THE INVARIANCE PRINCIPLE FOR ASSOCIATED PROCESSES

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In this paper we prove the invariance principle for associated processes, satisfying some moment conditions. No stationarity is required. Our results imply an extension to the nonstationary case of an invariance principle of Newman and Wright and an improvement of a central limit theorem of Cox and Grimmett.

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sequences of associated random variables * invariance principle * central limit theorem

1. Introduction and notation

Let \{X_j: j \in \mathbb{N}\} be a sequence of random variables on some probability space \((\Omega, \mathcal{F}, P)\) with \(EX_j = 0, \ EX_j^2 < \infty\). For \(n \in \mathbb{N}\) put
\[
S_n = \sum_{j=1}^{n} X_j, \quad \sigma_n^2 = ES_n^2,
\]
and define
\[
W_n(t) = \sigma_n^{-1} S_{[nt]}, \quad t \in [0, 1],
\]
where \(S_0 = 0\). Then \(W_n\) is a measurable map from \((\Omega, \mathcal{F})\) into \((D, \mathcal{B}(D))\), where \(D\) is the set of all functions on \([0, 1]\) which have left hand limits and are continuous from the right, and \(\mathcal{B}(D)\) is the Borel-\(\sigma\)-algebra induced by the Skorohod topology. \(\{X_j: j \in \mathbb{N}\}\) fulfills the invariance principle if \(W_n\) converges weakly to standard Brownian motion \(W\) on \(D\).

In this paper we investigate the invariance principle for sequences satisfying a condition of positive dependence called association. A finite collection \(\{X_1, \ldots, X_m\}\) of random variables is associated if for any two coordinatewise nondecreasing functions \(f_1, f_2\) on \(\mathbb{R}^m\) such that \(\hat{f}_i = f_i(X_1, \ldots, X_m)\) has finite variance for \(i = 1, 2\), there holds \(\text{Cov} (\hat{f}_1, \hat{f}_2) \geq 0\). An infinite collection is associated if every finite subcollection is associated (cf. Esary, Proschan and Walkup [3]). Many recent papers have been concerned with limit theorems for associated processes (see for example...
Newman [6]). Note that a sequence of associated random variables is not necessarily asymptotically independent. For example the sequence \( \{X_j = X_i : j \in \mathbb{N}\} \) is associated by \((P_3)\) and \((P_4)\) of Esary, Proschan and Walkup [3], but by its dependence structure it is not possible to obtain useful limit theorems. Hence, beside the property of being associated, we need additional conditions which ensure asymptotic independence. It has been proved that the independence structure of an associated process is highly determined by its covariance structure (cf. Newman [6]).

Newman and Wright [7] obtained an invariance principle for stationary sequences of associated random variables satisfying a simple and natural summability criterion on their covariances. Instead of \( W_n \) they considered the \( C[0,1] \)-valued random element
\[
\tilde{W}_n(t) = \sigma_n^{-1}(S_{[nt]} + (nt-[nt])X_{[nt]+1}), \quad t \in [0,1].
\]
But this difference presents no difficulties since, by Remark 2.11 of Herrndorf [4], \( W_n \) converges weakly to \( W \) if and only if \( \tilde{W}_n \) converges weakly to \( W \). Hence there holds:

**Theorem A** (Newman, Wright). Let \( \{X_j : j \in \mathbb{N}\} \) be a strictly stationary sequence of associated random variables with \( EX_j = 0, EX_j^2 < \infty \). Assume
\[
0 < \sigma^2 = \text{Cov}(X_1, X_1) + 2 \sum_{j=2}^{\infty} \text{Cov}(X_1, X_j) < \infty. \tag{1.1}
\]
Then \( \{X_j : j \in \mathbb{N}\} \) fulfills the invariance principle.

Cox and Grimmett [2] weakened the assumption of strict stationarity and replaced it by certain conditions on the moments of the random variables. Using the coefficient
\[
u(n) = \sup_{k \in \mathbb{N}} \sum_{j \in \mathbb{N}, |j-k| \leq n} \text{Cov}(X_j, X_k), \quad n \in \mathbb{N} \cup \{0\},
\]
they obtained the following central limit theorem:

**Theorem B** (Cox, Grimmett). Let \( \{X_j : j \in \mathbb{N}\} \) be a sequence of associated random variables with \( EX_j = 0, EX_j^2 < \infty \). Assume
\[
u(n) \to 0, \quad \nu(0) < \infty, \tag{1.2}
\]
\[
\inf_{j \in \mathbb{N}} \text{Var}(X_j) > 0, \tag{1.3}
\]
\[\sup_{j \in \mathbb{N}} E|X_j|^3 < \infty. \tag{1.4}\]

Then \( \{X_j : j \in \mathbb{N}\} \) satisfies the central limit theorem, that is \( \sigma_n^{-1}S_n \) is asymptotically normally distributed.
Note that for a wide sense stationary sequence of associated random variables condition (1.1) implies
\[ u(0) = \sigma^2, \quad u(n) = 2 \sum_{j=n+1}^{\infty} \text{Cov}(X_1, X_j), \quad n \in \mathbb{N}, \]
and hence (1.2) and (1.3) are automatically satisfied. Therefore in the stationary case Theorem B is the implicit central limit theorem of Theorem A except the superfluous third moment condition (1.4).

One aim of our paper is to extend Theorems A and B to an invariance principle for nonstationary associated processes. Theorem B still holds if condition (1.4) is replaced by the Lindeberg condition (cf. Theorem 3). If we add the condition
\[ \sigma_{nk}^2 \rightarrow k \quad \text{for } k \in \mathbb{N}, \]
then even the invariance principle holds (Corollary 1). This follows from a general invariance principle for associated processes (Theorems 1 and 2) which requires neither stationarity nor the finiteness of \( u(n) \). In place of that we assume a condition on the covariances of the process, namely
\[ \sigma_n^{-2} \text{Cov}(S_{nk}, S_{nl}) \rightarrow \min\{k, l\} \quad \text{for } k, l \in \mathbb{N}. \]
This condition is a weak form of stationarity and necessary for the invariance principle (see Remarks 1 and 2).

All results are stated in Section 2. The proofs of our theorems as well as some lemmas are given in Section 3.

2. Results

Theorem 1. Let \( \{X_j: j \in \mathbb{N}\} \) be a sequence of associated random variables with \( E X_j = 0, E X_j^2 < \infty \). Assume
\[ \sigma_n^{-2} E(S_{nk}S_{nl}) \rightarrow \min\{k, l\} \quad \text{for } k, l \in \mathbb{N}, \tag{2.1} \]
\[ \{\sigma_n^{-2}(S_{nk} - S_m)^2; m \in \mathbb{N} \cup \{0\}, n \in \mathbb{N}\} \text{ is uniformly integrable.} \tag{2.2} \]
Then \( \{X_j: j \in \mathbb{N}\} \) fulfills the invariance principle.

The following remark shows that condition (2.1) is necessary for the invariance principle whereas condition (2.2) cannot be deduced from the invariance principle.

Remark 1. (i) Let \( \{X_j: j \in \mathbb{N}\} \) be a sequence of associated random variables with \( EX_j = 0, EX_j^2 < \infty \). If \( \{X_j: j \in \mathbb{N}\} \) fulfills the invariance principle, then condition (2.1) holds.

(ii) There exists a sequence \( \{X_j: j \in \mathbb{N}\} \) of associated random variables with \( EX_j = 0, EX_j^2 < \infty \), which fulfills the invariance principle, but does not satisfy condition (2.2).
Proof. (i) Assume that the invariance principle is fulfilled. By Remark 2.3 of Herrndorf [4] it follows that \( \sigma_n^2 = nh(n) \), where \( h : \mathbb{R}_+ \to \mathbb{R}_+ \) is slowly varying. Hence,  
\[
\sigma_n^{-2} \sigma_{nk}^2 \to k \quad \text{for } k \in \mathbb{N}.
\]

By Lemma 2 (see Section 3) it remains to prove  
\[
\sigma_n^{-2} E((S_{[nt]} - S_{[ns]})(S_{[nv]} - S_{[nu]})) \to 0 \quad \text{for } 0 \leq s \leq t \leq u \leq v \leq 1.
\]

To (1): Let \( 0 \leq s \leq t \leq u \leq v \leq 1 \) be given. Since the invariance principle is fulfilled, \( \{\sigma_n^{-2} S_n^2 : n \in \mathbb{N}\} \) is uniformly integrable. Hence  
\[
\{\sigma_n^{-2} (S_{[nt]} - S_{[ns]})(S_{[nv]} - S_{[nu]}): n \in \mathbb{N}\}
\]

is uniformly integrable, according to Lemma 1. As  
\[
(W_n(t) - W_n(s), W_n(v) - W_n(u)) \to (W(t) - W(s), W(v) - W(u))
\]
in distribution, it follows that  
\[
\sigma_n^{-2}(S_{[nt]} - S_{[ns]})(S_{[nv]} - S_{[nu]}) \to (W(t) - W(s))(W(v) - W(u))
\]
in distribution. According to Theorem 5.4 of Billingsley [1],  
\[
\sigma_n^{-2} E((S_{[nt]} - S_{[ns]})(S_{[nv]} - S_{[nu]}))
\]
converges to  
\[
E((W(t) - W(s))(W(v) - W(u))).
\]

But  
\[
E((W(t) - W(s))(W(v) - W(u))) = E(W(t) - W(s))E(W(v) - W(u)) = 0,
\]
which proves (1).

(ii) Let \( \{X_j : j \in \mathbb{N}\} \) be a sequence of independent random variables, such that the distribution of \( X_j \) is  
\[
N(0, (j + 1) \log(j + 1) - j \log(j)).
\]
According to Theorem 2.1 of Esary, Proschan and Walkup [3], the process \( \{X_j : j \in \mathbb{N}\} \) is associated. Since the random variables are independent and normally distributed, the distribution of \( S_n \) is \( N(0, (n + 1) \log(n + 1)) \), and hence \( \{X_j : j \in \mathbb{N}\} \) fulfills the central limit theorem. Moreover, for \( k \leq l \in \mathbb{N} \),  
\[
\sigma_n^{-2} E(S_{nk}S_{nl}) = \sigma_n^{-2} \sigma_{nk}^2 \to k.
\]
According to Theorem 2, \( \{X_j : j \in \mathbb{N}\} \) fulfills the invariance principle. But for \( m_0 \in \mathbb{N} \)
there holds
\[
\sup_{m \in \mathbb{N} \cup \{0\}, n \in \mathbb{N}} \sigma_n^{-2} E(S_{n+m} - S_m)^2 \geq (2 \log(2))^{-1} E X_{m_0+1}^2
\]
\[
\geq (2 \log(2))^{-1} \log(m_0 + 2),
\]
and thus
\[
\sup_{m \in \mathbb{N} \cup \{0\}, n \in \mathbb{N}} \sigma_n^{-2} E(S_{n+m} - S_m)^2 = 0.
\]
Hence \( \{X_j : j \in \mathbb{N}\} \) does not satisfy condition (2.2).

Theorem 2 shows that the invariance principle still holds if condition (2.2) is replaced by the validity of the central limit theorem. Hence we obtain necessary and sufficient conditions for the invariance principle.

**Theorem 2.** Let \( \{X_j : j \in \mathbb{N}\} \) be a sequence of associated random variables with \( EX_j = 0 \), \( EX_j^2 < \infty \). Then the following assertions are equivalent:

(i) Condition (2.1) is fulfilled and

\( \{X_j : j \in \mathbb{N}\} \) satisfies the central limit theorem, \hfill (2.3)

(ii) \( \{X_j : j \in \mathbb{N}\} \) fulfills the invariance principle.

Remark 2 shows that condition (2.1) is a weak form of stationarity:

**Remark 2.** The following two conditions together are equivalent to condition (2.1):

\[
\sigma_n^{-2} \sigma_{nk}^2 \to k \quad \text{for} \quad k \in \mathbb{N}, \quad (2.4)
\]
\[
\sigma_{nl-nk} E(S_{nl} - S_{nk})^2 \to 1 \quad \text{for} \quad k < l \in \mathbb{N}. \quad (2.5)
\]

**Proof.** Putting \( k = l \), condition (2.4) follows from (2.1). Hence we may always assume that (2.4) is fulfilled. Then the assertion follows easily from the identity (for \( k < l \))

\[
\sigma_{nl-nk} E(S_{nl} - S_{nk})^2 = \sigma_{n(l-k)}^2 \sigma_{nl}^2 + \sigma_{n(l-k)}^{-2} \sigma_{nk}^2 - 2 \sigma_{n(l-k)}^{-2} E(S_{nk} S_{nl}).
\]

The following example shows that in our Theorems 1 and 2 condition (2.1) is not superfluous, even if the condition

\[
n^{-1} \sigma_n^2 \to \sigma^2 \in (0, \infty) \quad (2.6)
\]
is fulfilled (and hence (2.4) holds):

**Example.** Let \( X \) be a random variable whose distribution is \( N(0, 1) \). For \( j \in \mathbb{N} \) put

\[
a_j = \begin{cases} 1, & \sqrt{j} \in \mathbb{N}, \\ 0, & \sqrt{j} \notin \mathbb{N}. \end{cases}
\]
According to $(P_3)$ and $(P_4)$ of Esary, Proschan and Walkup [3], $\{X_j : j \in \mathbb{N}\}$ is a sequence of associated random variables with $E X_j = 0$, $EX_j^2 < \infty$. As $S_n = (\sum_{j=1}^{n} a_j)X$, we obtain

$$\sigma_n^2 = \left(\sum_{j=1}^{n} a_j\right)^2 = [\sqrt{n}]^2.$$ 

This shows that (2.6) is valid. It is easy to see that

$$\sigma_n^{-2}(S_{n+m} - S_m)^2 \leq CX^2$$

holds for all $m \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$. Hence condition (2.2) is fulfilled. Finally the sequence $\{X_j : j \in \mathbb{N}\}$ satisfies the central limit theorem, since $\sigma_n^{-1}S_n = X$ is normally distributed. But for $k, l \in \mathbb{N}$ there holds

$$\sigma_n^{-2}E(S_{nk}S_{nl}) \to \sqrt{k}\sqrt{l},$$

i.e. condition (2.1) is not valid. By Remark 1 $\{X_j : j \in \mathbb{N}\}$ cannot fulfill the invariance principle.

Theorem 2 shows that the central limit theorem is an important tool in establishing the invariance principle for associated processes. Hence it is desirable to look for conditions which imply the central limit theorem.

**Theorem 3.** Let $\{X_j : j \in \mathbb{N}\}$ be a sequence of associated random variables with $EX_j = 0$, $EX_j^2 < \infty$. Assume

$$u(n) \to 0, u(1) < \infty,$$  

$$\inf_{n \in \mathbb{N}} n^{-1}\sigma_n^2 > 0,$$  

$$\sigma_n^{-2}\sum_{j=1}^{n} E(X_j^21_{\{|X_j| \geq \varepsilon \sigma_n\}}) \to 0 \quad \text{for } \varepsilon > 0.$$  

Then $\{X_j : j \in \mathbb{N}\}$ satisfies the central limit theorem.

Note that Theorem 3 is an extension of Theorem B: (2.7) follows from (1.2). Since the random variables are nonnegatively correlated, (1.3) implies

$$n^{-1}\sigma_n^2 \geq n^{-1}\sum_{j=1}^{n} \operatorname{Var}(X_j)$$

$$\geq \inf_{j \in \mathbb{N}} \operatorname{Var}(X_j) > 0,$$

i.e. (2.8) is fulfilled. Obviously (2.8) and (1.4) imply condition (2.9).

From our theorems we get the following corollaries:

**Corollary 1.** Let $\{X_j : j \in \mathbb{N}\}$ be a sequence of associated random variables with $EX_j = 0$, $EX_j^2 < \infty$. If (2.4), (2.7), (2.8), and (2.9) are fulfilled, then $\{X_j : j \in \mathbb{N}\}$ satisfies the invariance principle.
Proof. According to Theorem 3, \( \{X_j: j \in \mathbb{N}\} \) satisfies the central limit theorem. Hence, by Theorem 2, it suffices to prove (2.1). Since (2.4) is fulfilled, (2.1) follows from
\[
\sigma_n^{-2} E((S_{nj} - S_{ni})(S_{nl} - S_{nk})) \to 0 \quad \text{for } i \leq j \leq k \leq l \in \mathbb{N} \cup \{0\},
\]
according to Lemma 2. But (1) is a simple consequence of the estimate
\[
0 \leq \sigma_n^{-2} E((S_{nj} - S_{ni})(S_{nl} - S_{nk})) \leq \sigma_n^{-2} \sum_{\nu=1}^{n(j-i)} u(\nu),
\]
and the assumptions (2.7) and (2.8).

Since (2.6) implies (2.7) and (2.8), we obtain:

Corollary 2. Let \( \{X_j: j \in \mathbb{N}\} \) be a sequence of associated random variables with \( \mathbb{E}X_j = 0 \), \( \mathbb{E}X_j^2 < \infty \). If (2.6), (2.7), and (2.9) are fulfilled, then \( \{X_j: j \in \mathbb{N}\} \) satisfies the invariance principle.

If \( \{X_j: j \in \mathbb{N}\} \) is stationary in the wide sense, condition (1.1) obviously implies (2.6) and (2.7). Hence we obtain:

Corollary 3. Let \( \{X_j: j \in \mathbb{N}\} \) be a wide sense stationary sequence of associated random variables with \( \mathbb{E}X_j = 0 \), \( \mathbb{E}X_j^2 < \infty \). If (1.1) and (2.9) are fulfilled, then \( \{X_j: j \in \mathbb{N}\} \) satisfies the invariance principle.

Corollary 3 immediately implies Theorem A.

3. Proof of the theorems

We need the following lemmas to derive conditions which are equivalent to (2.1) and (2.4):

Lemma 1. Let \( \{X_j: j \in \mathbb{N}\} \) be a sequence of associated random variables with \( \mathbb{E}X_j = 0 \), \( \mathbb{E}X_j^2 < \infty \). Then the following conditions are equivalent:

\begin{itemize}
  \item[(i)] \[ \sigma_n^{-2} \sigma_k \to k \quad \text{for } k \in \mathbb{N}, \]
  \item[(ii)] \[ \sigma_n^{-2} \sigma_t \to t \quad \text{for } t > 0. \]
\end{itemize}

Proof. It suffices to show (i) \( \Rightarrow \) (ii). First we consider the special case \( t = 1/p, p \in \mathbb{N} \). We prove (ii) for the subsequences
\[ \{mp + l: m \in \mathbb{N}\}, \quad 0 \leq l \leq p - 1, \]
that is,

\[(1) \quad \sigma_{mp+l}^2 \sigma_{[(mp+l)/p]}^2 = \sigma_m^2 \sigma_{m'}^2 \to 1/p.\]

To (1): Since the random variables are nonnegatively correlated, \(\sigma_n^2\) is nondecreasing. Hence, by (i),

\[(2) \quad \lim \sup_{m \in \mathbb{N}} \sigma_{mp+l}^2 \sigma_m^2 \leq \lim \sup_{m \in \mathbb{N}} \sigma_{mp}\sigma_m^2 = 1/p.\]

Let \(k \in \mathbb{N}, k \geq 2\), be fixed. It is easy to see that for \(m \geq (k-1)^2\) there holds

\[r(k-1) \leq m, \quad mp+l \leq rk,\]

where \(r = \lfloor m/(k-1) \rfloor\). Hence, by (i),

\[(3) \quad \lim \inf_{m \in \mathbb{N}} \sigma_{mp+l}^2 \sigma_m^2 \geq \lim \inf_{m \in \mathbb{N}} \sigma_{rk}\sigma_{r(k-1)}^2 = (1-1/k)1/p.\]

Since \(k \geq 2\) can be chosen arbitrarily, (2) and (3) imply (1). This proves (ii) for \(t = 1/p\). Now it is easy to obtain the general case of (ii): If \(t = p/q\) is rational, then (ii) follows directly from (i) and our special case. For arbitrary \(t > 0\) there exist rational \(u \uparrow t, v \downarrow t\). Then (ii) follows from the rational case, as \(\sigma_n^2\) is nondecreasing.

**Lemma 2.** Let \(\{X_j: j \in \mathbb{N}\}\) be a sequence of associated random variables with \(EX_j = 0, EX_j^2 < \infty\). Assume that (2.4) is fulfilled. Then the following conditions are equivalent:

(i) \(\sigma_n^{-2} E((S_j - S_i)(S_l - S_k)) \to 0 \quad \text{for } i \leq j \leq k \leq l \in \mathbb{N} \cup \{0\},\)

(ii) \(\sigma_n^{-2} E(S_{nk}S_{sl}) \to \min\{k, l\} \quad \text{for } k, l \in \mathbb{N},\)

(iii) \(\sigma_n^{-2} E(S_nS_{(nt)}) \to t \quad \text{for } t \in [0, 1],\)

(iv) \(\sigma_n^{-2} E((S_{[nt]} - S_{[ns]})(S_{[nu]} - S_{[nu]})) \to 0 \quad \text{for } 0 \leq s \leq t \leq u \leq v,\)

(v) \(\sigma_n^{-2} E((S_{[nt]} - S_{[ns]})(S_{[nu]} - S_{[nu]})) \to 0 \quad \text{for } 0 \leq s \leq t \leq u \leq v \leq 1.\)

**Proof.** According to Lemma 1, it holds that

\[(1) \quad \sigma_n^{-2} \sigma_{[nt]}^2 \to t \quad \text{for } t > 0.\]

(i) \(\Rightarrow\) (ii). (ii) follows from (i) and (1), putting \(i = 0\) and \(j = k\) (if \(k \leq l\)).

(ii) \(\Rightarrow\) (iii). Let \(t \in [0, 1]\) be given. Since the random variables are nonnegatively correlated, we have for \(n \geq n(t)\)

\[\sigma_{[nt]}^{-2} E(S_{[nt]}(1_{\lfloor 2/t \rfloor + 1})S_{[nt]} \geq \sigma_{[nt]}^{-2} E(S_nS_{[nt]} \geq 1.\)
Hence, by (ii),
\[ \sigma_{[nt]}^{-2} E(S_nS_{[nt]}) \to 1, \]
which together with (1) proves (iii).

(iii) \(\Rightarrow\) (iv). Let \(0 \leq s \leq t \leq u \leq v\) be given. Choose \(l \in \mathbb{N}\) with \(v \leq l\). Since the random variables are nonnegatively correlated, it follows that
\[ 0 \leq \sigma_{[nt]}^{-2} E((S_{[nt]} - S_{[ns]})(S_{[nv]} - S_{[nt]})) \leq \sigma_{[nt]}^{-2} E(S_{[nl]} - S_{[nlt]})). \]
Applying (iii) (with \(n_l\) instead of \(n\)) and (1), we get (iv).

(iv) \(\Rightarrow\) (v) is trivial.

(v) \(\Rightarrow\) (i). Let \(i \leq j \leq k \leq l \in \mathbb{N} \cup \{0\}\) be given. As
\[ \sigma_{[ni]}^{-2} E((S_{[ni]} - S_{[nl]})(S_{[nl]} - S_{[nk]})) = \sigma_{[ni]}^{-2} E((S_{[nij]} - S_{[nli]})(S_{[nl]} - S_{[nk]})), \]
(i) follows from (v) (with \(n_l\) instead of \(n\)) and (1).

**Proof of Theorem 1.** Condition (2.1) implies (2.4) and hence, by Lemma 1,
\[
(1) \quad \sigma_{[nt]}^{-2} \to t \quad \text{for } t > 0.
\]
According to Lemma 2, we have
\[
(2) \quad \sigma_{[nt]}^{-2} E((S_{[nt]} - S_{[ns]})(S_{[nv]} - S_{[nt]})) \to 0 \quad \text{for } 0 \leq s \leq t \leq u \leq v.
\]
We will apply Theorem 19.1 of Billingsley [1]. For this reason we prove for every \(\varepsilon > 0\)
\[
(3) \quad \limsup_{n \to \infty} P\{w(W_n, \delta) > \varepsilon\} \to 0 \quad \text{as } \delta \downarrow 0,
\]
where
\[ w(W_n, \delta) = \sup_{|s-t| < \delta} |W_n(s) - W_n(t)|. \]
First we show that (3) implies our assertion: Since \(W_n(0) = 0\), (3) and Theorem 15.5 of Billingsley [1] yield the tightness of the sequence \(\{W_n: n \in \mathbb{N}\}\). Let \(X\) be a limit in distribution of a subsequence of \(\{W_n: n \in \mathbb{N}\}\). Then \(P\{X \in C[0, 1]\} = 1\) by Theorem 15.5 of [1]. It suffices to show that \(X\) is distributed like \(W\). By (2.2) and (1) \(\{W^2_n(t): n \in \mathbb{N}\}\) and \(\{W_n(t): n \in \mathbb{N}\}\) are uniformly integrable for every \(t \in [0, 1]\). As
\[ W_n(t) \to X(t), \quad W^2_n(t) \to X^2(t) \]
in distribution (for a subsequence), Theorem 5.4 of Billingsley [1] and (1) imply
\[ EX(t) = 0, \quad EX^2(t) = t. \]
According to Theorem 19.1 of Billingsley [1], \( X \) is distributed like \( W \) if \( X \) has independent increments, that is

\[
X(t_1) - X(t_0), \ldots, X(t_k) - X(t_{k-1}) \text{ are independent for all } k \in \mathbb{N}, \quad 0 \leq t_0 \leq t_1 \leq \cdots \leq t_k \leq 1.
\]

To show (4), put

\[
U_n = W_n(t_i) - W_n(t_{i-1}), \quad 1 \leq i \leq k.
\]

Since

\[
(U_{n1}, \ldots, U_{nk}) \rightarrow_n (X(t_1) - X(t_0), \ldots, X(t_k) - X(t_{k-1}))
\]

in distribution (for a subsequence), and since the \( U_{ni} \) are associated by \((P_4)\) of Esary, Proschan and Walkup [3],

\[
X(t_1) - X(t_0), \ldots, X(t_k) - X(t_{k-1})
\]

are associated, according to \((P_5)\) of [3]. A similar argument as above (using Theorem 5.4 of Billingsley [1]) yields, for \( i \neq j \),

\[
\text{Cov}(X(t_i) - X(t_{i-1}), X(t_j) - X(t_{j-1})) = \lim_{n \to \infty} \text{Cov}(U_{ni}, U_{nj}) = 0,
\]

according to (2). Hence the \( X(t_i) - X(t_{i-1}) \) are associated and uncorrelated random variables and thus independent by Corollary 3 of Newman [6]. This proves (4).

**To (3):** Let \( \varepsilon > 0 \) be given. Using

\[
\{ x \in D[0, 1]: w(x, \delta) > \varepsilon \} \subset \bigcup_{i=1}^{r} \{ x \in D[0, 1]: \sup_{s \in [t_{i-1}, t_i]} |x(s) - x(t_{i-1})| > \varepsilon / 3 \}
\]

for

\[
0 = t_0 < t_1 < \cdots < t_r = 1 \quad \text{with} \quad t_i - t_{i-1} \geq \delta \quad (2 \leq i \leq r - 1),
\]

we obtain

\[
P\{ w(W_n, \delta) > \varepsilon \} \leq \sum_{i=0}^{[1/\delta]} P\left\{ \max_{[ni\delta] < r \leq [n(i+1)\delta]} |S_r - S_{[ni\delta]}| > \sigma_n \varepsilon / 3 \right\}.
\]

For fixed \( i \) Corollary 5 of Newman and Wright [8] yields

\[
(1 - (\sigma_n^2 \varepsilon^2 / 36)^{-1} E(S_{[n(i+1)\delta]} - S_{[ni\delta]})^2)
\]

\[
\cdot P\left\{ \max_{[ni\delta] < r \leq [n(i+1)\delta]} |S_r - S_{[ni\delta]}| \geq \sigma_n \varepsilon / 3 \right\}
\]

\[
\leq P\{ |S_{[n(i+1)\delta]} - S_{[ni\delta]}| \geq \sigma_n \varepsilon / 6 \}.
\]
Since the random variables are nonnegatively correlated, we have
\[ \sigma_n^{-2} E(S_{n(i+1)\delta} - S_{ni\delta})^2 \leq \sigma_n^{-2}(\sigma_{n(i+1)\delta}^2 - \sigma_{ni\delta}^2) \leq 2\delta \]
for \( n \geq n(i, \delta) \), according to (1). Hence by (5), (6) for \( \delta < \varepsilon^2/72 \), \( n \geq \max_{0 < i < [1/\delta]} n(i, \delta) \) it holds that

(7) \[ \Pr\{w(W_n, \delta) > \varepsilon\} \leq (1 - 72\delta/\varepsilon^2)^{-1} \sum_{i=0}^{[1/\delta]} \Pr\{|S_{n(i+1)\delta} - S_{ni\delta}| \geq \sigma_n \varepsilon/6\}. \]

Let \( \varepsilon_0 < \varepsilon/6 \), \( \delta < \varepsilon^2/72 \) and \( 0 < i < [1/\delta] \) be fixed. According to (1), there exists \( m(i, \delta, \varepsilon_0) \in \mathbb{N} \) such that for \( n \geq m(i, \delta, \varepsilon_0) \) we have
\[
\Pr\{|S_{n(i+1)\delta} - S_{ni\delta}| \geq \sigma_n \varepsilon/6\} \\
\leq \Pr\{|S_{n(i+1)\delta} - S_{ni\delta}| \geq \varepsilon_0 \delta^{-1/2}\}. \]

Then (7) yields
\[
\limsup_{n \to \infty} \Pr\{w(W_n, \delta) > \varepsilon\} \\
\leq \limsup_{n \to \infty} (1 - 72\delta/\varepsilon^2)^{-1} \sum_{i=0}^{[1/\delta]} \Pr\{|S_{n(i+1)\delta} - S_{ni\delta}| \geq \varepsilon_0 \delta^{-1/2}\} \cdot \sup_{m \in \mathbb{N} \cup \{0\}, n \in \mathbb{N}} E(\sigma_n^{-2}(S_{n+m} - S_{m})^2 1_{\{|S_{n+m} - S_{m}| \geq \sigma_n \varepsilon_0 \delta^{-1/2}\}}).
\]

Now assumption (2.2) implies (3), and the proof of Theorem 1 is complete.

**Proof of Theorem 2.** It suffices to show (i) \( \Rightarrow \) (ii). Like in the proof of Theorem 1 we obtain relations (1) and (2). From (2.3) and (1) it follows for \( t > 0 \)
(3) \[ \sigma_n^{-1} S_{[nt]} \to N(0, t) \text{ in distribution.} \]

We will prove, for \( 0 < s < t \),
(4) \[ \sigma_n^{-1}(S_{[nt]} - S_{[ns]}) \to N(0, t - s) \text{ in distribution.} \]

**To (4):** Let \( 0 < s < t \) be given. Then the sequence
\[ \{(\sigma_n^{-1} S_{[ns]}, \sigma_n^{-1} S_{[nt]}): n \in \mathbb{N}\} \]
is tight. Let \( Q \) be a probability measure on \( \mathbb{R}^2 \) such that for a subsequence
\[ (\sigma_n^{-1} S_{[ns]}, \sigma_n^{-1} S_{[nt]}) \to Q \text{ in distribution.} \]

Then we have
\[ (\sigma_n^{-1} S_{[ns]}, \sigma_n^{-1}(S_{[nt]} - S_{[ns]})) \to Q(\pi_1, \pi_2 - \pi_1)^{-1} \text{ in distribution}, \]
where \( \pi_i : \mathbb{R}^2 \to \mathbb{R} \), \( i = 1, 2 \), are the natural projections. Since the random variables \( \sigma_{n i}^{-1} S_{[n]} \) and \( \sigma_{n i}^{-1} (S_{[n]} - S_{[n-1]}) \) are associated by \( (P_4) \) of Esary, Proschan and Walkup [3], \( (P_3) \) of [3] implies that \( \pi_1 \) and \( \pi_2 - \pi_1 \) are associated with respect to \( Q \). According to (3), the sets

\[
\{ \sigma_{n i}^{-1} S_{[n]} : n \in \mathbb{N} \}, \quad \{ \sigma_{n i}^{-1} S_{[n]} : n \in \mathbb{N} \} \quad \text{and} \quad \{ \sigma_{n i}^{-2} S_{[n]} S_{[n-1]} : n \in \mathbb{N} \}
\]

are uniformly integrable. Hence, using Theorem 5.4 of Billingsley [1] and (2), we obtain

\[
\text{Cov}(\pi_1, \pi_2 - \pi_1) = \lim_{n \to \infty} \text{Cov}(\sigma_{n i}^{-1} S_{[n]}, \sigma_{n i}^{-1} (S_{[n]} - S_{[n-1]})) = 0.
\]

As associated and uncorrelated random variables, \( \pi_1 \) and \( \pi_2 - \pi_1 \) are \( Q \)-independent. Since \( Q \pi_1^{-1} = N(0, s), \ Q \pi_2^{-1} = N(0, t) \), this proves (4).

Using arguments of Herrndorf [4] (cf. the end of the proof of Theorem 2.2), it is now easy to complete the proof of Theorem 2 similarly to the proof of Theorem 1.

To prove Theorem 3, we shall use the following decomposition for \( S_n, n \in \mathbb{N} \): Let \( k = k(n) \in \mathbb{N}, \ p = p(n) \in \mathbb{N} \) with \( kp \leq n \). Let \( S_n = \sum_{i=0}^{k} \xi_i \), where

\[
\xi_i = \xi_{i,n} = \sum_{j=p+1}^{(i+1)p} X_j, \quad 0 \leq i < k,
\]

\[
\xi_k = \xi_{k,n} = \sum_{j=kp+1}^{n} X_j.
\]

**Lemma 3.** Let \( \{X_j: j \in \mathbb{N}\} \) be a sequence of random variables with \( E X_j = 0, \ EX_j^2 < \infty \). Assume

\[
k \to \infty,
\]

\[
\sigma_n^{-2} \sum_{i=0}^{k} \text{Var}(\xi_i) \to 1,
\]

and

\[
\left| E \exp(it\sigma_n^{-1} S_n) - \prod_{i=0}^{k} E \exp(it\sigma_n^{-1} \xi_i) \right| \to 0 \quad \text{for } t \in \mathbb{R}.
\]

If

\[
\sigma_n^{-2} \sum_{i=0}^{k} E(\xi_i^2 1_{\{|\xi_i| > \varepsilon \sigma_n\}}) \to 0
\]

for \( \varepsilon > 0 \), then \( \{X_j: j \in \mathbb{N}\} \) satisfies the central limit theorem.
Proof. The proof is similar to the proof of Lemma 3.1 of Withers [9] and will be omitted.

Special $k$ and $p$ are required to prove Theorem 3:

**Lemma 4.** Let $\{X_j : j \in \mathbb{N}\}$ be a sequence of random variables with $EX_j = 0$, $EX_j^2 < \infty$. Assume that the Lindeberg condition (2.9) is fulfilled. Then there exist $p = p(n) \in \mathbb{N}$, $n \in \mathbb{N}$, with $p \leq n$, $p \to \infty$, and $p/n \to 0$ such that for $\varepsilon > 0$ there holds

$$p^2 \sigma_n^{-2} \sum_{j=1}^n E(X_j^2 1_{\{|X_j| > \varepsilon \sigma_n/p\}}) \to 0.$$

**Proof.** Put $n_1 = 1$. According to (2.9), for every $k \in \mathbb{N}$, $k \geq 2$, there exists $n_k \in \mathbb{N}$ such that $2n_k \leq n_{k+1}$ and

$$\sigma_n^{-2} \sum_{j=1}^n E(X_j^2 1_{\{|X_j| > \varepsilon \sigma_n/k^2\}}) \leq 1/k^3$$

for $n \geq n_k$. Define $p(n) = k$ if $n_k \leq n < n_{k+1}$. Then it is easy to see that $p$ fulfills the assertion.

**Proof of Theorem 3.** Let $p = p(n)$, $n \in \mathbb{N}$, be as in Lemma 4 and put $k = k(n) = \lfloor n/p(n) \rfloor$. We will apply Lemma 3. According to Lemma 4, there holds $k \to \infty$. By Newman's inequality (cf. Theorem 1 of Newman [5]) we have for $t \in \mathbb{R}$

$$\left| E \exp(it \sigma_n^{-1} S_n) - \prod_{l=0}^{k-1} E \exp(it \sigma_n^{-1} \xi_l) \right| \leq (t^2/2)(1 - \sigma_n^{-2} \tau_n^2),$$

where $\tau_n^2 = \sum_{l=0}^{k} \text{Var}(\xi_l)$. Hence, by Lemma 3, it suffices to prove

1. $\sigma_n^{-2} \tau_n^2 \to 1$,

2. $\sigma_n^{-2} \sum_{j=0}^k E(\xi_j^2 1_{\{|\xi_j| > \varepsilon \sigma_n\}}) \to 0$ for $\varepsilon > 0$.

To (1): Since the random variables are nonnegatively correlated, it follows

$$0 \leq 1 - \sigma_n^{-2} \tau_n^2$$

$$= 2\sigma_n^{-2} \sum_{l=0}^{k-1} \text{Cov} \left( \xi_l, \sum_{m=l+1}^{k} \xi_m \right)$$

$$\leq 2k \sigma_n^{-2} \sum_{j=1}^{p} u(j)$$

$$\leq 2n \sigma_n^{-2} p^{-1} \sum_{j=1}^{p} u(j).$$

Using $p \to \infty$ and our assumptions (2.7) and (2.8), we obtain (1).
To (2): Let $\epsilon > 0$ be given. For fixed $0 \leq l < k$ there holds
\[ \xi_l^2 \leq p \sum_{j=lp+1}^{(l+1)p} X_j^2, \]
and hence
\[ E(\xi_l^2 1_{|\xi_l| > \epsilon \sigma_n}) \leq p E \left( \sum_{j=lp+1}^{(l+1)p} X_j^2 1_{\{X_j^2 > \epsilon^2 \sigma_n^2/p\}} \right). \]
For $\omega \in \Omega$ let $j_0 = j_0(\omega) \in \{lp+1, \ldots, (l+1)p\}$ be such that
\[ |X_{j_0}(\omega)| = \max_{lp+1 \leq j \leq (l+1)p} |X_j(\omega)|. \]
Then
\[ \sum_{j=lp+1}^{(l+1)p} X_j^2(\omega) 1_{\{X_j^2 > \epsilon^2 \sigma_n^2/p\}}(\omega) \leq p X_{j_0}^2(\omega) 1_{\{X_{j_0}^2 > \epsilon^2 \sigma_n^2/p\}}(\omega) \]
\[ \leq p \sum_{j=lp+1}^{(l+1)p} X_j^2(\omega) 1_{\{X_j^2 > \epsilon^2 \sigma_n^2/p\}}(\omega). \]
This proves
\[ E(\xi_l^2 1_{|\xi_l| > \epsilon \sigma_n}) \leq p^2 \sum_{j=lp+1}^{(l+1)p} E(X_j^2 1_{|X_j| > \epsilon \sigma_n/p}). \]
In the same way, using $n-kp \leq p$, we get
\[ E(\xi_k^2 1_{|\xi_k| > \epsilon \sigma_n}) \leq p^2 \sum_{j=lp+1}^{n} E(X_j^2 1_{|X_j| > \epsilon \sigma_n/p}). \]
(3) and (4) imply
\[ \sigma_n^{-2} \sum_{l=0}^{k} E(\xi_l^2 1_{|\xi_l| > \epsilon \sigma_n}) \leq p^2 \sigma_n^{-2} \sum_{j=1}^{n} E(X_j^2 1_{|X_j| > \epsilon \sigma_n/p}). \]
According to Lemma 4, this proves (2) and completes the proof of Theorem 3.

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References