x) \geqslant 0$ for all $x \geqslant 0$.

One says that F_+ is *normal* if there exists a neighbourhood basis $\{V\}_V$ of zero in F such that

$$V = (V + F_+) \cap (V - F_+).$$

Such neighbourhoods are said to be *full*. For properties of normal cones we refer to [28] where it is proved that most classical ordered topological vector spaces are normal.

We adjoin an abstract greatest element infinity and a lowest one to F and we shall write $F^\bullet = F \cup \{+\infty\}$ and $\bar{F} = F \cup \{-\infty, +\infty\}$.

We shall say that F is *order-complete* if every nonempty subset with an upper bound in F has a supremum in F.

A mapping $f\colon E \to F^\bullet$ is *convex* if

$$f(sx + ty) \leqslant (sf(x) + tf(y) \tag{1.1}$$

whenever x, y lie in E and s and t are positive numbers with $s + t = 1$. We recall that the domain $\operatorname{dom} f$ and the epigraph $\operatorname{epi} f$ of f are defined by

$$\operatorname{dom} f = \{x \in E\colon f(x) \in F\} \qquad \text{and} \qquad \operatorname{epi} f = \{(x, y) \in E \times F\colon f(x) \leqslant y\}.$$

The *subdifferential* or *subgradient* for f at $a \in E$ is defined by

$$\partial f(a) = \{T \in L(E, F)\colon T(x - a) \leqslant f(x) - f(a), \forall x \in E\}$$

whenever $a \in \operatorname{dom} f$ and $\partial f(a) = \varnothing$ whenever $a \notin \operatorname{dom} f$

2. Mappings which are at Each Point Limit of Continuous Affine Minorants

Before defining the notion of V-subdifferentials we introduce a class of convex operators which will appear very useful for proving some important properties of V-subdifferentials.

2.1. Definition. Let $f\colon E \to F^{\bullet}$ be a convex mapping. We shall say that $f \in A(E)\ F)$ if for each $x \in \operatorname{dom} f$ there exists a directed set J_x and a collection $(A_j(\cdot) + b_j)_{j \in J_x}$ of continuous affine mappings (here $A_j \in L(E, F)$ and $b_j \in F$) with

$$A_j(y) + b_j \leqslant f(y) \qquad \text{for every } y \in \operatorname{dom} f$$

and

$$f(x) = \lim_{j \in J_x} (A_j(x) + b_j).$$

Remark. Obviously $f + g \in A(E, F)$ and $tf \in A(E, F)$ whenever f, g lie in $A(E, F)$ and t is a positive real number.

2.2. Definition (see [28]). One says that F is a topological vector lattice if $\sup(x, y)$ exists for all $x, y \in F$ and if F has a neighbourhood basis $\{V\}_V$ of zero such that

$$V = \bigcup_{x \in V} \{ y \in F\colon |y| \leqslant |x| \}$$

where $|x| = \sup(-x, x)$.

The following lemma will be used in the next proposition. Although its proof is similar to the one of Theorem 6 in Valadier [37], we did not find the statement below in the literature. In fact, the classical formulation is that for each (continuous) sublinear mapping $f\colon E \to F$ we have $f(x) = \max\{T(x)\colon T \in \partial f(0)\}$ whenever F is order complete (and normal) (see, e.g., Valadier [37], Rubinov [29], and Kutateladze [19]).

2.3. Lemma. *Assume that F is order complete and F_+ is normal. Let f be a continuous sublinear mapping (i.e., positively homogeneous and convex) from E into F. Then for each $x \in E$ we have*

$$\partial f(0)(x) = [-f(-x), f(x)],$$

where

$$\partial f(0)(x) = \{T(x) : T \in \partial f(0)\}$$

and

$$[-f(-x), f(x)] = \{y \in F : -f(-x) \leqslant y \leqslant f(x)\}.$$

Proof. If $T \in \partial f(0)$ we have $T(x) \leqslant f(x)$ for every $x \in E$ and hence $-f(-x) \leqslant -T(-x) = T(x) \leqslant f(x)$ for every $x \in X$, which proves that

$$\partial f(0)(x) \subset [-f(-x), f(x)].$$

Let us show the reverse inclusion. Let $a \in E$ and $b \in [-f(-a), f(a)]$. Consider the linear mapping T from $\mathbb{R} \cdot a$ into F defined by $T(ta) = tb$ for every $t \in \mathbb{R}$. Then for every real number $t \geqslant 0$ we have

$$T(ta) = tb \leqslant tf(a) = f(ta)$$

and

$$T(-ta) = t(-b) \leqslant tf(-a) = f(-ta)$$

and hence by the generalized Hahn Banach extension theorem for order complete vector space (see [11]) there exists a linear mapping $\bar{T}$ from E into F such that

$$\bar{T}(x) \leqslant f(x) \qquad \text{for all } x \in E$$

and this relation implies that $\bar{T}$ is continuous since f is continuous at zero. Therefore $\bar{T} \in \partial f(0)$ and $b \in \partial f(0)(a)$, which completes the proof of the lemma. ∎

Remark. If one considers the algebraic subdifferential it is enough to assume that F is order complete.

2.4. PROPOSITION. *Let F be an order complete topological vector lattice.*

(1) *For $x, y \in F$ there exist l_i, k_i in $L_+(F, F)$ with $l_1 + l_2 = k_1 + k_2 = \mathrm{Id}_F$ (here Id_F denotes the identity mapping) such that*

$$\sup(x, y) = l_1 x + l_2 y \qquad \textit{and} \qquad \inf(x, y) = k_1 x + k_2 y.$$

(2) *If $x \leqslant y$, then*

$$[x, y] = \{l_1 x + l_2 y : l_i \in L_+(F, F) \textit{ and } l_1 + l_2 = \mathrm{Id}_F\}$$

(3) *For each finite family* $(x_k)_{k \in K}$ *there exists a finite family* $(l_k)_{k \in K}$ *of elements of* $L_+(F, F)$ *such that*

$$\sup_{k \in K} x_k = \sum_{k \in K} l_k(a_k) \qquad \textit{and} \qquad \sum_{k \in K} l_k = \mathrm{Id}_F.$$

Proof. Let $f: F \to F$ defined by $f(x) = x^+$, where $x^+ = \sup(x, 0)$. The mapping f is sublinear and continuous since F is a topological lattice. Then by Lemma 2.3

$$\partial f(0)(x) = [-f(-x), f(x)]$$

and hence

$$\partial f(0)(x) = [-x^-, x^+], \qquad \text{where } x^- = \sup(-x, 0).$$

Moreover making use of the relation $f(x) = \sup(\mathrm{Id}_F(x), 0)$ it is not difficult to show that

$$\partial f(0) = \{l \in L(F, F): 0 \leqslant l \leqslant \mathrm{Id}_F\}.$$

Therefore

$$[-x^-, x^+] = \{lx: l \in L(F, F), 0 \leqslant l \leqslant \mathrm{Id}_F\}. \tag{2.1}$$

Consider now $x, y \in F$. By relation (2.1) there exists $l \in L(F, F)$ with $0 \leqslant l \leqslant \mathrm{Id}_F$ such that

$$\begin{aligned} \sup(x, y) &= \sup(x - y + y, y) = (x - y)^+ + y \\ &= l(x - y) + y = l(x) + (\mathrm{Id}_F - l)(y). \end{aligned} \tag{2.2}$$

In the same way, if $x \leqslant y$ we have again by relation (2.1),

$$\begin{aligned} [x, y] &= x + [0, y - x] = x + [-(y - x)^-, (y - x)^+] \\ &= x + \{l(y - x): l \in L(F, F), 0 \leqslant l \leqslant \mathrm{Id}_F\} \\ &= \{(\mathrm{Id}_F - l)\, x + ly: l \in L(F, F), 0 \leqslant l \leqslant \mathrm{Id}_F\}. \end{aligned}$$

So the proof is complete since statement (3) and relation $\inf(x, y) = k_1 x + k_2 y$ are direct consequences of relation (2.2). ∎

Following Moreau [23] we shall denote by $\Gamma(E, F)$ the set of all mappings $f: E \to F^{\bullet}$ such that f is the pointwise supremum on $\operatorname{dom} f$ of the collection of all continuous affine minorants for f.

We recall that F is a *Dini space* (see [3] and [26] for many examples of Dini spaces) if every increasing net with a supremum converges to that supremum.

2.5. PROPOSITION. *If F is a Dini order complete topological vector lattice, then $\Gamma(E, F) \subset \Lambda(E, F)$.*

Proof. Let $f \in \Gamma(E, F)$ and $a \in \operatorname{dom} f$. Let us denote by $(A_i(\ ) + b_i)_{i \in I}$ the collection of all continuous affine mappings pointwise majorized by f. If $\mathscr{S}$ denotes the set of all finite subsets of I and if we put for each $K \in \mathscr{S}$

$$f_K(x) = \sup_{k \in K} (A_k(x) + b_k) \qquad \text{for every } x \in E,$$

then the family $(f_K(a))_{K \in \mathscr{S}}$ is upper bounded in F and is an increasing net with respect to the order defined by the inclusion relation on $\mathscr{S}$ and as

$$f(a) = \sup\{f_K(a) : K \in \mathscr{S}\}$$

we have

$$f(a) = \lim_{K \in \mathscr{S}} f_K(a)$$

since F is Dini. By Proposition 2.4, for each $K \in \mathscr{S}$ there exists a finite family $(l_k)_{k \in K}$ of elements in $L_+(F, F)$ satisfying

$$\sum_{k \in K} l_k = \mathrm{Id}_F \qquad \text{and} \qquad f_K(a) = \sum_{k \in K} (l_k \circ A_k(a) + l_k(b_k)).$$

If we put

$$T_K(x) = \sum_{k \in K} l_k \circ A_k(x) \qquad \text{and} \qquad c_K = \sum_{k \in K} l_k(b_k)$$

then

$$f_K(a) = T_K(a) + c_K$$

and for each $x \in \operatorname{dom} f$ we have

$$\begin{aligned} T_K(x) + c_K &= \sum_{k \in K} l_k(A_k(x) + b_k) \\ &\leqslant \sum_{k \in K} l_k(f(x)) \\ &= f(x). \end{aligned}$$

Therefore $T_K(\cdot)+c_K$ is a continuous affine minorant for f and

$$f(a)=\lim_{K\in\mathscr{S}}(T_K(a)+c_K),$$

which completes the proof. ▮

To give another class of mappings in $A(E, F)$ let us recall the following notion.

A subset C of a vector space X is said to be *lineally closed* if the intersection of C with every line in X is a closed set in the natural topology of the line.

2.6. Corollary. *Let f be a convex mapping from E into $F^{\bullet}$ with* $\operatorname{int}(\operatorname{dom} f)\neq\varnothing$. *Assume F is a Dini order complete topological vector lattice,* epi f *is lineally closed in $E\times F$ and f is continuous on* $\operatorname{int}(\operatorname{dom} f)$. *Then* $f\in A(E, F)$.

Proof. This is a direct consequence of Proposition 2.5 and Theorem 2.2 in [7] which holds only with the assumption that epi f is lineally closed but non-necessarily topologically closed. ▮

3. V-Subdifferentials

We introduce in this section the notion of V-subdifferentials.

3.1. Definition. Let $f: E\to F^{\bullet}$ be a convex operator. For a subset S of F containing zero, the *S-subdifferential* $\partial_S f(a)$ of f at a is the set

$$\partial_S f(a)=\{T\in L(E, F): T(x-a)\in f(x)-f(a)+S-F_+,\ \forall x\in\operatorname{dom} f\}$$

if $a\in\operatorname{dom} f$, and $\partial_S f(a)=\varnothing$ if $a\notin\operatorname{dom} f$.

Remarks. (1) Obviously if $0\in S'\subset S$, then

$$\partial f\subset\partial_{S'} f\subset\partial_S f.$$

(2) If F_+ is closed and if $\mathscr{N}$ denotes a neighbourhood basis of zero in F, then for every $a\in\operatorname{dom} f$

$$\partial f(a)=\bigcap_{V\in\mathscr{N}}\partial_V f(a).$$

Until now the notion of V-subdifferential has not been defined. Borwein [6], Kutateladze [20], and Thera [32] have only studied the notion of ε-

subdifferential ($\varepsilon \in F_+$) of a convex operator, a notion which is a direct transcription of the usual definition of ε-subdifferentials of real-valued convex functions.

The notion of V-subdifferential is equivalent to that of ε-subdifferential when $F = \mathbb{R}$ by Remark 2 following Definition 3.2. However, this is not true whenever $\operatorname{int}(F_+) = \varnothing$ and many results established in this paper for V-subdifferentials are not true for ε-subdifferentials.

3.2. Definition. Let $f: E \to F^\bullet$ be a convex mapping and $\varepsilon \in F_+$. The ε-*subdifferential* of f at a point $a \in \operatorname{dom} f$ is the set

$$\partial^\varepsilon f(a) = \{T \in L(E, F): T(x-a) \leqslant f(x) - f(a) + \varepsilon, \forall x \in \operatorname{dom} f\}.$$

Remarks. (1) The reader will note the notation used for ε-subdifferentials to reserve the usual notation for V-subdifferentials when F is normed and V is the closed ball around zero of radius ε, see Corollary 4.4.

(2) For $\varepsilon \in F_+$, $\partial^\varepsilon f = \partial_{[-\varepsilon,\varepsilon]}$, where $[-\varepsilon, \varepsilon] = (-\varepsilon + F_+) \cap (\varepsilon - F_+)$.

(3) For every $\varepsilon \in F_+$ and every neighbourhood V of zero in F there exists a real number $t > 0$ such that $\partial^\varepsilon f \subset \partial_{tV} f$. ∎

Let us give a first remarkable property of V-subdifferentials.

3.3. Proposition. *Let $f \in A(E, F)$. If $a \in \operatorname{dom} f$ and if V is a neighbourhood of zero in F, then $\partial_V f(a) \neq \varnothing$.*

Proof. As $f \in A(E, F)$, there exists a directed set J and a collection $(A_j(\cdot) + b_j)_{j \in J}$ of continuous affine minorants for f such that

$$f(a) = \lim_{j \in J} [A_j(a) + b_j]$$

and hence there exists $i \in J$ such that

$$A_i(a) + b_i \in f(a) - V.$$

Therefore for every $x \in \operatorname{dom} f$ we have

$$A_i(x-a) = A_i(x) + b_i - A_i(a) - b_i \in f(x) - F_+ - f(a) + V$$

and hence $A_i \in \partial_V f(a)$. ∎

Before stating the next proposition and its corollary, which give a second remarkable property of V-subdifferentials, let us recall that a family $(f_i)_{i \in I}$ of mappings from E into $F^\bullet$ is said to be *equicontinuous* at a point

$a \in \bigcap_{i \in I} \operatorname{dom} f_i$ if for every neighbourhood V of zero in F there exists a neighbourhood X of zero in E such that

$$f_i(a+X) - f_i(a) \subset V \qquad \text{for all } i \in I.$$

3.4. PROPOSITION. *Let $(f_i)_{i \in I}$ be a family of convex mappings from E into $F^\bullet$ equicontinuous at a point $a \in \bigcap_{i \in I} \operatorname{dom} f_i$. Then for every neighbourhood V of zero in F there exist a neighbourhood V' of zero in F and a neighbourhood X of zero in E such that*

$$\partial_{V'} f_i(a+x) \subset \partial_V f_i(a) \qquad \textit{for all } x \in X \textit{ and } i \in I.$$

Proof. Let V be any neighbourhood of zero in F. Let us choose a circled neighbourhood V' of zero in F satisfying

$$V' + V' + V' + V' + V' \subset V.$$

By equicontinuity, there exists a neighbourhood X' of zero in E such that $f_i(a+X') - f_i(a) \subset V'$ for all $i \in I$. Let us choose a neighbourhood X of zero in E with $X + X \subset X'$ and let us show that

$$\partial_{V'} f_i(a+x) \subset \partial_V f_i(a) \qquad \text{for all } x \in X \text{ and } i \in I.$$

Consider any point $x \in X$, any point $i \in I$ and any element $T \in \partial_{V'} f_i(a+x)$. For every $y \in \operatorname{dom} f$ we have

$$\begin{aligned} T(y-a) = T(x) + T(y-a-x) &\in f_i(a+x+x) - f_i(a+x) \\ &\quad + V' + f_i(y) - f_i(a+x) + V' - F_+ \end{aligned}$$

and hence

$$\begin{aligned} T(y-a) &\in f_i(y) - f_i(a) + 2(f_i(a) - f_i(a+x)) \\ &\qquad + (f_i(a+x+x) - f_i(a)) + V' + V' - F_+ \\ &\subset f_i(y) - f_i(a) + V' + V' + V' + V' + V' - F_+ \\ &\subset f_i(y) - f_i(a) + V - F_+. \end{aligned}$$

Therefore $T \in \partial_V f_i(a)$ and the proof is complete. ∎

3.5. COROLLARY. *Let $(f_i)_{i \in I}$ be a family of convex mappings from E into $F^\bullet$ equicontinuous at a point $a \in \bigcap_{i \in I} \operatorname{dom} f_i$. Then for every neighbourhood V of zero in F there exists a neighbourhood X of zero in E such that*

$$\partial f_i(a+x) \subset \partial_V f_i(a) \qquad \textit{for all } x \in X \textit{ and } i \in I$$

Proof. This is a direct consequence of Proposition 3.4 and Remark 2 following Definition 3.1. ▮

4. Equicontinuity of V-Subdifferentials

In this section we shall study equicontinuity properties of V-subdifferentials as subsets of $L(E, F)$.

Let us begin by recalling the following result which has been established in [16, Proposition 2.5].

4.1. Propostition. *Let $(f_i)_{i \in I}$ be a family of convex mappings from E into $F^{\bullet}$. Then the family is equicontinuous at a point a if and only if it is equilipschitzian around a in the sense that for each closed circled neighbourhood W of zero in E there exists a neighbourhood X of zero and a closed circled neighbourhood V of zero in E such that*

$$\rho_W(f_i(a+x)-f_i(a+x')) \leqslant \rho_V(x-x') \qquad \textit{for all } x, x' \in X \textit{ and } i \in I,$$

where

$$\rho_W(y) = \inf\{t > 0 \colon y \in tW\}.$$

The following proposition gives a first result on equicontinuity of subdifferentials.

4.2. Proposition. *Let $(f_i)_{i \in I}$ be a family of convex mappings from E into $F^{\bullet}$ equicontinuous at a point a. If F_+ is normal, then for any topologically bounded subset S of F containing zero, there exists a neighbourhood X of zero in E such that $\bigcup_{i \in I} \partial_S f_i(a+X)$ is equicontinuous in $L(E, F)$, where $\partial_S f_i(a+X) = \bigcup_{x \in X} \partial_S f_i(a+x)$.*

Proof. Let W be a full-circled neighbourhood of zero in F and W_0 a closed-circled neighbourhood of zero satisfying $W_0 + W_0 \subset W$. By Proposition 4.1 there exists a neighbourhood U of zero in E such that

$$f_i(a+x)-f_i(a+x') \in W_0 \qquad \textit{for all } i \in I \textit{ and } x, x' \in U.$$

Choose a real number $t \in]0, 1]$ with $tS \subset W_0$ and a circled neighbourhood X of zero in E with $X + X \subset U$. We may assume that $\bigcup_{i \in I} \partial_S f(a+X)$ is nonempty. Then for each $y \in X$, each $x \in X$, each $i \in I$ with $\partial_S f_i(a+x) \neq \varnothing$ and each $T_i \in \partial_S f_i(a+x)$ we have

$$T_i(y) \in f_i(a+x+y) - f_i(a+x) + S - F_+ \subset W_0 + S - F_+$$

and hence

$$T_i(ty) \in tW_0 + tS - F_+ \subset W_0 + W_0 - F_+ \subset W - F_+.$$

Therefore we have

$$T_i(tX) \subset (W - F_+) \cap (W + F_+) = W$$

and hence $\bigcup_{i \in I} \partial_S f_i(a + X)$ is equicontinuous in $L(E, F)$. ∎

As a first immediate corollary we have:

4.3. COROLLARY. *Let $f\colon E \to F^\bullet$ be a convex mapping continuous at a point $a \in \operatorname{dom} f$. If F_+ is normal, then for every topologically bounded subset S of F containing zero there exists a neighbourhood X of zero in E such that $\partial_S f(a + X)$ is equicotinuous in $L(E, F)$.*

If F is normed and if B_F is the closed unit ball around zero of F, then for every real number $r > 0$ we shall denote the rB_F-subdifferential of f at a by $\partial_r f(a)$.

4.4. COROLLARY. *Assume F is normed and normal. Let $(f_i)_{i \in I}$ be a family of convex mappings from E into $F^\bullet$ equicontinuous at a point a. Then for every real number $r > 0$ there exists a neighbourhood X of zero in E such that $\bigcup_{i \in I} \partial_r f_i(a + X)$ is equicontinuous in $L(E, F)$.*

Reciprocally to Proposition 4.2 we have the following result.

4.5. PROPOSITION. *Assume that F_+ is normal. Let $(f_i)_{i \in I}$ be a family of convex mappings from E into $F^\bullet$ for which there exist a neighbourhood X of zero in E, a topologically bounded subset S of F containing zero and an equicontinuous family $(T_{i,x})_{(i,x) \in I \times X}$ in $L(E, F)$ satisfying $T_{i,x} \in \partial_S f_i(a + x)$ for each $i \in I$ and each $x \in X$. Then the family $(f_i)_{i \in I}$ is equicontinuous at a.*

Proof. Let W be a full circled neighbourhood of zero in F and W_0 a circled neighbourhood of zero in F with $W_0 + W_0 \subset W$. Choose a real number $t \in]0, 1]$ with $tS \subset W_0$ and a circled neighbourhood U of zero such that $U \subset X$ and

$$T_{i,x}(U) \subset W_0 \qquad \text{for all } i \in I \text{ and } x \in X.$$

Then for each $x \in U$ and each $i \in I$ we have

$$f_i(a) - f_i(a + x) \in T_{i,x}(-x) - S + F_+$$

and hence

$$f_i(a+x)-f_i(a)\in T_{i,x}(x)+S-F_+\subset W_0+S-F_+.$$

Therefore for each $x\in U$ and each $i\in I$ we have by the convexity of f_i

$$f_i(a+tx)-f_i(a)\in t[f_i(a+x)-f_i(a)]-F_+\subset tW_0+tS-F_+$$

and hence

$$f_i(a+tx)-f_i(a)\in W-F_+.$$

By the convexity of f_i once again we have for each $x\in U$ and each $i\in I$

$$f_i(a+tx)-f_i(a)\in -[f_i(a-tx)-f_i(a)]+F_+\subset W+F_+$$

and hence

$$f_i(a+tU)-f_i(a)\subset (W-F_+)\cap(W+F_+)=W$$

for all $i\in I$ and the proof is complete. ▮

Making use of the class of convex mappings $\Lambda(E,F)$ we can give the following necessary and sufficient condition for equicontinuity of families of mappings in $\Lambda(E,F)$.

4.6. COROLLARY. *Assume F is normed and normal. Let $(f_i)_{i\in I}$ be a family of mappings in $\Lambda(E,F)$. Then this family is equicontinuous at a point $a\in\bigcap_{i\in I}\operatorname{dom} f_i$ if and only if there exists a real number $r>0$ and a neighbourhood X of zero in E such that $\bigcup_{i\in I}\partial_r f_i(a+X)$ is equicontinuous in $L(E,F)$.*

Proof. Since by Proposition 3.3 $\partial_r f_i(x)\neq\varnothing$ for each $x\in\operatorname{dom} f_i$, the corollary follows from Proposition 4.5 and Corollary 4.4. ▮

5. DIFFERENTIABILITY AND V-SUBDIFFERENTIALS

In this section following the way opened by Asplund and Rockafellar [2] for real-valued convex functions (see also [13]) we shall study the relationship between the differentiability and the V-subdifferentials of convex operators.

Let $\mathscr{B}$ be a family of *bounded subsets* of E such that $E=\bigcup\{B: B\in\mathscr{B}\}$ and for any $B\in\mathscr{B}$ the set $-B=\{-b: b\in B\}$ belongs to $\mathscr{B}$. A mapping

$f: E \to F^{\bullet}$ is said to be $\mathscr{B}$-differentiable at a point $a \in \operatorname{dom} f$ if there exists a continuous linear mapping $T \in L(E, F)$ such that for each $B \in \mathscr{B}$

$$\lim_{t \downarrow 0} t^{-1}[f(a+tb)-f(a)] = T(b)$$

uniformly with respect to $b \in B$.

If $\mathscr{B}$ consists of singleton (resp. compact or bounded) subsets of E, f is said to be *Gateaux* (resp. *Hadamard* or *Fréchet*) differentiable at a.

Obviously if f is $\mathscr{B}$-differentiable at a and F_+ is closed then $\partial f(a) = \{A\}$, where A is the $\mathscr{B}$-differential of f at a. So we shall always assume in the sequel that F_+ is *closed.*

5.1. PROPOSITION. *Assume that F_+ is normal. If a convex mapping $f: E \to F^{\bullet}$ is $\mathscr{B}$-differentiable at a point $a \in \operatorname{dom} f$, then for each $B \in \mathscr{B}$ and each neighbourhood W of zero in F there exists a neighbourhood V of zero in E such that*

$$\partial_V f(a)(b) \subset \partial f(a)(b) + W \qquad \textit{for every } b \in B,$$

where

$$\partial_V f(a)(b) = \{T(b) : T \in \partial_V f(a)\}$$

Proof. Put $\partial f(a) = \{A\}$. Let $B \in \mathscr{B}$ and W any neighbourhood of zero in F. Choose a full circled neighbourhood W' of zero in F with $W' \subset W$ and a neighbourhood V of zero in F with $V + V \subset W'$. By differentiability there exists a real number $t > 0$ such that

$$t^{-1}[f(a+tb)-f(a)] \in A(b) + V \qquad \text{for every } b \in B \cup (-B).$$

Then for any $T \in \partial_V f(a)$ and any $b \in B \cup (-B)$ we have

$$T(b) \in t^{-1}[f(a+tb)-f(a)] + V - F_+ \subset A(b) + V + V - F_+$$

and hence

$$T(b) \in A(b) + W' - F_+.$$

Therefore for any $T \in \partial_V f(a)$ and any $b \in B$ we have

$$T(b) \in A(b) + (W' - F_+) \cap (W' + F_+) = A(b) + W' \subset A(b) + W$$

and the proof is complete. ∎

The following result states a kind of continuity of the subdifferential mapping. Borwein [3] has also established a similar property in the context of normed vector spaces.

5.2. COROLLARY. *Assume that F_+ is normal. If a convex mapping $f\colon E \to F^\bullet$ is continuous at a point $a \in \operatorname{dom} f$ and $\mathscr{B}$-differentiable at a, then for each $B \in \mathscr{B}$ and each neighbourhood W of zero in F there exists a neighbourhood V of zero in F and a neighbourhood X of zero in E such that*

$$\partial_V f(a+x)(b) \subset \partial f(a)(b) + W$$

for all $b \in B$ and $x \in X$.

Proof. This is a direct consequence of Propositions 3.4 and 5.1. ∎

Remark. If the topology of E is metrizable and $\mathscr{B}$ contains the set of all compact subsets of E, then the continuity assumption is redundand (see, e.g., Proposition 1.7.1 in [38]).

In [36] it is proved that for each neighbourhood V of zero in F the V-subdifferential mapping $\partial_V f$ is locally Lipschitz (on the set of points at which f is continuous and subdifferentiable) with respect to the Haussdorff metric on the set of subsets of $L(E, F)$ whenever E and F are normed. This property which is one of the strongest properties which can be required on a multifunction is satisfied neither by the subdifferential multifunction ∂f nor by the ε-subdifferential multifunction $\partial^\varepsilon f$ for $\varepsilon \in F_+$ and $\operatorname{int} F_+ = \emptyset$. We now proceed to show that, in fact, the continuity of the subdifferential multifunction ∂f characterizes the Fréchet-differentiability of f.

5.3. PROPOSITION. *Assume F_+ is normal and $f\colon E \to F^\bullet$ is a convex mapping with $a \in \operatorname{dom} f$. If f is subdifferentiable at a* (i.e., $\partial f(a) \neq \emptyset$) *and if for each $B \in \mathscr{B}$ and each neighbourhood W of zero in F there exists a neighbourhood X of zero in E such that*

$$\partial f(a)(b) \subset \partial f(a+x)(b) + W$$

for all $x \in X$ and $b \in B$, then f is $\mathscr{B}$-differentiable at a.

Proof. Choose $A \in \partial f(a)$. Let B be in $\mathscr{B}$ and W any full circled neighbourhood of zero in F. Choose a neighbourhood X of zero in E such that

$$\partial f(a)(b) \subset \partial f(a+x)(b) + W \qquad \text{for all } x \in X \text{ and } b \in B.$$

Let r be a positive number with $sB \subset X$ for every $s \in]0, r]$. Consider any point $t \in]0, r]$ and any point $b \in B$ and choose $T_{t,b} \in \partial f(a+tb)$ such that $(A - T_{t,b})(b) \in W$. Then

$$t^{-1}[f(a+tb) - f(a)] - A(b) \in F_+ \tag{5.1}$$

and

$$t^{-1}[f(a+tb)-f(a)] = -t^{-1}[f(a+tb-tb)-f(a+tb)] \in T_{t,b}(b) - F_+. \tag{5.2}$$

Therefore by relation (5.2),

$$t^{-1}[f(a+tb)-f(a)] - A(b) \in T_{t,b}(b) - A(b) - F_+ \subset W - F_+$$

and hence by relation (5.1)

$$t^{-1}[f(a+tb)-f(a)] - A(b) \in (W - F_+) \cap F_+ \subset W,$$

which proves that f admits A as $\mathscr{B}$-differential at the point a. ∎

5.4. COROLLARY. *Assume F_+ is normal and $f: E \to F^{\bullet}$ is a convex mapping which is continuous at a point a and subdifferentiable on a neighbourhood of a. Then f is $\mathscr{B}$-differentiable at a if and only if for each $B \in \mathscr{B}$ and each neighbourhood W of zero in F there exists a neighbourhood X of zero in E such that*

$$\partial f(a)(b) \subset \partial f(a+x)(b) + W \qquad \text{for all } x \in X \text{ and } b \in B.$$

Proof. The assertion is obviously sufficient by Proposition 5.3. Suppose now f is $\mathscr{B}$-differentiable at a. Then if $B \in \mathscr{B}$ and W is a neighbourhood of zero in F there exists by Corollary 5.2 a neighbourhood X' of zero in E such that

$$\partial f(a+x)(b) \subset \partial f(a)(b) - W \qquad \text{for all } b \in B \text{ and } x \in X'. \tag{5.3}$$

Choose a neighbourhood of zero $X \subset X'$ such that f is subdifferentiable on $a + X$. Then for each $x \in X$ and each $b \in B$ we have by relation (5.3)

$$\partial f(a)(b) \subset \partial f(a+x)(b) + W$$

since $\partial f(a)$ is a singleton and $\partial f(a+x)$ is nonempty, and hence the proof is finished. ∎

Remark. For a list of conditions ensuring subdifferentiability of f on a neighbourhood of a we refer the reader to the papers of Borwein [3, 5].

In the next corollary E and F will be normed and $L(E, F)$ will be endowed with the topology defined by the norm $\|T\| = \{\|T(x)\|: \|x\| \leqslant 1\}$. The closed unit ball around zero of radius $r > 0$ in $L(E, F)$ or F will be denoted by L_r or F_r respectively.

5.5. COROLLARY. *Assume E and F are normed and F_+ is normal. Let $f: E \to F^\bullet$ be a convex mapping continuous at a point a and subdifferentiable on a neighbourhood of that point. Then f is Fréchet-differentiable at a if and only if the subdifferential multifunction ∂f is Hausdorff continuous at a in the sense that for each real number $r > 0$ there exists a neighbourhood X of zero in E such that*

$$\partial f(a+x) \subset \partial f(a) + L_r \qquad \text{and} \qquad \partial f(a) \subset \partial f(a+x) + L_r$$

for every $x \in X$.

Proof. It is easy to see that the inclusion $\partial f(a) \subset \partial f(a+x) + L_r$ implies that $\partial f(a)(b) \subset \partial f(a+x)(b) + F_r$ for every b in the closed unit ball around zero in E and hence by Corollary 5.4 the condition is sufficient. Assume now that f is Fréchet-differentiable at a and consider a real number $r > 0$. By Corollary 5.2 there exists a neighbourhood X of zero in E such that f is subdifferentiable on $a + X$ and

$$\partial f(a+x)(b) \subset \partial f(a)(b) + F_r$$

for all x in X and b in the closed unit ball around zero in E. Then as $\partial f(a)$ is a singleton it follows that

$$\partial f(a+x) \subset \partial f(a) + L_r \qquad \text{for every } x \in X.$$

Invoking the fact that $\partial f(a)$ is a singleton once again and the nonvacuity of $\partial f(a+x)$ one sees that the second inclusion of the statement of the corollary is equivalent to the first one and hence the proof is complete. ∎

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