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An improved estimate of black hole entropy in the quantum geometry approach

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Abstract

A proper counting of states for black holes in the quantum geometry approach shows that the dominant configuration for spins are distributions that include spins exceeding one-half at the punctures. This raises the value of the Immirzi parameter and the black hole entropy. However, the coefficient of the logarithmic correction remains $-1/2$ as before.

The quantum geometry approach to a quantum theory of gravity is reasonably well established now: see [1] for reviews. In [2] a general framework for the calculation of black hole entropy in this approach was proposed. A lower bound for the entropy was worked out on the basis of the association of spin one-half to each *puncture* and found to be proportional to the area of the horizon. The proportionality constant involves what is known as the Immirzi parameter, which can be chosen so that the entropy becomes a quarter of the area.

Recently, this lower bound was sharpened in [3] to include a logarithmic correction $-\frac{1}{2}\ln A$. Subsequently, it was found [4] that the dominant term in the entropy is somewhat higher by taking spins higher than one-half into account, though the logarithmic correction is unaffected in this calculation. In the present Letter we investigate the modification of the lower bound of [3] in view of this development and are led to a further increase in the leading term.

Let a generic configuration have s_j punctures with spin j, $j = 1/2, 1, 3/2, 2, \ldots$. Note that

$$2\sum_j s_j\sqrt{j(j+1)} = A, \tag{1}$$

where A is the horizon area in units where $4\pi\gamma\ell_P^2 = 1$, γ being the Immirzi parameter and ℓ_P the Planck length. Following [2], we shall treat the punctures as distinguishable, see also [5]. We shall count states of the physical Hilbert space considering both j and its projection m as quantum numbers. The difference be-

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tween this procedure and the calculation carried out in [4] will be commented on later.

If we ignore the *zero spin projection constraint* ($\sum m = 0$, the sum extending over all punctures) initially, the total number of states is given by

$$N = \frac{(\sum_j s_j)!}{\prod_j s_j!} \prod_j (2j+1)^{s_j}, \tag{2}$$

where one has to sum over all non-negative s_j consistent with the given value of A. We will estimate the sum by maximizing the above expression with respect to the variables s_j subject to a fixed value of A.

Using Stirling's formula, we see that

$$\ln N = \sum_j s_j \left[\ln(2j+1) - \ln s_j\right] + \left(\sum_j s_j\right) \ln\left(\sum_j s_j\right). \tag{3}$$

Hence,

$$\delta \ln N = \sum_j \delta s_j \left[\ln(2j+1) - \ln s_j + \ln \sum_k s_k\right], \tag{4}$$

so that with some Lagrange multiplier λ to implement the area constraint, we can set

$$\ln(2j+1) - \ln s_j + \ln \sum_k s_k - \lambda\sqrt{j(j+1)} = 0. \tag{5}$$

Thus,

$$s_j = (2j+1) \exp\left[-\lambda\sqrt{j(j+1)}\right] \sum_k s_k. \tag{6}$$

Summing over j, we obtain the relation

$$\sum_j (2j+1) \exp\left[-\lambda\sqrt{j(j+1)}\right] = 1, \tag{7}$$

which determines $\lambda \simeq 1.72$. It may be mentioned that (7) was noted as a mathematical possibility in [4], and was derived with a somewhat different motivation in [6].

Substituting the expression for s_j one easily gets the entropy to be

$$S = \ln N = \lambda A/2. \tag{8}$$

This means that the Immirzi parameter has to be set at $\lambda/(2\pi) \simeq 0.274$. Note that the summation over s_j may raise this value while the imposition of the zero projection constraint is expected to lower it slightly.

The higher spins clearly raise the leading term, as in [4], but our expression is even larger than that of [4]. The difference arises from the fact that we have allowed all values $m = -j, \ldots, j$ for all j, whereas [4] did not distinguish states with the same values of m but different j. It is interesting to notice that their equation

$$\sum_j 2 \exp\left[-\tilde{\gamma}\sqrt{j(j+1)}\right] = 1, \tag{9}$$

which they got instead of (7), would have been obtained by us if we had restricted $m = \pm j$ for each j. This shows that although they wanted to count states characterized by only the quantum numbers m and satisfying $2\sum\sqrt{|m|(|m|+1)} \leqslant A$, allowing for $|m| \leqslant j$, their result is the same as though they were interested only in states with $|m| = j$ and area *equal* to A. States with lower values of $|m|$ appear to be negligibly fewer in comparison.

Note further that if one allows m to have all its $2j+1$ values for each j, their first recursion relation (with the zero projection constraint ignored) would get altered to

$$N(A) = \sum_j (2j+1) N\left(A - 2\sqrt{j(j+1)}\right) + \sqrt{A^2+1}, \tag{10}$$

which is satisfied by our estimate $N(A) = \exp(\lambda A/2)$ with λ satisfying (7) above. Our expression for the entropy thus agrees with the solution obtained from the *modified* recursion relation when the zero projection constraint is ignored.

We shall now impose the constraint of zero angular momentum projection. The number of configurations will be reduced somewhat, and a correction is expected to emerge. Let $s_{j,m}$ punctures carry spin j and projection m, i.e., $s_j = \sum_m s_{j,m}$. Since at each puncture (j,m) assigns a unique state the total number of states N equals the number of ways s_j and $s_{j,m}$ can be distributed among themselves,

$$N = \frac{(\sum_j s_j)!}{\prod_j s_j!} \prod_j \frac{s_j!}{\prod_m s_{j,m}!} = \frac{(\sum_{j,m} s_{j,m})!}{\prod_{j,m} s_{j,m}!}, \tag{11}$$

subject to the constraints $\sum_{j,m} m s_{j,m} = 0$ and (1). A lower bound is obtained by replacing $s_{j,m}$ for each

m by $s_j/(2j+1)$ for the corresponding j. This maximizes the number of combinations $s_j!/\prod_m s_{j,m}!$ for each j and also ensures zero total spin projection for each j, hence for the sum. In Stirling's approximation, the main departure from $(2j+1)^{s_j}$ occurs as the denominator contains a factor $[s_j/(2j+1)]^{j+1/2}$, leading to a correction $-(j+1/2)\ln[s_j/(2j+1)]$ (cf. [3]) in $\ln N$. As $s_j/(2j+1) \propto A\exp[-\lambda\sqrt{j(j+1)}\,]$, this correction can be expressed as

$$-\sum_j \bigl[\ln A - \lambda\sqrt{j(j+1)}\,\bigr](j+1/2), \tag{12}$$

which appears to be divergent. This happens because all s_j have been assumed to be large, although for large j, s_j in the expression given above goes to zero. So we restrict the sum to j for which s_j is greater than unity. Taking the largest j to be $n/2$, we see that

$$\exp\bigl[-\lambda\sqrt{n(n+2)/4}\,\bigr]A \simeq 1, \tag{13}$$

so that

$$n \simeq 2\ln A/\lambda. \tag{14}$$

Now

$$\sum_j j = n(n+1)/4 \simeq (\ln A)^2/\lambda^2. \tag{15}$$

Therefore the $\ln A$ piece yields a $(\ln A)^3$ correction. The piece $-\lambda\sqrt{j(j+1)}$ also has to be taken into account, using the sum

$$\sum_j j^2 \simeq n^3/12 \simeq 2(\ln A)^3/\bigl(3\lambda^3\bigr). \tag{16}$$

The total correction comes to $-(\ln A)^3/(3\lambda^2)$: the total entropy is bounded by the contribution of these configurations:

$$S \geqslant \lambda A/2 - (\ln A)^3/\bigl(3\lambda^2\bigr). \tag{17}$$

This is our new lower bound.

It must be noted that this bound has been derived by assuming a specific distribution of spins and spin projections to give the largest number of combinations. Summing over different s_j is expected to increase the number of configurations. Note that there also are additional non-leading terms in the expressions used above which have been neglected, but these are much smaller in magnitude than $(\ln A)^3$ and yield $(\ln A)^2$ and $\ln A$ pieces.

Let us now estimate the entropy, which as mentioned above is expected to be higher than the above bound because of summation over different configurations. In view of the zero spin projection constraint, the number of configurations may be written by explicitly summing over $s_{j,m}$ for each j as (see [3])

$$N_{\text{corr}} = \frac{(\sum_j s_j)!}{\prod_j s_j!} \int_{-2\pi}^{2\pi} \frac{d\omega}{4\pi} \prod_j \Bigl[\sum_{m_j} \exp(i m_j \omega)\Bigr]^{s_j}. \tag{18}$$

This can be rewritten as

$$N_{\text{corr}} = \int_{-2\pi}^{2\pi} \frac{d\omega}{4\pi} N(\omega), \tag{19}$$

where

$$N(\omega) = \frac{(\sum_j s_j)!}{\prod_j s_j!} \prod_j \Bigl[\sum_{m_j} \exp(i m_j \omega)\Bigr]^{s_j}. \tag{20}$$

To maximize N_{corr}, we regard s_j as functions $s_j(\omega)$ subject to the area constraint and maximize $N(\omega)$. The result is a simple modification of the one obtained above,

$$N(\omega) = \exp\bigl(\lambda(\omega)A/2\bigr), \tag{21}$$

where $\lambda(\omega)$ satisfies

$$1 = \sum_j \exp\bigl[-\lambda(\omega)\sqrt{j(j+1)}\,\bigr] \sum_{m=-j}^{j} \exp(i\omega m). \tag{22}$$

This equation differs from that of [4] in m going over $-j, \ldots, j$, whereas their m goes over $\pm j$ as before. The modified recursion relation for $N(A,p)$, which is the number of configurations satisfying the area constraint (1) and the relation $\sum m = p$, is

$$N(A,p) = \sum_j \sum_{m=-j}^{j} N\bigl(A - 2\sqrt{j(j+1)}, p-m\bigr) + \theta\bigl(A - 2\sqrt{|p|(|p|+1)}\bigr), \tag{23}$$

and gives rise to the above equation for $\lambda(\omega)$.

For $\omega = 0$, (22) resembles (7), so $\lambda(0) = \lambda$. This yields the dominant contribution $\exp(\lambda A/2)$ seen above. For small ω, $\lambda(\omega)$ falls quadratically, and the ω integral becomes a Gaussian, which is readily seen to be proportional to $A^{-1/2}$ by appropriate scaling. Thus,

$$S_{\text{corr}} = \ln N_{\text{corr}} \sim \ln\left[\frac{\exp(\lambda A/2)}{A^{1/2}}\right] = \lambda A/2 - \frac{1}{2}\ln A. \qquad (24)$$

This is exactly as in [3,4], indicating that the $(\ln A)^3$, $(\ln A)^2$ terms do not survive when summed over configurations.

One can see this directly by approximating the sums over configurations (i.e., sums over $s_{j,m}$) by integrals: variation of N in (11) with $s_{j,m}$ leads to a factor $\exp[-(\delta s_{j,m})^2/2s_{j,m}]$. Denominator factors of $(2\pi s_{j,m})^{1/2}$ coming from Stirling's approximation are cancelled by similar factors in the numerator coming from this Gaussian integration:

$$\int_{-\infty}^{\infty} d(\delta s_{j,m}) \exp\left[-\frac{(\delta s_{j,m})^2}{2s_{j,m}}\right] = (2\pi s_{j,m})^{1/2}. \qquad (25)$$

Each $s_{j,m}$ is proportional to A. The area constraint and the spin projection constraint, which may be thought of as reducing the number of summations, reduce the number of factors of $\sqrt{A}$ by two. But the numerator too has such a factor through $(\sum s)^{1/2}$. An overall factor $1/\sqrt{A}$ is thus left, as above, leading to the logarithmic correction with a coefficient $-1/2$.

In conclusion, we have estimated the entropy of a black hole in the quantum geometry approach by allowing spins of all non-zero values at different punctures and regarding both j and m as relevant quantum numbers. It was noted in [2] that the entropy in the leading order is the same whether one considers j as relevant or not, with spin one-half assumed to yield the counting. However, the dominant configuration, with the largest contribution to the number of states, contains spins higher than one-half, so that the assumption made in [2] has to be relaxed. The counting done in [4] treated j as irrelevant and our result is different from theirs in the leading order, although, somewhat surprisingly, the coefficient of the logarithmic term remains the same. The reason why j has at times been disregarded in the state counting is that this quantum number appears only in the volume Hilbert space and *not* in the surface Hilbert space [2], while it is the surface Hilbert space which is considered to be the space of quantum states of the isolated horizon. However, although the area of a classical isolated horizon is defined intrinsically on the surface, the area operator of a quantum isolated horizon is defined only to act on the volume Hilbert space. In fact, the area of the horizon is determined by the 'volume' quantum numbers j. So, in our view, j cannot be regarded as a hidden quantum number in characterizing the states of a quantum isolated horizon. In the end, then, the reason for the difference in our results from [4] is due to this difference in the definition of black hole states. Which definition is more appropriate may be fixed either by making an independent estimate of the Immirzi parameter or by performing some other semiclassical calculations from quantum isolated horizons.

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