Normal Subobjects and Abelian Objects in Protomodular Categories

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INTRODUCTION

Recent works about Mal’cev categories [6, 7, 9–11], a notion arising from considerations about Mal’cev varieties, led the way for an alternative approach to problems that confronted universal algebra. This approach focuses on semantical aspects of the theories rather than the syntactical ones, and singles out some new points of views.

Actually the notion of a Mal’cev category is a link in a chain of notions which are organized, the existence of finite limits being assumed, by the following implications [3]

\[
\begin{align*}
\text{essentially affine} \ [2] & \Rightarrow \text{Naturally Mal’cev} \ [8] \\
\downarrow & \\
\text{protomodular} \ [2] & \Rightarrow \text{Mal’cev} \ [6, 7]
\end{align*}
\]

They have all the particularity to be classified by a unique fibration, namely the fibration of pointed objects [3]: \( p: \text{Pt}(\mathbb{E}) \to \mathbb{E} \).

Before going into the details of this chain, let us say that probably the most meaningful way of defining protomodularity, at least when the basic category \( \mathbb{E} \) has a zero object, is to require that the split short five lemma holds: a morphism between two split exact sequences with the same ends is necessarily an isomorphism. Of course this property refers to the category \( \text{Grps} \) of groups as the leading and guiding example of this notion. And, in
effect, some very familiar properties came out from this single axiom:

- A map is a monomorphism if and only if its kernel is zero [2],
- A reflexive relation is an equivalence relation ("protomodular" implies "Mal’tsev") [3],
- An internal category is a groupoid [2], and
- Mainly, a regular epimorphism is the cokernel of its kernel [2, prop. 14 and corollary], which gives a plain meaning to the notion of exact sequence (see also [4]) and seems to make protomodular categories convenient for nonabelian homological algebra [5].

This paper will be devoted to the proof that a protomodular category fulfills another very familiar property of the category of groups: any equivalence relation is completely determined by the class of the unit element, and consequently it will be devoted to the notion of normal subobject. From that, we shall derive a co-universal property for the product of two objects, the fact that when a morphism \( f: X \to Y \) in \( \mathcal{E} \) is split by a normal monomorphism, then \( X \) is isomorphic to \( \text{Ker} f \times Y \), and the characterization of the abelian objects \( X \) in \( \mathcal{E} \), as those which have their diagonal \( X \mapsto X \times X \) normal, this last point, we must emphasize, being obtained without any assumption of a right exactness property and in particular without the existence of coequalizers.

The notion of normal monomorphism still holds when \( \mathcal{E} \) does not have a zero object. In this case the fact that the diagonal of \( X \) is normal characterizes the existence on \( X \) of a (unique) Mal’tsev operation \( \pi: X \times X \times X \to X \).

Finally, there is hidden in the proof of the previous co-universal property a means for characterizing the existence of a Mal’tsev operation not only in a protomodular category, but also in any Mal’tsev category, by a normalization and commutation condition, which is obtained, once more, without any assumption of right exactness conditions. This gives a direct way to identify the abelian objects in Mal’tsev varieties [12].

The paper is organized along the following lines:

1. protomodularity and related notions,
2. equivalence relations and normal monomorphisms,
3. stability properties, examples, the Chasles relation associated with a Mal’tsev operation,
4. normal monomorphisms in protomodular categories,
5. the case of the pointed protomodular categories,
6. characterization of abelian objects in Mal’tsev categories.
1. PROTOMODULARITY AND RELATED NOTIONS

We shall assume, for the basic category $\mathbb{E}$, the existence of pullbacks of any split epimorphism along any map. We shall denote by $Pt(\mathbb{E})$ the category whose objects are the split epimorphisms in $\mathbb{E}$, with a given splitting, and whose morphisms are the commutative squares between such data, and by $\pi: Pt(\mathbb{E}) \to \mathbb{E}$ the forgetful functor assigning to each split epimorphism its codomain. It is clear, according to our assumption, that $\pi$ is a fibration we shall call the fibration of pointed objects in $\mathbb{E}$; cf [2]. Clear too is the fact that each fiber $Pt(x)$, above an object $X$ in $\mathbb{E}$, has a zero object (i.e., is pointed).

In certain circumstances, this fibration is very simple. When $\mathbb{E}$ is additive, the classical result following which the domain of a split epimorphism $f: X \to Y$ is isomorphic to $\text{Ker} f \oplus Y$ means exactly that the change of base functor along the initial map $0 \to Y$ in $\mathbb{E}$ is an equivalence of categories and consequently implies that the change of base functor along any map is an equivalence too, and so that $\pi$ is trivial.

**Definition 1** (see [2]). A category $\mathbb{E}$ is said to be essentially affine when $\pi$ is trivial.

**Examples.** Not only is an additive category $\mathbb{A}$ with kernels essentially affine but also any slice $\mathbb{A}/A$ and any coslice $A\backslash\mathbb{A}$ of $\mathbb{A}$.

The major consequence of this axiom is that, when $\mathbb{E}$ has finite products, the fibration $\pi$ is additive [2, Prop. 5]; i.e., each fiber is additive and each change of base functor is additive. But this last fact is equivalent to the fact that $\mathbb{E}$ is naturally Mal’cev [3, Theorem 7], where:

**Definition 2** (see [8]). A category $\mathbb{E}$ with finite products is naturally Mal’cev when every object $X$ in $\mathbb{E}$ is endowed with a natural Mal’cev operation $p: X \times X \times X \to X$.

Now coming back to the case of an additive category, the change of base functor along the initial map $0 \to Y$ is an equivalence if and only if, as any equivalence:

1. it reflects the isomorphisms,
2. its left adjoint $- \oplus Y$ is a right inverse.

Of course the first condition is still meaningful in the category of groups since it is precisely the split short five lemma, whence the following definition:

**Definition 3** (see [2]). A category $\mathbb{E}$ is said to be protomodular when $\pi$ is such that any change of base functor reflects the isomorphisms.
EXAMPLES. These include, of course, the category of groups; the category of rings, and more generally any variety of $\Omega$-groups; the variety of Mal’cev operations on a non-empty set, provided they are weakly right associative $p(x, y, p(y, x, z)) = z$; the category of Heyting algebras and, as a consequence, the dual of any topos; and, as we have seen, any additive or any essentially affine category. On the other hand, when $E$ is left exact, the category $\text{Grps}_E$ of internal groups in $E$ is protomodular.

**DEFINITION 4** (see [6, 7]). A left exact category $E$ is Mal’cev whenever any reflexive relation $R$ in $E$: $R \rightarrow X \times X$ is an equivalence relation.

It is shown in [3] that a left exact category $E$ is Mal’cev if and only if the fibers of $\pi: \Pr(E) \rightarrow E$ are unital, where a unital category is a category $C$ with finite products, a zero object, and such that for each pair $(X, Y)$ of objects in $C$, the pair

$$X \xrightarrow{i_X} X \times Y \xleftarrow{i_Y} Y$$

is jointly strongly epic, i.e., whenever a monomorphism $j: Z \rightarrow X \times Y$, which is such that its pullbacks along $i_X$ and $i_Y$ are isomorphisms, is itself an isomorphism. This means that the product $X \times Y$ is “generated” by $i_X(X)$ and $i_Y(Y)$.

Thus clearly, when $E$ is left exactly, “naturally Mal’cev” implies “Mal’cev.” That “protomodular” implies “Mal’cev” is a little less straightforward [3].

2. EQUIVALENCE RELATIONS AND NORMAL MONOMORPHISMS

We shall assume, for sake of simplicity, that the basic category $E$ is left exact. We shall denote by $\text{Rel}_E$ the category whose objects are the pairs $(X, R)$ of an object $X$ in $E$ and an equivalence relation $R$ on $X$, and whose maps between $(X, R)$ and $(X', R')$ are morphisms $f: X \rightarrow X'$ in $E$, such that there is a (unique) factorization $\bar{f}$.

$$\begin{array}{ccc}
R & \xrightarrow{f} & R' \\
\downarrow{[d_0, d_1]} & & \downarrow{[d_0, d_1]} \\
X \times X & \xrightarrow{f \times f} & X' \times X'
\end{array}$$

Let $b: \text{Rel}_E \rightarrow E$ be the functor assigning $X$ to $(X, R)$. This functor $b$ is a fibration, a morphism in $\text{Rel}_E$ being cartesian if and only if the previous square is a pullback. In this case, we shall often denote $R$ by $f^{-1}(R')$. 

Actually the fibration $b$ is left exact as a fibration: each fiber is left exact and each change of base functor is left exact. Not only does the fiber over $X$ have a terminal object $\text{gr} X$, the coarse relation on $X$, which is given by the identity map $X \times X \to X \times X$, but also an initial object $\text{dis} X$, the discrete relation, given by the diagonal $s_0: X \to X \times X$. Given $R$ and $S$, two equivalence relations on $X$, we shall classically denote by $R \cap S$ the relation on $X$ determined by the following pullback in $\text{Rel} \mathbb{E}$.

$$
\begin{array}{ccc}
R \cap S & \longrightarrow & R \\
\downarrow & & \downarrow \\
S & \longrightarrow & \text{gr} X
\end{array}
$$

**Definition 5.** A morphism $f: (X, R) \to (X', R')$ in $\text{Rel} \mathbb{E}$ is called fibrant, when the following square is a pullback.

$$
\begin{array}{ccc}
R & \xrightarrow{f} & R' \\
\downarrow{d_0} & & \downarrow{d_0} \\
X & \xrightarrow{f} & X'
\end{array}
$$

Of course, the class of fibrant morphisms is, as well as the class of cartesian morphisms, stable by composition, pullback, and product in $\text{Rel} \mathbb{E}$.

Given a map $g: Y \to Z$ in $\mathbb{E}$ the kernel equivalence $R[g]$ of $g$ is obtained by the following pullback in $\text{Rel} \mathbb{E}$.

$$
\begin{array}{ccc}
R[g] & \longrightarrow & \text{dis} Z \\
\downarrow & & \downarrow \\
\text{gr} Y & \xrightarrow{g} & \text{gr} Z
\end{array}
$$

Of course $g$ is a monomorphism if and only if $R[g]$ is isomorphic to $\text{dis} Z$ or, equivalently, if and only if the map $R[g] \to \text{dis} Z$ is fibrant. More generally we have the following lemma:

**Lemma 1.** If a map $f: (X, R) \to (X', R')$ is cartesian and fibrant in $\text{Rel} \mathbb{E}$, then the underlying map $f: X \to X'$ in $\mathbb{E}$ is a monomorphism.

**Proof.** Let us consider the following diagram in $\text{Rel} \mathbb{E}$.

$$
\begin{array}{ccc}
R[f] & \longrightarrow & \text{dis} X' \\
\downarrow & & \downarrow \\
R & \xrightarrow{f} & R' \\
\downarrow{gr f} & & \downarrow{gr f} \\
\text{gr} X & \xrightarrow{gr f} & \text{gr} X'
\end{array}
$$
The lower square is a pullback since $f$ is cartesian; then $\mathcal{R}[f]$ factors through $R$ and the upper square is a pullback. Now $f: (X, R) \to (X', R')$ is fibrant too; then the map $\mathcal{R}[f] \to \text{dis } X'$ is fibrant and $f: X \to X'$ is a monomorphism.

**Remark.** When $f: X \to X'$ is an isomorphism, then of course $\text{gr } f: \text{gr } X \to \text{gr } X'$ is fibrant. Conversely if $\text{gr } f$ is fibrant, then with $\text{gr } f$ always being cartesian, the map $f: X \to X'$ is a monomorphism. But it is not an isomorphism in general.

However, if we suppose that $X$ has a universally global support, i.e., that the terminal map $\tau_X: X \to 1$ is a universal regular epimorphism (a coequalizer of its kernel equivalence which, as such, is stable by pullback), then we do have the converse.

**Proposition 2.** When $X$ has a universally global support, then $f: X \to X'$ is an isomorphism if and only if $\text{gr } f$ is fibrant in $\text{Rel } \mathbb{E}$.

**Proof.** Let us consider the following diagram in $\mathbb{E}$.

$$
\begin{array}{ccc}
X \times X & \xrightarrow{p_0} & X \\
| & \searrow & | \\
X \times X' & \xrightarrow{p_X} & X \\
| & \downarrow & | \\
X' \times X' & \xrightarrow{p_0} & X'
\end{array}
$$

The lower square always being a pullback, the map $\text{gr } f$ is fibrant if and only if the map $X \times f$ is an isomorphism in $\mathbb{E}$. Now considering the following diagram and the fact that $\tau_X$ is a universal regular epimorphism, then $p_X$ coequalizes $p_0 \times X'$ and $p_1 \times X'$, as well as $p_1$ coequalizes $p_1$ and $p_2$. Consequently $f$ is the factorization through the quotients of the isomorphism $X \times f$ and thus is itself an isomorphism.

$$
\begin{array}{ccc}
X \times X \times X & \xrightarrow{p_1} & X \times X \xrightarrow{p_1} X \\
\xrightarrow{X \times \times f} & \searrow & \downarrow \text{f} \\
X \times X \times X' & \xrightarrow{p_0 \times X'} & X \times X' \xrightarrow{p_X} X'
\end{array}
$$

**Remark 1.** Of course the condition of Proposition 2 is fulfilled when the terminal map $\tau_X$ is split. In particular, the condition holds for any object $X$ when $\mathbb{E}$ has a zero object, and then a map $f$ in $\mathbb{E}$ is an isomorphism, if and only if $\text{gr } f$ is fibrant in $\text{Rel } \mathbb{E}$. 

**Remark 2.** As a corollary of the previous proposition, we have the following result which gives a direct proof and weakens the hypothesis of the result (6.10) in [1] (we do not require that $g$ is a regular epimorphism).

**Corollary 2.** Consider the following diagram in any left exact category $\mathbb{E}$:

$$
\begin{array}{ccc}
R[f] & \overset{p_0}{\longrightarrow} & X \\
\downarrow \alpha & \quad & \downarrow \alpha' \\
R[g] & \overset{p_0}{\longrightarrow} & Y \\
\end{array}
$$

If the left hand square with the $p_0$ is a pullback and the map $f$ is a universal regular epimorphism (for instance, when $f$ is split), then the right hand square is a pullback. When, furthermore, $\alpha$ is a monomorphism, then $\alpha'$ is a monomorphism.

**Proof.** Let us consider the pullback $\bar{g}$ of $g$ along $\alpha'$, and $\beta$, the factorization of $\alpha$ through $\bar{g}$. Then the map $\beta$ is a map in the slice category $\mathbb{E}/X'$. Now the condition that the square with the $p_0$ is a pullback implies that $gr\beta$, calculated in $\mathbb{E}/X'$, is fibrant. On the other hand, the map $f$, being a universal regular epimorphism in $\mathbb{E}$, is an object in $\mathbb{E}/X'$ which has a universally global support. Then applying Proposition 2 to the map $\beta$ in $\mathbb{E}/X'$, this map is an isomorphism and the right hand square a pullback.

Now consider the following diagram.

$$
\begin{array}{ccc}
R[\alpha] & \overset{\bar{f}}{\longrightarrow} & R[\alpha'] \\
\downarrow p_0 & \quad & \downarrow p_0 \\
X & \overset{f}{\longrightarrow} & X' \\
\downarrow \alpha & \quad & \downarrow \alpha' \\
Y & \overset{g}{\longrightarrow} & Y'
\end{array}
$$

Since the lower square is a pullback, the upper square with the $p_0$ is a pullback and $\bar{f}$ is a regular epimorphism since $f$ is a universal regular epimorphism. Now if $\alpha$ is a monomorphism, then the left hand side $p_0$ is an isomorphism. The maps $f$ and $\bar{f}$ being regular epimorphisms, the right hand side $p_0$ is an isomorphism as a factorization of the left hand side $p_0$. Consequently $\alpha'$ is a monomorphism. $\blacksquare$
We are now in a position to introduce the main definition of this paper:

**Definition 6.** Given an equivalence relation \((X', R')\), a map \(f: X \to X'\) is said to be normal to \(R'\) when:

1. \(f^{-1}(R') = \text{gr } X\),
2. the cartesian map \(\text{gr } X \to R'\) is also fibrant.

**Remarks.**
1. Then according to Lemma 1, the map \(f\) is necessarily a monomorphism.
2. \(X\) being seen as a part of \(X'\), condition 1 means

\[\forall x_1, x_2 \in X, x_1 R' x_2\]

and condition 2 means

\[\forall x \in X, \forall x' \in X', x R' x' \Rightarrow x' \in X.\]

In other words, Definition 6 is an intrinsic way to define an equivalence class of \(R'\).

### 3. Stability Properties, Examples

1. The normal morphisms are stable by pullback. If the square

\[
\begin{array}{cc}
Y & \xrightarrow{h} & X \\
\downarrow{g} & & \downarrow{f} \\
Y' & \xrightarrow{h'} & X'
\end{array}
\]

is a pullback and if \(f\) is normal to \(R'\), then \(g\) is normal to \(h'^{-1}(R')\).

2. They are stable by products, if \(f\) is normal to \(R\) and \(\tilde{f}\) normal to \(\tilde{R}\), then \(f \times \tilde{f}\) is normal to \(R' \times \tilde{R'}\).

3. They are stable by intersection. When \(f_1: X_1 \to X'\) is normal to \(R'_1\) and \(f_2: X_2 \to X'\) is normal to \(R'_2\), then the diagonal \(f_1 \Delta f_2\) in the following pullback is normal to \(R'_1 \cap R'_2\).

\[
\begin{array}{ccc}
X_1 \cap X_2 & \twoheadrightarrow & X_1 \\
\downarrow & \searrow & \downarrow{f_1} \\
X_2 & \underset{f_2}{\twoheadrightarrow} & X'
\end{array}
\]
(4) When $E$ has a zero object, then the kernel $K[h]$ of a map $h: Y \to Z$ is defined by the following pullback.

$$
\begin{array}{ccc}
K[h] & \rightarrow & Y \\
\downarrow & & \downarrow h \\
0 & \rightarrow & Z
\end{array}
$$

Then clearly $k[h]$ is normal to the kernel equivalence $R[h]$ of $h$. The converse is true as well:

**Proposition 3.** When $E$ is left exact and has a zero object, if $f: X \to X'$ is normal to an effective equivalence relation $R' = R[h]$, then $f$ is the kernel of $h$.

*Proof.* Let us consider the following diagram.

$$
\begin{array}{ccc}
R[f] & \rightarrow & X \\
\downarrow f & & \downarrow 0 \\
R[h] & \rightarrow & Z
\end{array}
$$

The square with the $p_0$ is a pullback since $f$ is normal to $R[h]$ and the terminal map $\tau_X: X \to 0$ is split. Then according to Corollary 2 the right hand square is a pullback and $f$ the kernel of $h$. 

(5) **Proposition 4.** When $E$ has a zero object, any equivalence relation $(Z, R)$ determines a normal monomorphism.

*Proof.* Let $R$ be an equivalence relation on $Z$.

$$
\begin{array}{ccc}
R \times_Z R & \rightarrow & R \\
\downarrow & & \downarrow d_0 \\
R & \rightarrow & Z \\
\downarrow p_1 \downarrow & & \downarrow s_0 \\
R & \rightarrow & Z \\
\downarrow p_2 \downarrow & & \downarrow d_1
\end{array}
$$

The pair $(p_0, p_1)$ makes $R \times_Z R$ the kernel equivalence $R[d_0]$ of the map $d_0$ and the pair $(d_1, p_2)$ determines a fibrant map between $R[d_0]$ and $R$ in $\text{Rel} E$. Now take the kernel $k[d_0]$ of $d_0$; then $j = d_1 \cdot k[d_0]$ is normal to $R$, 

according to the following diagram and the fact that the fibrant maps are stable by composition.

\[
K[d_0] \times K[d_0] \xrightarrow{\tilde{k}[d_0]} R[d_0] \xrightarrow{p_2} R
\]

\[
K[d_0] \overset{k[d_0]}{\downarrow} \xrightarrow{R[d_0]} \overset{d_1}{R} \xrightarrow{d_0} Z
\]

\[
0 \xrightarrow{} Z
\]

**Remark.** Of course, the map \( j = d_1 \cdot k[d_0] \) can be obtained directly from the pullback

\[
K[d_0] \xrightarrow{d_1 \cdot k[d_0]} R \xrightarrow{} Z \\
\overset{Z, Z}{\downarrow} \xrightarrow{\cdot s_1 Z} \overset{Z \times Z}{\downarrow}
\]

where \( s_1 Z \), according to the simplicial notations, is \([\omega_2, Id_z]\) and \( \omega_z : Z \to Z \) is the zero morphism. Conversely if \( f : X \to Z \) is normal to \( R \), then, considering the following diagram, the map \( \bar{f} = \bar{f} \cdot s_1 X \) is the kernel of \( d_0 : R \to Z \) and, of course, we have \( d_1 \cdot \bar{f} = d_1 \cdot \bar{f} \cdot s_1 X = f \cdot p_1 \cdot s_1 X = f \):

\[
X \overset{\cdot s_1 X}{\downarrow} X \times X \xrightarrow{f} R \\
\quad \overset{\cdot s_1 X}{\downarrow} \quad \overset{\cdot s_1 X}{\downarrow} \xrightarrow{f} Z
\]

Consequently when \( \mathbb{E} \) has a zero object, any equivalence relation determines a unique, up to isomorphism, normal monomorphism.

\( 6 \) Mal’cev operations and the associated Chasles relations: Let us suppose that \( \mathbb{E} \) no longer has a zero object. Given an internal ternary operation on an object \( X \) of \( \mathbb{E} \),

\[
p : X \times X \times X \to X,
\]

then the pair \( (\pi_1, p_2) : \pi_1(x, y, z) = (x, p(x, y, z)), p_2(x, y, z) = (y, z) \) defines an internal relation \( Ch[p] \) on \( X \times X \):

\[
\overset{\pi_1}{X \times X \times X \xrightarrow{\implies} X \times X} \overset{p_2}{X \times X}
\]
We shall call this relation the Chasles relation associated to \( p \), where 

\[(x, t) \in \text{Ch}[(y, z) \iff t = p(x, y, z)].\]

The relation \( \text{Ch}[p] \) is reflexive if and only if \( p(x, x, y) = y \).

It is symmetric if and only if \( p(x, y, p(y, x, z)) = z \).

It is transitive if and only if \( p(x, y, p(y, z, t)) = p(x, z, t) \) \( \star \)

We shall need another ingredient:

**Definition 7.** An equivalence relation \((X \times X, R)\) on \( X \times X \) is said to be normalized when the diagonal \( s_0: X \to X \times X \) is normal to \( R \).

The three previous identities being assumed, the equivalence relation \( \text{Ch}[p] \) is normalized if and only if \( p(x, y, y) = x \).

In other words, any Mal’cev operation which is right associative, i.e., satisfying \( \star \), determines a normalized equivalence relation \( \text{Ch}[p] \) on \( X \times X \).

(7) It is clear that in general a single equivalence relation generates many normal monomorphisms. We have, however, the following expected information:

**Proposition 5.** If \( f_1: X_1 \to X' \) and \( f_2: X_2 \to X' \) are two monomorphisms normal to the same equivalence relation \((X', R')\), then, as soon as \( X_1 \cap X_2 \) has a universally global support, there is a unique isomorphism \( \gamma: X_1 \to X_2 \), that \( f_2 \cdot \gamma = f_1 \).

**Proof.** Let us consider the following pullback.

\[
\begin{array}{ccc}
X_1 \cap X_2 & \xrightarrow{\gamma} & X_1 \\
\downarrow & & \downarrow f_1 \\
X_2 & \xrightarrow{f_2} & X'
\end{array}
\]

Then \( f_1 \cdot \gamma = f_1 \Lambda f_2 \) is normal to \( R' \cap R' = R' \). Consequently, in this diagram, the left hand side part with the \( p_0 \) is a pullback,

\[
\begin{array}{ccc}
(X_1 \cap X_2) \times (X_1 \cap X_2) & \xrightarrow{\gamma \times \gamma} & X_1 \times X_1 \\
\downarrow p_0 \downarrow p_1 & & \downarrow p_0 \downarrow p_1 \\
X_1 \cap X_2 & \xrightarrow{f_1} & X'
\end{array}
\]

and consequently \( \text{gr} \gamma \) is fibrant in \( \text{Rel} \). But according to Proposition 2, \( X_1 \cap X_2 \) having a universally global support, \( \gamma \) is an isomorphism and \( X_1 \) is canonically isomorphic to \( X_2 \).
4. NORMAL MONOMORPHISMS IN PROTOMODULAR CATEGORIES

It is clear too that, in general, in the category of sets for instance, a single morphism can be normal to different equivalence relations. The main point of this paper is that, in any protomodular category, in the same way as in the category of groups, a morphism is normal to at most one equivalence relation.

**Theorem 6.** Let $\mathcal{E}$ be left exact and protomodular. If a map $f: X \to X'$ is normal to $(X', R')$, then $R'$ is unique up to isomorphism.

**Proof.** Let $(X', R')$ be another equivalence relation to which $f$ is also normal. Then $f^{-1}(R' \cap R) = f^{-1}(R') \cap f^{-1}(R) = \text{gr} X \cap \text{gr} X = \text{gr} X$. Furthermore the following diagram in $\text{Rel} \mathcal{E}$ is a pullback since the two horizontal maps are cartesian and the two vertical ones are inside a fiber:

\[
\begin{array}{ccc}
\text{gr} X & \to & R' \cap R \\
\downarrow & & \downarrow j \\
\text{gr} X & \to & R'
\end{array}
\]

Now, the lower map is fibrant since $f$ is normal to $(X', R')$ and therefore the upper one is fibrant too. Consequently the following inside and outside squares are pullbacks in $\mathcal{E}$:

\[
\begin{array}{ccc}
X \times X & \to & R' \cap R \\
\downarrow p_0 & & \downarrow j \\
X & \to & X'
\end{array}
\]

But each vertical map is split. This means that $f^*(j)$ is an isomorphism in $\Pr(X)$ and, $\mathcal{E}$ being protomodular, $j$ is itself an isomorphism. Consequently $R' \cong R' \cap R = R$. □

Thus, in the presence of the protomodularity condition, being normal is no more a condition relative to a given equivalence relation but becomes an intrinsic property.

For instance, in the category of groups, the notion of a normal monomorphism corresponds to the notion of a normal subgroup. In the category of rings, it corresponds to the notion of a two-sided ideal. It is possible to show that the only normal monomorphisms in the category of Heyting algebras are the isomorphisms.

As a general consequence, in a protomodular category, the normality of the diagonal $s_0: X \to X \times X$ characterizes the existence of a Mal’cev operation on $X$. 
**Proposition 7.** In any protomodular category with finite products, the diagonal $s_0: X \to X \times X$ is normal if and only if $X$ is endowed with a Mal’cev operation.

**Proof.** The basic category $\mathbb{E}$ being protomodular, any Mal’cev operation on $X$ is right associative ($\star$ in Section 3.6), and according to Example 3.6 the diagonal is normal to $\text{Ch}[p]$.

Conversely let $(X \times X, R)$ be the equivalence relation to which the diagonal is normal. Let us denote by $\varphi: R \to X \times X \times X$ the map $[p_0 \cdot d_0, p_0 \cdot d_1, p_1 \cdot d_1]$ which, internally speaking, assigns the triple $(x, y, z)$ to the pair $((x, t), (y, z))$ of equivalents objects of $X \times X$. Now let us consider the following diagram.

\[
\begin{array}{ccc}
X \times X & \xrightarrow{s_0} & X \\
\downarrow{s_0} & & \downarrow{s_0} \\
X & \xrightarrow{R} & X \times X \times X \\
\end{array}
\]

We have

\[\begin{align*}
p_2 \cdot \varphi &= [p_0 \cdot d_1, p_0 \cdot d_1] = d_1 \\
\varphi \cdot s_0 &= [p_0 \cdot d_0 \cdot s_0, p_0 \cdot d_1 \cdot s_0, p_1 \cdot d_1 \cdot s_0] = [p_0, p_0, p_1] = s_1 \\
\varphi \cdot \tilde{s}_0 &= [p_0 \cdot d_0 \cdot \tilde{s}_0, p_0 \cdot d_1 \cdot \tilde{s}_0] = [p_0 \cdot s_0 \cdot d_0, p_0 \cdot s_0 \cdot d_1, p_1 \cdot s_0 \cdot d_1] \\
&= [d_0, d_1, d_1] = s_0.
\end{align*}\]

The left hand downward square is a pullback since $s_0$ is normal to $R$. The outside downward square is a pullback in any category, so that $s_0^*(\varphi)$ is an isomorphism in $\text{Pt}(X)$. Now $\mathbb{E}$ is protomodular, and $\varphi$ is an isomorphism in $\text{Pt}(X \times X)$. Consequently $p = p_1 \cdot d_0 \cdot \varphi^{-1}: X \times X \times X \to X$ is a ternary operation on $X$ such that $p(x, x, y) = y$ and $p(x, y, y) = x$. And $R$ is the Chasles relation associated to $p$. $\square$

We recalled in Section 1 that an essentially affine category is protomodular. In this case, every monomorphism is normal as it is familiar in any additive category.

**Proposition 8.** When $\mathbb{E}$ is left exact and essentially affine, every monomorphism is normal.

**Proof.** Let $f: X \to X'$ be a monomorphism. The change of base functor $f^*: \text{Pt}(X') \to \text{Pt}(X)$ being an equivalence, let us denote by $(\pi_0, \sigma_0), \pi_0: X' \to X'$ the unique (up to isomorphism) split epimorphism

\[
\begin{array}{ccc}
X' & \xrightarrow{\pi_0} & X \\
\downarrow{\pi_0} & & \downarrow{\pi_0} \\
X & \xrightarrow{f} & X'
\end{array}
\]
such that the following square is a pullback.

\[
\begin{array}{c}
\begin{array}{ccc}
X \times X & \xrightarrow{f} & X' \\
\downarrow{\sigma_0} & & \downarrow{\pi_0} \\
X & \xrightarrow{f} & X'
\end{array}
\end{array}
\]

(1)

Actually the upward square is a pushout [2, prop. 4]. Consequently the map \( p_1': X \times X \rightarrow X \) produces a map \( \pi_1': X' \rightarrow X \) such that \( \pi_1 \cdot \sigma_0 = 1_X \) and \( \pi_1 \cdot f = f \cdot p_1 \). Thus we have now a reflexive graph \( R \) on \( X' \).

Let us consider the following diagram.

\[
\begin{array}{c}
\begin{array}{ccc}
X \times X & \xrightarrow{f} & X' \\
\downarrow{X \times f} & & \downarrow{[\pi_0, \pi_1]} \\
X \times X' & \xrightarrow{f \times X'} & X' \times X' \\
\downarrow{p_X} & & \downarrow{p_0} \\
X & \xrightarrow{f} & X'
\end{array}
\end{array}
\]

The lower square and the total vertical square being pullbacks, the upper square is a pullback. But \( X \times f \) is a monomorphism, and \( E \) being protomodular, the pullback functors reflect the monomorphism [2, prop. 9]. Consequently \( [\pi_0, \pi_1] \) is a monomorphism and \( R \) is a reflexive relation on \( X' \). But \( E \), being protomodular, is Mal’cev, and any reflexive relation is an equivalence relation. Clearly \( f^{-1}(R) = \text{gr} X \) and the map \( \text{gr} X \rightarrow R \) is fibrant since the square (1) is a pullback. Consequently \( f \) is normal to \( R \).

\[ \blacksquare \]

Remark. Conversely if \( E \) is left exact, protomodular, and such that every monomorphism is normal, then, every diagonal being normal, every object \( X \) in \( E \) has a canonical Mal’cev operation which, of course, is natural in the sense of [8] and then \( E \) is naturally Mal’cev. But it seems difficult to say more.

5. THE CASE OF THE POINTED PROTOMODULAR CATEGORIES

The most familiar situation of the category of groups will be recovered when \( E \) will be supposed pointed (i.e., having a zero object).

First, any initial map \( \alpha_0: 0 \rightarrow X \) is clearly normal (to \( \text{dis} X \)) and, the notion being pullback stable, any canonical injection \( i_Y = Y \rightarrow X \times Y \) is normal too.
Second, thanks to Proposition 4 and Theorem 6, the class of normal monomorphisms (up to isomorphism) with codomain $X'$ is in one to one correspondence with the class of equivalence relations on $X'$ (up to isomorphism). More precisely let us denote by $\text{Norm } E$ the category whose objects are the normal monomorphisms in $E$ and whose morphisms are the commutative squares between them. Of course the previous remark and Proposition 4 determine an equivalence of categories

$$\text{Rel } E \rightarrow \text{Norm } E,$$

where the cartesian maps in $\text{Rel } E$ correspond to the squares in $\text{Norm } E$ which are pullbacks and the fibrant maps to those squares in $\text{Norm } E$ whose upper map $h$ is an isomorphism.

Third, we have the following characterization of the abelian objects:

**Proposition 9.** An object $X$ in a left exact pointed protomodular category $E$ is endowed with a group structure (which is unique and abelian) if and only if the diagonal $s_0: X \rightarrow X \times X$ is normal.

**Proof.** When $X$ has a group structure in $E$, it is abelian and $s_0$ is the kernel of $u = p_1 - p_0: X \times X \rightarrow X$ and thus is normal.

Conversely, if the diagonal is normal, according to Proposition 7, $X$ is endowed with a Mal’cev operation. But $E$ is pointed and this Mal’cev operation actually comes from a binary operation on $X$ which can be recovered from the Mal’cev operation by $x \cdot x' = p(x, 1, x')$.

This result enlarges the one of [5] obtained in the presence of coequalizers. As a corollary we have the following characterization of additive categories.

**Corollary 9.** A pointed left exact category $E$ is additive if and only if it is protomodular and such that every monomorphism is normal.

**Proof.** We saw that an additive category is essentially affine and thus is protomodular, and such that any monomorphism is normal (Proposition 8).

Conversely if $E$ is pointed, protomodular, and has every monomorphism normal, then every diagonal is normal and every object $X$ is canonically abelian. Thus $E$ is equivalent $\text{Ab}(E)$ and consequently is additive.

Finally, we observed in Section 3.4 that in a pointed category $E$, any kernel map is normal and that conversely any map normal to an effective
An equivalence relation is a kernel (Proposition 3). Consequently, the familiar property, in the category of groups, following which every normal monomorphism is a kernel, is thus equivalent to the fact that every equivalence relation is effective. More generally, we do have:

**Proposition 10.** When \( E \) is left exact, pointed, and protomodular, every normal monomorphism is a kernel if and only if every equivalence relation is effective.

**Proof.** Straightforward.

A co-universal property for the product

We are now reaching the second major point of this paper. Let \( j: X \times Y \to Z \) be a monomorphism in a pointed category; then obviously the following outside square is a pullback.

\[
\begin{array}{ccc}
0 & \to & X \\
\downarrow & & \downarrow \iota_x \\
Y & \to & X \times Y \\
\downarrow \iota_y & & \downarrow j \\
\quad & & Z
\end{array}
\]

On the other hand, when \( E \) is protomodular, we observed that the maps \( \iota_x \) and \( \iota_y \) are normal. We have actually a couniversal property for the product:

**Theorem 11.** Given a pair \((x, y), x: X \to Z \) and \( y: Y \to Z \) of normal monomorphisms such that \( X \cap Y = 0 \), there is a unique normal monomorphism \( g: X = Y \to Z \) such that \( j \circ i_x = x \) and \( j \circ i_y = y \).

**Proof.** The unicity is a consequence of the fact that the pair \((\iota_x, \iota_y)\) is jointly epic [2, prop. 11].

Let \( R \) and \( S \) be the equivalence relations on \( Z \) corresponding to the normal monomorphisms \( x \) and \( y \). Now the equivalence of categories \( \text{Rel} E \cong \text{Norm} E \) tells us that the condition \( X \cap Y = 0 \) corresponds to \( R \cap S = \text{dis} Z \).

Let us define \( R \sqcup S \) by the following pullback.

\[
\begin{array}{c}
R \sqcup S \\
\downarrow \\
S \times Z_{[d_0, d_1]_{[d_0, d_1]}} \to Z \times Z \times Z \times Z
\end{array}
\]

It corresponds to the subobject of \( Z^4 \) consisting of the quadruples \((x, x', y, y')\) such that \( xSx', ySy', xRy, x'Ry'\) and determines a double rela-
The condition $R \cap S = \text{dis} Z$ implies that the following square is a pullback

$$
\begin{array}{ccc}
R \square S & \xrightarrow{d_0} & R \\
\downarrow{d_0} & & \downarrow{d_1} \\
S & \xleftarrow{d_i} & Z \\
\end{array}
$$

The condition $R \cap S = \text{dis} Z$ implies that the following square is a pullback

$$
\begin{array}{ccc}
R \square S & \xleftarrow{s_0} & R \\
\downarrow{d_0} & & \downarrow{d_0} \\
S & \xleftarrow{s_0} & Z \\
\end{array}
$$

Indeed, when we have $x = x'$ in a quadruple of $R \square S$, then $xRy$ and $xRy'$ imply $yRy'$. Now $ySy'$, thus, $R \cap S$ being $\text{dis} Z$, $y$ is necessarily equal to $y'$.

Since the category $\mathcal{E}$ is protomodular, this implies [2, prop. 7] that the following square is a pullback and more generally any commutative square of the diagram (2).

$$
\begin{array}{ccc}
R \square S & \xrightarrow{d_0} & R \\
\downarrow{d_0} & & \downarrow{d_0} \\
S & \xrightarrow{d_0} & Z \\
\end{array}
$$

Consequently $R \square S$ is underlying an equivalence relation on $Z$. More precisely we have $R \square S = R \circ S = S \circ R$, where $\circ$ denotes the usual composition of relations. Now let us consider the following diagram.

$$
\begin{array}{ccc}
X \times Y & \xrightarrow{\varphi} & R \square S \\
\downarrow{d_1} & & \downarrow{d_1} \\
R & \xrightarrow{s_0} & Z \\
\end{array}
$$

We have $\tilde{x} = \tilde{x} \cdot s_1 x$ and $\tilde{y} = \tilde{y} \cdot s_1 y$, as in the remark of Section 3.5, which are the kernels of the maps $d_0$, whence a unique factorization $\varphi: X \times Y$
→ \( R \square S \), making \( \varphi \) the kernel of \( d_0 \cdot d_0 : R \square S \to Z \). Now, according to Proposition 4, \( j = d_1 \cdot d_1 \cdot \varphi \) is normal to \( R \square S \).

Furthermore \( \varphi \cdot i_X = [\tilde{x}, \omega_{\tilde{x}}] \), where \( \omega_{\tilde{x}} \) is the zero map \( X \to S \). Then

\[
j \cdot i_X = d_1 \cdot d_1 \cdot \varphi \cdot i_X = d_1 \cdot d_1 \cdot [\tilde{x}, \omega_{\tilde{x}}] = d_1 \cdot \tilde{x} = \tilde{x}
\]

(Remark, prop. 4). The equality \( j \cdot i_Y = y \) is checked in the same way.

Whence a characterization of the product:

**Proposition 12.** When an epimorphism \( f : X \to Z \) is split by a normal monomorphism \( s \), then \( X \) is isomorphic to \( K[f] \times Z \).

**Proof.** Let us consider the following pullback.

\[
\begin{array}{ccc}
K[f] & \xrightarrow{k[f]} & X \\
\downarrow & & \downarrow f \\
0 & \xrightarrow{\alpha_Z} & Z
\end{array}
\]

Then \( k[f] \) and \( s \) are normal and such that \( K[f] \cap Z = 0 \), whence a map \( j : K[f] \times Z \to X \) such that \( j \cdot i_{K[f]} = k[f] \) and \( j \cdot i_Z = s \). Now the pair \((i_{K[f]}, i_Z)\) being jointly epic [2, prop. 11], we have \( f \cdot j = p_Z \) since

\[
\begin{align*}
f \cdot j \cdot i_{K[f]} &= f \cdot k[f] = \omega = p_Z \cdot i_{K[f]} \\
f \cdot j \cdot i_Z &= f \cdot s = 1_Z = p_Z \cdot i_Z.
\end{align*}
\]

Accordingly \( j \) is a map in \( Pr(Z) \). Clearly \( \alpha_Z^*(j) \) is an isomorphism in \( Pr(0) \), and therefore \( j \) is an isomorphism in \( Pr(Z) \).

6. CHARACTERIZATION OF ABELIAN OBJECTS IN MAL’CEV CATEGORIES

Now let us consider \( \mathcal{E} \) a Mal’cev category. Considering the following pullback and proposition 8 in [3], the pair \((s_0, s_1)\): \( X \times X \Rightarrow X \times X \times X \) is jointly epic

\[
\begin{array}{ccc}
X \times X \times X & \xrightarrow{s_1} & X \times X \\
p_0 \downarrow & & \downarrow p_1 \\
X \times X & \xrightarrow{s_0} & X
\end{array}
\]

A Mal’cev operation on \( X \) being a map \( p : X \times X \times X \to X \) such that \( p \cdot s_0 = p_0 \) and \( p \cdot s_1 = p_1 \), there is at most one Mal’cev operation on \( X \) in a Mal’cev category. Furthermore it is well known that this Mal’cev opera-
tion, when it exists, is autonomous, i.e., commutes with itself:

\[ p(p(x_1, x_2, x_3), p(y_1, y_2, y_3), p(z_1, z_2, z_3)) = p(p(x_1, y_1, z_1), p(x_2, y_2, z_2), p(x_3, y_3, z_3)). \]

An object \( X \) in \( \mathcal{E} \) is said to be *abelian* when it is endowed with its unique Mal’cev operation, as in the case of the Mal’cev varieties.

It is clear that the autonomous property implies the right associativity (\( \star \)) of Section 3.6, and that therefore, when the object \( X \) is abelian, the diagonal is normal to the Chasles relation associated with \( p \). We saw by Proposition 7 that this is a characteristic condition when \( \mathcal{E} \) is protomodular. Some further condition will be necessary to give a characteristic condition in the Mal’cev case, which is given by the following definition.

**Definition 8.** Given any left exact category \( \mathcal{E} \) and \( \mathcal{R} \) a relation on the object \( X \times X \), this relation is said to be parallelistic when \( (x, y) \mathcal{R}(x', y') \Rightarrow (x, x') \mathcal{R}(y, y') \).

If we denote by \( \nu w : X \times X \rightarrow X \times X \) the twisting isomorphism \( \nu w(x, y) = (y, x) \), then \( \mathcal{R} \) is parallelistic if and only if there is a (unique) factorization

\[
\begin{array}{ccc}
R & \rightarrow & R \\
\downarrow & & \downarrow \\
X^4 & \xrightarrow{\nu w \times \nu w} & X^4
\end{array}
\]

**Example.** The Chasles relation \( \text{Ch}[p] \) associated with a ternary operation is parallelistic if and only if \( p \) satisfies the following condition:

\[ p(x, p(x, y, z), z) = y(\star \star \star). \]

The autonomous condition implies the last one, and consequently, in any Mal’cev category, the Chasles relation associated to any Mal’cev operation is parallelistic and also normalized (Definition 7).

**Proposition 13.** Let \( \mathcal{E} \) be a left exact category and suppose a relation \( \mathcal{R} \) on \( X \times X \) which is normalized and parallelistic. Then considering \( p_0 : X \times X \rightarrow X \), we have

\[ R \cap R[p_0] = \text{dis}(X \times X). \]

**Proof.** Suppose \( (x, y) \mathcal{R}(x', y') \) and \( (x, y)R[p_0](x', y') \). The second condition means \( x = x' \). Now \( (x, y)\mathcal{R}(x', y') \) implies \( (R \text{ parallelistic}) (x, x)\mathcal{R}(y, y') \). Since \( R \) is also normalized and the diagonal an equivalence class, then \( y = y' \).
**Theorem 14.** In any Mal’cev category $\mathcal{E}$, an object $X$ is abelian if and only if there is a reflexive relation $R$ on $X \times X$ which is normalized and parallellistic.

**Proof.** We observed that it is the case when $X$ is abelian. Conversely let $R$ be a reflexive, normalized, and parallellistic relation on $X \times X$. Being reflexive in a Mal’cev category, it is an equivalence relation. According to Proposition 13, we have also $R \cap R[p_0] = \text{dis}(X \times X)$.

Now let us consider $R \boxtimes R[p_0]$ as in the proof of Theorem 11.

\[
\begin{array}{ccc}
R \boxtimes R[p_0] & \xrightarrow{d_0} & R \\
d_1 & \downarrow & \downarrow d_1 \\
R[p_0] & \xrightarrow{d_0} & X \times X
\end{array}
\]

Again the condition $R \cap R[p_0] = \text{dis}(X \times X)$ implies that the square

\[
\begin{array}{ccc}
R \boxtimes R[p_0] & \xrightarrow{d_0} & R \\
\downarrow & & \downarrow d_0 \\
R[p_0] & \xleftarrow{s_0} & X \times X
\end{array}
\]

is a pullback and that, for symmetrical reasons, the following is a pullback too.

\[
\begin{array}{ccc}
R \boxtimes R[p_0] & \xrightarrow{d_0} & R \\
s_0 & \downarrow & \downarrow s_0 \\
R[p_0] & \xleftarrow{s_0} & X \times X
\end{array}
\]

From that, we shall conclude again that, $\mathcal{E}$ being Mal’cev, the following square is a pullback.

\[
\begin{array}{ccc}
R \boxtimes R[p_0] & \xrightarrow{d_0} & R \\
\downarrow & & \downarrow d_0 \\
R[p_0] & \xleftarrow{d_0} & X \times X
\end{array}
\]

In order to show it, let us consider the factorization $\alpha$ from $R \boxtimes R[p_0]$ to the pullback of $d_0$: $R \to X \times X$ along $d_0$: $R[p_0] \to X \times X$ which is actually a product in the fiber $P(X \times X)$. Thus $\alpha$ is a map in $P(X \times X)$, which is a monomorphism since $R \cap R[p_0] = \text{dis}(X \times X)$. But $P(X \times X)$ is unital (see Section 1) since $\mathcal{E}$ is Mal’cev. Now the fact that the squares
(3) and (4) are pullbacks means that the pullbacks of $\alpha$ along the two maps $s_0$ are isomorphisms. Consequently $\alpha$ is an isomorphism and the square (5) is a pullback.

Therefore $R \Box R[p_0]$ represents the composition $R \circ R[p_0]$, as well as $R[p_0] \circ R$, of the two relations. Now, given any triple $(x, y, z)$ of elements of $X$, we have always

$$(x, x) R(y, y) R[p_0](y, z)$$

since $R$ is normalized. Consequently there exists a unique element $t$ of $X$ such that

$$(x, x) R[p_0](x, t) R(y, z).$$

To set $t = p(x, y, z)$ is to define the Mal’cev operation on $X$.

Remark. In this last step, we derived the Mal’cev operation uniquely from the equality $R \cap R[p_0] = \text{dis}(X \times X)$ and the fact that $R$ is normalized. Then, in any Mal’cev category $\mathbb{E}$, given a relation $R$ on $X \times X$ which is reflexive and normalized, we have $R \cap R[p_0] = \text{dis}(X \times X)$ if and only if $R$ is parallelistic.

REFERENCES