Renaissance notions of number and magnitude

Antoni Malet

Universitat Pompeu Fabra, Departament d'Humanitats, c. Ramon Trias Fargas 25, 08005 Barcelona, Spain

Available online 3 March 2005

Abstract

In the 16th and 17th centuries the classical Greek notions of (discrete) number and (continuous) magnitude (preserved in medieval Latin translations of Euclid’s *Elements*) underwent a major transformation that turned them into continuous but measurable magnitudes. This article studies the changes introduced in the classical notions of number and magnitude by three influential Renaissance editions of Euclid’s *Elements*. Besides providing evidence of earlier discussions preparing notions and arguments eventually introduced in Simon Stevin’s *Arithmétique* of 1585, these editions document the role abacus algebra and Renaissance views on the history of mathematics played in bridging the gulf between discrete numbers and continuous magnitudes.

© 2005 Elsevier Inc. All rights reserved.

Résumé

Pendant le seizième et dix-septième siècles, les notions classiques de nombre (toujours sous-entendu discrète) et de grandeur (continue), bien conservées dans les éditions latines médiévales des *Éléments* d’Euclide, ont connu une transformation majeure au bout de laquelle on trouve les deux notions confondues sous la nouvelle notion de grandeur mesurable ou quantifiée. Cet article étudie les modifications introduites dans les notions classiques de nombre et magnitude par trois éditions des *Éléments* d’Euclide parues pendant le seizième siècle et largement utilisées. Ces éditions nous montrent comment elles ont amorcé de notions et d’arguments finalement introduits dans l’*Arithmétique* de Simon Stevin de 1585. Elles nous enseignent aussi le rôle joué par l’algèbre des abacistes

© 2005 Elsevier Inc. All rights reserved.

1. Introduction

In spite of the centrality of the notions of number and magnitude, we know little about how they evolved from their classical Greek forms, rather faithfully preserved in medieval translations of Euclid’s *Elements*, into the modern notion of real number. We do know that numbers and magnitudes underwent a major transformation during the 16th and the 17th centuries [Bos, 2001, 135–143; Neal, 2002]. Besides, it has always been assumed that this transformation was closely connected to the growing importance of algebraic methods in 16th- and 17th-century mathematics [Giusti, 1990]. In the present article we aim to provide evidence about the ways in which an informal numerical understanding of continuous magnitude originating in abacus books crept into authoritative 16th-century versions of Euclid’s *Elements*. We shall first introduce the basic notions our article focuses on before briefly examining critically Jacob Klein’s well-known account of Viète’s influence on the changing notion of number and magnitude in the 16th century. Then the article turns to the notions of number and magnitude as they appear in three influential Renaissance editions of Euclid’s *Elements*. In our account these changes prepared the way to notions and arguments eventually introduced in Simon Stevin’s 1585 *L’Arithmétique*. As we shall see, finally, there is strong evidence suggesting that humanistic views on the history of mathematics also played a role in bridging the gulf between discrete numbers and continuous magnitudes.

2. Classical notions of number and magnitude

As is well known, Euclid’s *Elements* contains no equivalent to the notion of the “length” of a segment, nor are there measures for geometrical objects in general. The rectangle “built” on two segments of straight lines at right angles plays a role in an indirect and clumsy way related to the modern notion of product, but Euclidean magnitudes cannot be multiplied or divided. Something similar is to be said about powers and roots. They do not exist as such for Euclidean magnitudes, although they are related to the notion of continued proportion between magnitudes. On the other hand, a Euclidean number is always what we call a “natural” or positive integer number, and the numerical operations involving them (for instance, the determination of proportional means) were severely restricted to ensure that the answers were acceptable—that is to say, that the answers were positive integer numbers.\footnote{Bos [2001, 120–127] offers a neat, detailed discussion of the differences between the classical meaning of numbers and magnitudes, as well as between arithmetical operations and geometrical constructions in Euclidean terms.} As far as we know, not only was the neat and consistent separation between the Euclidean notions of numbers and magnitudes preserved in Latin medieval translations (see below), but these notions were still regularly taught in
the major schools of Western Europe in the second half of the 15th century. By the second half of the 17th century, however, the distinction between the classical notions of (natural) numbers and continuous geometrical magnitudes was largely gone, as were the notions themselves.

If we hardly know how this conceptual shift took place it is largely because it happened, so to speak, in a highly unmathematical way, almost by stealth. The 16th and 17th centuries saw, to be sure, a few discussions about the relations between arithmetic and geometry, some mentioned in Giovanni Crapulli’s classical study of the notion of *mathesis universalis* [Crapulli, 1969]. However, we cannot find in the 16th century a theory of numbers and magnitudes that confronts the classical notions head on—not until we get to Simon Stevin’s *Arithmétique* of 1585.

In Arabic medieval mathematics the classical notions of number and magnitude come very close together, when as a matter of course magnitudes are handled through their numerical measures. In particular, Arabic medieval algebra takes for granted that arithmetical notions and operations can be applied to geometrical magnitudes. To do so, some specific resources are needed, such as rules for computations with numerical roots (or “surd” numbers), but Arabic algebra does provide all the necessary numerical algorithms. In this way some (but by no means all) of the obstacles preventing the use of equations within geometrical problems were first removed (see also [Bos, 2001, 135–136, 143]).

Such a fundamental shift was neither theoretically elaborated nor properly grounded. To be sure, it hardly had any reflection in medieval Latin editions of the *Elements*. As is well known, while medieval Latin translators and commentators did not grasp some of the subtler points in the *Elements*, they faithfully preserved its structure, notions, and results, with the exception of the definitions of ratio and proportionality in Book V [Malet, 1990; Molland, 1968–1969; Murdoch, 1963, 1968]. The conceptual shift so prominent in Arabic algebra came to the Latin West mostly by way of the applied mathematics taught in *abacus* schools and explained in *abacus* manuscripts, the obvious heirs of Arabic algebras. The mathematical works written by the most competent *abacus* teachers established, in ways that we do not fully know yet, the European textual tradition from which emerged the 16th-century contributions of Cardano, Bombelli, Stevin, and others.

3. Klein on Viète and the changing notion of number

In his classical and influential study, Jacob Klein pointed to François Viète’s redefinition of algebra into an “analytical art” as a crucial factor in the transformation of the *arithmos* concept [Klein, 1968]. In Klein’s view—very much shaped by his neo-Kantian leanings—Viète was indebted to the “modern way of conceptualization” typical of the Renaissance. By reinterpreting within the “modern” intellectual landscape of the Renaissance the concept of *eidos* he found in Pappus and Diophantus, Viète was able to transform it into the symbolic concept of *species*. In this account, such a concept is the precondition for a symbolic understanding of *quantitas* in which *arithmos* and *megethos* would collapse into a new and numerical understanding of magnitudes. Klein’s insightful analysis of the classical notion of number (first published in German in 1934–1936) can only be neglected at one’s peril, and yet it is on more than one count dated. We have now a quite different understanding of the Renaissance. We know in much more detail the social context in which early modern mathematics thrived. Above all, we distrust a historical explanation that posits an idealist realm in which mathematical notions inhabit and evolve and impose themselves on historical actors regardless of historical circumstance.
One major oddity in Klein’s volume is the place given to Simon Stevin (1548–1620), who first published a cogent criticism of the old numerical concepts and then set up a new notion of *grandeur* that was numerical through and through. While Stevin’s indebtedness to the cossist tradition seems clear, the sources of his discussion, published in his *Arithmétique* [1585], are far from evident. Nowadays we see him as one of the major figures in Renaissance algebra before Viète turned it into a tool of “analysis.” Yet, in Klein’s account, Stevin’s innovative numerical notions hinge strangely enough on Viète’s symbolic understanding of *species* [Klein, 1968, 186–197]. To stress the implausibility of this account, besides obvious problems of chronology and communication (it is hardly possible that Stevin knew of Viète’s innovations in the early 1580s), notice that Stevin’s understanding of algebra incorporated no symbolic breakthrough comparable to Viète’s, not even after Viète’s works began to circulate in the 1590s [Dijksterhuis, 1970, 21–35; Freguglia, 1989, 1992; Malet, forthcoming].

For antecedents of Stevin’s innovative account of numerical magnitude we shall turn in what follows to three influential 16th-century editions of Euclid’s *Elements*, Tartaglia’s Italian one (first printed in 1543, followed by many other editions), Billingsley’s English one famously prefaced by John Dee (1570), and the Jesuit Clavius’s Latin one (first printed in 1574, followed by very many editions). As we shall see now, 16th-century editions of the *Elements* offer puzzling but valuable evidence. Although they maintain distinctly separated the classical notions of discrete whole numbers and continuous magnitudes, they do not fail to incorporate key features of the practical and algebraic tradition. In fact, in some books of the *Elements* we even find the distinction between numbers and magnitudes blurred. Crucially for the purposes of this article, we shall see that all the texts examined here commit themselves to a numerical understanding of continuous magnitude and in practice “translate” most geometrical results into what we would call the language of “geometrical algebra.” In particular, Clavius traces such an understanding to Regiomontanus, whom he mentions as an authority. We shall see as well that these versions of the *Elements* contain excursions in mathematical history, usually to explain why some facet of the *Elements* was in a poor state. History and philosophy therefore helped Renaissance mathematicians to set forth a view of mathematics in which magnitudes are to be understood as numbers.

4. Tartaglia

In the encomium of the mathematical sciences that prefaces Tartaglia’s *Elements* we find a strongly Platonic understanding of knowledge and mathematics combined with a strongly utilitarian approach to mathematics. The uses of mathematics run the gamut from the designing of fortifications and war machines, to the building of bridges and churches, to the practice of almost any art and science. This includes the law, philosophy, and theology—where Nicholas of Cusa is brought in as an authority. The metaphysical importance of mathematics derives from God’s use of geometry (meaning the results that reveal specific proportions among the parts of figures): “God, Himself the measure of everything, does not govern himself without [geometry] in shaping the parts of the human body.”2 If the most skillful architects shape public and private buildings in proportion to the human body, it is for no other reason that the human body is “built by the Greatest Architect in its due measures.”

---

2 Dedicatory letter to Gabrielle Tadino, Abbot of Barletta, in Tartaglia [1543, 2r–4r, f. 3r]. According to Debus, this idea is already found in Agrippa’s *De occulta philosophia* (1533), and then is also elaborately set forth in John Dee’s Preface as foundation for the art of “Anthropographie”; see [Dee, 1975, 12, [33]].
Mathematics, and geometry in particular, is about measurable objects that are dealt with in the mind, “undressed of matter.” According to Tartaglia, mathematical objects are so universally useful because the mind “sees” the true forms through the visual observation of visible forms. Figures cannot be true in matter because of its imperfection, but they tell the mind how the forms are “in themselves.” The mind apprehends them in an abstract, direct, and pure way that cannot but convey the truth. This Platonic, immediatist understanding of mathematical forms (lines, triangles, pyramids, and so on) allows Tartaglia to reassert Plato’s claim about the capital relevance of geometry for philosophy in general [Tartaglia, 1543, f. 2v]. Interestingly, it seems as well to ground Tartaglia’s notion that the “quantity” or measure of mathematical forms belongs to them in an essential way and therefore is directly and immediately apprehended by the mind. The “innate” quantitative nature of Tartaglia’s mathematical forms is most apparent in the changes he introduces in the definitions of point, line, surface, ratio, and proportionality, but also in his rendering of Book II.

Mathematical figures, according to Tartaglia, are but “forms of immobile continuous quantity” (forme della quantita continua immobile) [Tartaglia, 1543, [5]r, 6r]. That geometrical figures primarily are quantities has consequences for Tartaglia’s understanding of the notion of point and for his definition of straight line and plane surface. To explain the difference between a “mathematical” consideration of lines and a “natural” one, Tartaglia takes two lines of the same measurable length, 2 yards (passi), and imagines them embodied in measuring sticks made of woods of different kind—the sticks are “equal” for the mathematician while they are not so “in nature” [Tartaglia, 1543, f. 6v]. It is therefore the measure of magnitudes that provides them with their essential mathematical nature. It cannot come as a surprise now that Tartaglia defines a straight line as the shortest path (“extension”) between two points, rather than by the Euclidean definition that requires that its points lie “equally” or “evenly.” The same happens when it comes to defining a plane surface [Tartaglia, 1543, f. 7v]. Tartaglia’s definition of a point does not modify the letter of the definition (Il Ponto e’ quello che non ha parte). The commentary accompanying it, however, shows that his main concern is to argue that points are border marks “imagined by the mind” (con la mente imaginato) out of those made “by art” (dall’arte). Tartaglia stresses that points cannot be compared to anything measurable (like a simple voice or sound). This would imply or suggest that points are continuous indivisible quantities, which, says Tartaglia, is a contradiction [Tartaglia, 1543, f. 6r].

Tartaglia’s commitment to a quantitative understanding of continuous magnitudes also shows itself throughout his commentary to Book II. His comment to Definition 1, “Ogni parallelogrammo rettangolo è detto contenersi sotto alle due linee che circondono l’angolo retto” (Every rectangle is said to be contained by the two lines about the right angle), turns rectangles into the product of their sides openly and unambiguously:

It must be noted that such a rectangle is usually called by many other names or expressions. For instance, be ab and cd the two right lines, I say that saying any of the following means or imports the same thing.

That which comes from leading ab into cd,

---

3 “Il naturale è differente dal mathematico in questo, che lui considera le cose vestite, il mathematico nude d’ogni materia sensibile.” [Tartaglia, 1543, [5]v; sic, the folio is actually numbered “III”].

4 “La linea retta è la breuissima estensione da uno ponto ad un’altro” [Tartaglia, 1543, 7r]. Clavius, who is drawing from the same sources as Tartaglia, has “Recta linea est, quae ex aequo sua interiacet puncta” [Clavius, 1611, 14]. In his commentary, however, Clavius stresses the equivalence of both notions.
The rectangle of \(ab\) into \(cd\),

The product that comes from leading \(ab\) into \(cd\),

The multiplication of \(ab\) into \(cd\), …

The rectangular surface contained by \(ab\) and \(cd\). [Tartaglia, 1543, f. 29r]

That Tartaglia cannot conceive magnitudes without measures is further suggested by his awkward rendering of the definition of ratio (Book V, Definition 3): “Ratio is a certain relation between two quantities of the same kind, of one to the other, no matter how large they are” (stress added). ⁵ The words underlined are a blatant, useless interpolation that is all the more remarkable because Tartaglia is the first Renaissance commentator to correct Campanus’s faulty reading of the difficult definition of proportionality, or equality of ratios. In fact Tartaglia’s edition of the Elements first provides a sound interpretation of Euclid’s Definition 5 (7, in Tartaglia’s version), Book V [Drake, 1973; Sasaki, 1985]. Tartaglia’s comments to Definition 3 evince that ratios can only exist among things quantified: “[things that] participano la natura & la proprieta della quantita” [Tartaglia, 1543, 59v, also 60r]. Therefore he does not need to mention that the comparison be made “in reference to quantity” or “size.”

According to Tartaglia, no faithful version of the mathematical Elements that Euclid actually wrote down is available. He assumes that mathematics was (in Plato’s and Euclid’s time) the “foundation of all Philosophy and Wisdom”—in fact the title of Euclid’s work would make reference to this role: “Opera di principii: quasi volendo dire Opera di principii e fondamenti della Sapientia, ouer Philosophia” [Tartaglia, 1543, f. 2v]. This work was originally organized with the “highest and most admirable order imaginable,” which was preserved for a long time. However, the “moderns” have “not only corrupted but annihilated” it, to the point that arithmetic and geometry “are almost lost,” and the two translations now available have “disordered” and “debased” Euclid’s works. ⁶ Tartaglia does not elaborate very much on the reasons for such decadence other than to stress the many consequences of the language barrier separating the moderns from the ancients [Tartaglia, 1543, f. 3r]. In Tartaglia’s version of the Elements it is therefore implied that he is not attempting to be faithful to an old model. He is just trying to make the best possible sense of the inherited textual tradition in the light of what he knows. Notice that by assuming that the text of the Elements was at best an approximation to the mathematical knowledge of the learned ancients, Tartaglia could freely interpret and even modify the main versions of Euclid’s work available. His encomium of the mathematical sciences, filled up with historical references, includes a long list of past mathematicians where just Diophantus’s name is missing among the usual ones, but begins with and highlights Hermes Trismegistus, “philosopho, sacerdote, & Re d’Egitto.”⁷ As we shall see Tartaglia was not alone in putting history at the service of an innovative exegesis of Euclid’s text.

⁵ “La proportione e la conuenientia certa de due quantita de vno medesimmo genere dell’una all’altra siano de quanta grandezza si uoglia” [Tartaglia, 1543, 59v]. Again, compare with Clavius’s more faithful rendering: “Ratio est duarum magnitudinem eiusdem generis mutua quaedam secundum quantitatem habitudo” [Clavius, 1611, 167]. In Heath’s translation: “A ratio is a sort of relation in respect of size between two magnitudes of the same kind” [Heath, 1956, II, 114].

⁶ “Et questo ordine antico fu osseruato & mantenuto gran tempo. Ma al presente da moderni no solamente è stato corrotto, ma totalmente annullato, che le dette due scientie sono quasi deperdite” [f. 2v]. Tartaglia’s views on the corruption of the Elements then available are stressed in Rose [1975, 152].

⁷ “Philosopher, priest, and King of Egypt.” The list includes Hermes, Pythagoras, Plato, Plotinus, Aristotle, Averroes, Hypocrates, Euclid, Ptolemy, Archimedes, Apollonius, Jordanus, and Vitruvius, mentioned in this order [Tartaglia, 1543 [4]r; sic, the page is actually numbered “III”].
5. Clavius (and Regiomontanus too)

Clavius’s Latin edition of the *Elements* (first published in 1574) also contains plenty of historical references, including its own longer list of past mathematicians (neither Hermes nor Diophantus is mentioned here). Following Commandino, whom he mentions appreciatively, Clavius reviews in particular the historical evidence showing Euclid the mathematician and Euclid the Platonic philosopher of Megara to be different men living in different times and places. Stressing the poor textual tradition on which the *Elements* stand, Clavius also approaches creatively the elucidation of the basic notions of the *Elements* and the refurbishing of its demonstrations [Clavius, 1611, 10]. Indeed the changes Clavius introduces in the understanding of magnitudes and whole numbers are more fully explained than Tartaglia’s, and he grounds algebraic techniques on the new numerical notions more explicitly than Tartaglia does. Another obvious continuity between Tartaglia and Clavius is the Platonic theme. Indeed, praise of Plato and of his philosophy is even more persistent and exaggerated in Clavius’s *Elements* than in Tartaglia’s [Clavius, 1611, 6]. While there is praise here for the practical applications of mathematics, yet Clavius emphasizes above all the educational uses and the philosophical relevance deriving from the intermediate position between “metaphysics” and “natural philosophy” (*scientia naturalis*) that mathematics occupies in the hierarchy of knowledge [Clavius, 1611, 5–6].

Tartaglia’s new numerical or quantitative understanding of magnitudes did not entail major changes in the arithmetical Books VII, VIII, and IX of the *Elements*, neither in their basic notions nor in their results and demonstrations. Clavius’s version of the definitions opening Book VII, however, include several extraordinary interpolations, all of them aiming to introduce the theory of operations with fractions or (what we now call) positive rational numbers—“*numeris fractis, et integris cum fractis*,” in Clavius’s words. In his long commentary to Definition 15, of what it means for one number “to multiply” another, Clavius includes the definition of “division” of numbers, and then extends all the operations among whole numbers to operations among fractions [Clavius, 1611, 308–309, 311, 312]. This is done in an ad hoc fashion throughout his commentary on the arithmetical books. Moreover, at the end of them Clavius includes as an appendix a mini-treatise devoted to dealing in a systematic, demonstrative way with the operations of fractional numbers: “*Minutiarum sive numerorum fractorum demonstrationes*” [Clavius, 1611, 381–394]. In the language of Jacob Klein, Clavius openly incorporates theoretical logistics into Euclid’s arithmetical books.

It is well known that Clavius offers a kinetic understanding of curves in which lines are “drawn” by the steady movement or flowing (*fluxum*) of a point [Clavius, 1611, 13, 14]. Clavius does not claim to be introducing a new understanding of curves—in fact it was already mentioned (not as a novelty) 4 years earlier in Henry Billingsley’s English edition of the *Elements* (see below). On the contrary, Clavius mentions it as a description of lines usually used by “mathematicians” to explain to a layperson what is a mathematical line [Clavius, 1611, 13]. Clavius extends such a kinetic approach to the notion of parallelogram, which he describes as arising from “the imaginary motion of one of their sides on the other.”

---

8 Clavius’s lists begins with Pythagoras and includes Anaxagoras, Hypocrates, Plato, Oenopides, Zenodorus, Brito, Antipho, Theodorus, Thetetetus, Aristarchus, Eratosthenes, Architas, Euclid, Serenus, Hypsicles, Archimedes, Apollonius, Theodosius, Mileus, Menelaus, Theon, Ptolemy, Eutocius, Pappus, and Proclus [Clavius, 1611, 4–5]. His elucidation of Euclid’s identity is on pp. 6–7. On Commandino’s views, see Rose [1975, 206].

9 “Ex motu imaginario unius lineae in alteram huius modi parallelogrammum conficitur. Si namque animo concipiatur recta AB, deorsum secundum rectam AD, moveri in transversum, ita ut semper angulum rectum cum AD, constituat, donec punc-
According to Clavius, this is the best way to understand what Euclid means in the already mentioned Definition 1, Book II (“Omne parallelogrammum rectangulum contineri dicitur sub rectis duabus lineis, quae rectum comprehendant angulum” [Clavius, 1611, 82]). Thus, this physical way of visualizing the generation of figures gives meaning to the “combination” or “operation” of two right segments via the notion of rectangle.

At the same time Clavius reshapes geometrical magnitudes according to the numerical understanding already found in Tartaglia. That is to say, not only Clavius assumes throughout that straight lines and geometrical objects have numerical measures, but also that it is legitimate to “arithmetically” operate with geometrical objects by making, for instance, rectangles “equal” or “equivalent” to the product of its sides, and so on. Clavius emphasizes the equivalence of the parallel structures and results obtaining between the operations of numbers and the “operations” of magnitudes when he deals with both pure numbers, in Books VII to IX [Clavius, 1611, 367–370], and straight lines, in Book II. In Book II he speaks about the “deep affinity” (magnam affinitatem) between figures and numbers in reference to their interchangeable nature and to the equivalence of such notions as multiplication of numbers and rectangles of lines [Clavius, 1611, 83]. He also plays with the meanings of the verb duco (“to lead,” in its more usual meaning) and the kinetic understanding of figures mentioned above. The product of numbers can be expressed by saying that it arises “ex ductu” of one number in another. Nevertheless, Clavius stresses that rectangles can also be said to be generated “ex ductu” of one of its sides into the other.10 It is perhaps worthwhile remembering that G. de Saint-Vincent, a Jesuit pupil of Clavius, introduced in the first decades of the 17th century a powerful technique of quadrature and cubature called “ductus plani in planum.” It was based on the notion that figures are generated “ex ductu” of a line or figure that changes according to some given law along (or in) a given line or axis [Whiteside, 1961, 316]. Let us finally stress that Clavius claims the numerical understanding of rectangles he defends to be already “well known by arithmeticians and geometers, and was demonstrated by Regiomontanus” in Book I of his De triangulis [Clavius, 1611, 83].

The first 19 theorems of Book I of Regiomontanus’s De triangulis contain nothing but elementary results. All of them “blend,” so to speak, results from the arithmetical Books VII to IX of the Elements with results from Books II and VI, therefore justifying the application of numerical results to geometrical configurations whose magnitudes are assumed to be numerically known (see [Bos, 2001, 136–138]). Among them we find Theorem 16, mentioned by Clavius in support of his numerical interpretation of Book II of the Elements. In a convoluted way, Regiomontanus’s Theorem 16 just “demonstrates” that the rectangle contained by two lines “is known” by the multiplication of the numbers that measure their sides.11 Let us stress that Regiomontanus’s De triangulis contains the most extraordinary mix of old and new notions, arithmetical as well as geometrical. In its opening definitions all magnitudes are measured or are measurable lengths; that is to say, all magnitudes are “quantities,” and quantities are only known when

---

10 “Vt nonnulli dicant, parallelogrammum rectangulum gigni ex ductu duarum linearum circa angulum rectum unius in alteram” [Clavius 1611, 83].
11 Regiomontanus’s argument is convoluted because he needs to “build up” a square unit surface out of the units of length of the sides; see Regiomontanus [1533, 17–18].
“measured by a known measure according to a known number.” \(^{12}\) It is one of Regiomontanus’s axioms (\textit{communes animi conceptiones}) that “equal quantities are those that measure the same” \cite{Regiomontanus, 1533, 8}. On the other hand, Regiomontanus’s numbers seem to be just the old \textit{arithmoi} (sets of discrete whole units), because any number is supposed to be known when “the intellect is able to discriminate the unities within it” \cite{Regiomontanus, 1533, 7}. It would therefore seem that Regiomontanus’s “world” is made up of pre-Eudoxian, rational measures. This impression is just reinforced by his definitions of ratio and proportionality, since the equality of ratios is defined through the equality of their “denominations,” and by the axiom that every ratio can be expressed in numbers.\(^{13}\)

When we leave the opening definitions, however, things become more complex. For instance, take Theorem 2, where Regiomontanus demonstrates that “The side of a known square is not unknown.” It all comes down to show that if the number measuring the square, \(L\), is known, so is its side. If Regiomontanus needs a theorem for such a trivial conclusion it is because he does not feel entitled to apply to squares what he knows about whole numbers. So, assuming first that the number \(L\) is a perfect square, he argues via Book VI, Proposition 1 (triangles and parallelograms under the same height are as their bases) of \textit{Elements} that the square root of \(L\) measures the side of the square. The interesting part of the proof comes when \(L\) is not assumed to be a perfect square. The argument is the same as before, except that Regiomontanus here, after a moral reflection on the limits of human knowledge (in many matters we cannot get the whole truth, but only an approximate version of it), redefines his notion of “known quantity” by allowing for the substitution of approximate values for such numbers that cannot be know accurately (\textit{praecise}). To assuage any misgivings of ambiguity the reader may feel before such considerations, Regiomontanus adduces again a moral lesson: “it is better to know something close to the truth than to ignore the truth completely, for it is praiseworthy (\textit{virtuti debitur}) not only to reach the goal but also to get close to it” \cite{Regiomontanus, 1533, 9}. All of this serves Regiomontanus to argue that when the measure of the square, \(L\), is a whole number or a fractional number that is not a square (fractions appear here without forewarning), then the theorem holds if in place of \(L\) we take the “closest” fraction or number that is a perfect square, and then proceed as before \cite{Regiomontanus, 1533, 9}. Notice that, committed to a numerical understanding of magnitudes, and taking for granted that numerical operations extend to continuous magnitudes, Regiomontanus is led to substitute rational approximations for what we would call today real numbers. Of course real numbers did not exist as yet, but Regiomontanus’s text suggests that any competent mathematician was already thinking in terms of all the possible numerical or quantified lengths, and when needed handled them by means of fractional approximations.

Clavius’s reference to Regiomontanus as an authority in this matter meant a mutual endorsement for the two authors. On the one hand, Regiomontanus’s highly influential text, written in the early 1460s,

\(^{12}\) “Cognita uocabitur quantitas, quam mensura famosa, aut pro libido sumpta secundum numerum metitur notum” \cite{Regiomontanus, 1533, 7}. Unfortunately B. Hughes’s translation is unreliable and must be used with extreme caution. Unless otherwise noted, all translations are mine.

\(^{13}\) “Proportiones aequales sunt, quibus una communis est denominatio” \cite{Regiomontanus, 1533, 7}. As is well known, the “denomination” of a ratio is a notion popularized in medieval mathematics that does not belong to the tradition of Euclid’s \textit{Elements} proper. The denomination was the name that according to an elaborate set of rules was given to ratios between whole numbers: \textit{dupla} and \textit{subdupla}, for instance, were the denominations of the ratios \((8 : 4)\) and \((4 : 8)\), \textit{sesquitertia} was the denomination of \((4 : 3)\), and so on; see \cite{Mahoney, 1978; Malet, 1990; Molland, 1968–1969; Murdoch, 1963}. By defining the equality of ratios through their denominations, Regiomontanus seems to exclude all ratios that cannot be expressed as ratios between whole numbers. In fact such a view is unambiguously stated in the guise of an axiom: “Every given ratio is expressed in numbers” (\textit{Omnem proportionem datam in numeris reperiri}, p. 8).
first published in 1533, and republished many times in the 16th century, was a fitting source to be quoted as an authority in such matters. Regiomontanus not only had been preeminent in astronomy, the mixed science par excellence where geometrical lines and angles cannot but be handled numerically, but he was also the foremost representative of the rebirth and renovation of mathematical thought in the West. Clavius was therefore drawing on an accepted practice then at least one hundred years old. On the other hand Clavius, himself a learned and highly competent mathematician, was assuming and endorsing the practical way in which mathematical practitioners were solving the difficulty of numerically handling geometrical magnitudes, a difficulty that did not have as yet a full, satisfactory solution.

In the opening paragraphs of his commentary on Book II Clavius seems not only aware of some of the theoretical difficulties Renaissance mathematicians were facing, but also suggests that Book II may solve some of them. According to him, the results in this Book are useful in geometry as well as in “human practical affairs,” and foremost among its benefits is that it provides demonstrations “for the worthy rules of Algebra.” Algebra is here extolled as one of the more distinguished creations of the human mind, able to solve difficult problems and to greatly empower the human natural abilities [Clavius, 1611, 83]. Clavius also stresses that some propositions of the same Book provide demonstrations for the rules of the addition, subtraction, multiplication, and division of radical numbers, which “cannot be expressed in any way”—“not even by Divine power (potentiam) can [such numbers] be expressed in numbers, because such a thing implies a contradiction” [Clavius, 1611, 83]. Book II, therefore, appears to Clavius as an obvious example of geometrical algebra. The authors to be examined next, writing almost contemporaneously with Clavius, show a strikingly similar appreciation of the relationship between algebra and Book II, and of course are fully committed to the numerical understanding of continuous magnitude.

6. Billingsley and Dee

Sir Henry Billingsley’s influential English edition of the Elements [1570], prefaced by the notorious magus John Dee, contains a highly articulated synthesis of the major motivations shaping the two editions previously considered, for it brilliantly allies practical usefulness, philosophical significance, and religious and moral value. As if in reflection to such a complex synthesis, we also find in it clear evidence of the ambiguous relationship between arithmetical and geometrical notions in these editions, and in particular of the tensions it caused within the deductive structure of the Elements.

It has been abundantly noticed that Dee’s Preface presents a typical view of how mathematics was understood from a neo-Platonic and Hermetic perspective [Clulee, 1988; Dee, 1975, “Introduction”; French, 1972; Harkness, 1999; Heilbron, 1978; Walton and Walton, 1997]. Mathematical objects are, according to Dee, of a “middle” nature between purely intellectual and spiritual objects (immortal, simple, incorruptible, apprehensible by the mind only, where “science” and certainty is possible) and sensible ones (with the opposite characteristics, perceived by the senses, where probability and conjecture reign). They are therefore neither so absolute and excellent as the former ones, nor so crude and corruptible as things “natural” [Dee, 1975, [2]]. (Notice the resemblance between such views and those of Clavius on the intermediate nature of mathematics between metaphysics and “natural science” [Clavius, 1611, 5].

14 Jean-Marc Mandosio’s insightful article on Dee and Euclid’s Elements [Mandosio, 2003], pointing out among other things Dee’s substantial influence on Billingsley’s edition of Euclid, came to my attention when this article had already been accepted for publication in July 2004.
In many other places the resemblance between the views of Billingsley and Dee (printed in 1570) and Clavius’s (in 1574) is so strong that it suggests some kind of connection between the two texts. The “marvellous neutrality” and “strange participation” of mathematical objects in the characteristics of both kinds of objects, intellectual and sensible, give them not only the status of “general forms” but also their many capabilities in matters practical, philosophical, and moral. In particular, Dee’s neo-Pythagorean understanding of numbers, which finds translation in Billingsley’s rendering of Euclid (see below), turns arithmetic into a science second only to theology in being “divine, most pure, most ample and general, most profonde, most subtile, most commodious and most necessary” [Dee, 1975, [12]]. In an analogous way, Dee argues that geometrical “contemplations” are most useful to and needed for “Christian men” because they train their imaginations “to conceiue, discourse, and conclude of things Intellectual, Spirituall, aeternal, and such as concerne our Blisse everlasting” [Dee, 1975, [15]].

On the other hand, Dee devotes the bulk of his Preface to extol applied mathematics and the manifold practical disciplines that profit from the applications of mathematics. Interestingly he includes all kinds of disciplines, from the most ancient and well established (astronomy, optics, or perspective, for instance) to the ones he had just devised (“hypogeiodie” or underground topography); from the most general (architecture, statics, pneumatics) to the more narrow in subject (“trochilike” or the art of designing wheels, “helicosophie” or the art of screws, “menadrie” or the art of cranes); from the strictly mathematical (stereometry) to “thaumaturgike” (defined as the art of building marvelous mechanical and optical contrivances) and “archemastrie” or the experimental study of nature; and many others. Contrary to what has been said, however, the emphasis of the Preface does not seem to lie in the long and verbose list of mathematical applications. Rather it seems to lie in the deep unity of it all, mathematics and philosophy and religion and engineering. Dee’s philosophical viewpoint allows him not only to overcome any opposition between Platonizing geometrical contemplations and engineering practice, or between Boethius’s numerology and commercial arithmetic and algebra, but also to integrate all of them in a single enterprise. In relation to the subject matter of this article, it is remarkable that Dee’s magical and religious views supported a renovated perspective on the nature and scope of the mathematical sciences. In particular, it seems not to have been noticed previously that Dee’s views are behind the main conceptual innovations we find in Henry Billingsley’s version of the Elements.

Billingsley’s Elements is faithful to the letter of most of Euclid’s definitions and results, but it shows the tensions arising from the modifications being introduced into the notions of number and magnitude. Billingsley, as Tartaglia and Clavius do, takes for granted that magnitudes are measurable, and their measures are brought in when needed without further ado. Interestingly the kinetic understanding of lines and surfaces already mentioned provides Billingsley with an argument to demonstrate the deep relationship between rectangles and the product of their sides:

The parallelogramme is imagined to be made by the draught or motion of one of the lines into the length of the other. As if two numbers should be multiplied the one into the other. [Billingsley, 1570, f. 60r]

Billingsley further explains his views by suggesting that this motion allows one to imagine the side that moves as erected upon every point of the side along which is being moved—in the same way as in the product of whole numbers represented by dots the dots of one of the numbers are set up as many times as dots are found in the other. This would be the reason that the rectangle is equal to the product of its sides [Billingsley, 1570, f. 60r–60v]. He reiterates the consequences of identifying magnitudes with their measures when he introduces Book VI as one that is specially relevant for the purposes of “use and
practice,” that is to say, for the theory of all measuring instruments, such as the astrolabe, the quadrant, the staff, etc.:

On the Theoremes and Problemes of this Booke depend for the most part, the composition of all instruments of measuring length, breadth, or deepeness, and also the reason of the use of the same instruments. [Billingsley, 1570, f. 153r]

In general Billingsley takes for granted that geometrical results can be solved by and applied to arithmetical results and contrariwise, and his allusions to algebra abound in explicit references to the “agreement” between arithmetic and geometry, or numbers and magnitudes (more about this below). Yet Billingsley also recognizes that such agreement is not perfect because numbers and magnitudes do differ in nontrivial aspects. In fact Billingsley provides contradictory views on the generality and relative importance of discrete numbers and arithmetic versus continuous magnitudes and geometry.

When he introduces the first arithmetical Book, Billingsley proclaims the superiority of arithmetic over geometry, and he does so by closely following arguments found in Dee’s Preface. Geometry needs arithmetic, whose “excellency and worthiness . . . is above Geometrie.” Arithmetic rests in itself:

Arithmeticke is of itself, sufficient and neadeth not at all any ayde of Geometrie, but is absolute and persists in itselfe, and may well be taught and attayned unto without it. [Billingsley, 1570, f. 183r]

The superiority of arithmetic is predicated on an epistemological basis, since it provides principles and grounds “to all other sciences and artes.” Furthermore, geometry deals with objects (lines, figures, bodies) that are apprehended by the senses while numbers are purer and more immaterial, and speak to the mind only:

[geometrical objects] offer themselves to the sences, . . . are sene and iudjed to be such as they are, by the sight: but nomber, which is the subject and matter of Arithmeticke, falleth under no sence, nor is represented by any shape, . . . but only by consideration of the minde . . . Now thinges sensible are farre under in degree then are thinges intellectuall and are of nature much more grosse then they. [Billingsley, 1570, f. 183r]

The superiority of arithmetic is also argued from authority, claiming such a view to have been held by the “wisest and best learned philosophers that have bene, as Pithagoras, Timeus, Plato and their followers,” who explained everything by number. Billingsley also quotes Boethius’s view that number is “the principall . . . patron in the minde of the creator” as well as Plato’s about the soul being composed of harmonic numbers [Billingsley, 1570, f. 185r].

In an apparently seamless argument, Billingsley also underscores the role of arithmetic as the most useful and most needed part of mathematics in the “common life of any man of any condition” [Billingsley, 1570, f. 185r]. In doing so, Billingsley only touches briefly on a new expanded view of the nature of arithmetic that Dee explains at length in the Preface. Here arithmetic is introduced as the “science that demonstrateth the properties, of Numbers, and all operations, in numbers to be performed.” Numbers, however, are no longer restricted to the Euclidean arithmoi, but include fractions as well as the numbers derived from root extraction, that is to say, the numbers that use “the common Logist, Reckenmaster, or
Arithmeticien.” According to Dee, therefore, there is one arithmetic of fractions and another of radical numbers that are not only legitimate but also fundamental parts of mathematics [Dee, 1975, [5]].

The superiority of arithmetic is also predicated on the role numbers play in handling ratios and proportionality. When Billingsley introduces Book V, he recognizes that Euclid “semeth to speake of [proportion and proportionality] onely Geometrically, as they are applied to quantitie continuall,” and yet he sets forth the view that ratios and proportionality “chiefly” pertain to numbers, and are to be applied to geometry “but in respect of number and by number” [Billingsley, 1570, f. 125v]. Billingsley’s examples show that once again he is not thinking only in terms of whole numbers. Some ratios are rational (which means they are certain and known) but others are not. For instance the ratio between the side of a square (Billingsley’s measures 4 in.) and its diagonal (which therefore measures $\sqrt{32}$) is “irrational, confused, unknowen, uncertaine, and surd” because it can only be expressed through the number $\sqrt{32}$, which “cannot be expressed by any determinate and certaine number, but only by this manner of circumlocution Roote square of 32” [Billingsley, 1570, f. 127r]. When he comes to Book X, he stresses that the only difference between numbers and magnitudes is precisely this one, that while there are no two incommensurable numbers, yet in magnitudes it is otherwise [Billingsley, 1570, f. 228r, 231r]. According to Billingsley all this is consistent with the superiority of arithmetic and discrete numbers over geometry and continuous magnitudes.

In other parts of Billingsley’s edition of the *Elements*, however, the notions and principles of geometry appear to be more general and fundamental than those of arithmetic. This view seems to dominate Billingsley’s commentary to Book II, which opens by introducing geometry as the foundation of arithmetical and algebraic operations:

The Arithmetician also out of this booke gathereth many compendious rules of reckoning, and many rules also of Algebra, with the equations therein used. The groundes also of those rules are for the most part by this second booke demonstrated. [Billingsley, 1570, f. 60r]

The first 10 theorems of this second book Billingsley illustrates with references to algebraic rules or results that can be deduced from them [Billingsley, 1570, f. 60r, 67v, 69r]. His goal is for the reader to perceive the intimate connection between arithmetic and geometry, or in his words:

the agreement of this art of Geometry with the science of Arithmetique, and how nere and deare sisters they are together, so that the one cannot without great blemish be without the other. [Billingsley, 1570, f. 62r]

Billingsley stresses, however, that this agreement is not quite an identity, since some problems can be solved in geometry that cannot in arithmetic, such as the finding of the golden section:

And doubtless it is wonderful to see how these two contrary kynd of quantity, quantity discrete or number, and quantity continual or magnitude… should have in them one and the same propieties common to them both in very many points, and affections, although not in all. For a line may in such sort be divided, that what proportion the whole hath to the greater parte the same shall the greater part have to the lesse. But that cannot be in number.” [Billingsley, 1570, f. 62r]

Billingsley seems to be suggesting here not only that geometry provides foundations for many rules of arithmetic and algebra, but also that it is more general because it solves more problems than arith-
metric does. Billingsley’s Euclid, therefore, offers very good evidence of the difficulties caused by the introduction of a numerical understanding of magnitude into the corpus of the Elements.

The three versions of the Elements examined so far show how important the input from algebra and the practical uses of mathematics was for the changing notions of number and magnitude in 16th-century editions of the Elements. At the same time they show remarkable coincidences in their taking for granted that geometry and arithmetic can mix in ways alien to classical Greek mathematics, but also in their numerical understanding of magnitudes and in their open-mindedness toward new sorts of numbers. Yet they all preserve the old definitions, axioms and postulates, which are the source of substantial problems, as is particularly evident in Billingsley’s text. The first attempt to tackle such problems by explicit criticism of the classical notions was published by Simon Stevin.

7. Stevin

Stevin’s comprehensive work on arithmetic and algebra, L’Arithmétique, was first published in 1585, and then reprinted in 1625 and 1634. As E. J. Dijksterhuis remarked many years ago, most of Stevin’s works (including this one) are textbooks rather than specialized scientific monographs [Dijksterhuis, 1970, 16]. This may partly explain the informal style in which Stevin introduces his new notion of number, particularly as he first criticizes some characteristic features of the old notion. On the other hand, if his book was written as a textbook and therefore intended not for the sophisticated mathematician, then Stevin’s discussion tells us something about what was then “common arithmetical knowledge.”

Stevin’s point of departure is twofold. First, he stresses that the origins of the concept of “number” cannot be different from the origins of other concepts. Thus, as in the past “men” (les Hommes) created the generic concept “bird” to subsume the species eagle, pigeon, or mockingbird, so they also created the generic concept “number” to subsume one, two, three, one half, one third, etc. [Stevin, 1585, 2]. Second, he identifies all possible numbers with all possible measures: “Nombre est cela, par lequel s’explique la quantité de chascune chose” [Stevin, 1585, 1]. He develops this Definition 1 of Arithmétique through different theses and commentaries. Among the most significant of them we find strong critiques of the old principles that “unity is not number” and that “numbers are discontinuous quantities,” and then the defense of radical quantities as fully legitimate numbers. In fact Stevin does not justify his first definition (“Number is that by which one can tell the quantity of anything”), since to him the kindred character and similarity, “quasi identity,” of number and magnitude (grandeur) are obvious and derive from a direct intuition of their nature. This intuition in turn rests on a strongly physical understanding of magnitudes that Stevin couches in his well-known “water-and-wetness” metaphor:

number is in magnitude like wetness in water, since it happens with number as it does with wetness, that as wetness pervades all and every part of the water, so the number assigned (destiné) to any magnitude pervades all and every part of its magnitude. [Stevin, 1585, 3]

Notice the power of such a metaphor in Stevin’s intellectual milieu, where “wetness” was one of the defining traits of the nature of the Aristotelian element “water” (water was wet and cold, earth was dry and cold, and so on). Through such a philosophical reference Stevin effectively suggests that number

15 For recent accounts of Stevin’s discussion of the notion of number see Bos [2001, 138–141] and Neal [2002, 36].
essentially belongs in the nature of magnitude. This is the key argument that appears in his attack on the old belief that unity is not a number but only its first principle or origin—a thesis faithfully reproduced in Tartaglia’s, Billingsley’s, and Clavius’s editions. According to Stevin, since one unit is a measure, it cannot but be a number. He deals in a similar way with the one major, time-honored feature of classical numbers, that they were “discontinuous quantities.” Again, such a categorization of quantities in general and of numbers in particular we find in Tartaglia, Billingsley, and Clavius. No number is discontinuous or discrete, says Stevin, because numbers are measures, and all measures are continuous necessarily: “everything that is but quantity is not discontinuous quantity.” His “water-and-wetness” metaphor helps him again: “as well as to continuous water corresponds a continuous wetness, so to a continuous magnitude corresponds a continuous number” [Stevin, 1585, 3]. Any one of Stevin’s whole numbers may be subdivided in many ways, exactly as it happens with any segment or measure, and therefore it cannot be understood as a collection of discrete units.

Stevin’s discussion of the status of irrational numbers aims to remove from them the stigma of being “absurd,” “less perfect,” “inexpressible,” or in any sense a quantity not to be considered a “full” or complete or legitimate number. That “any root whatsoever is number” [Stevin, 1585, 8] is also a consequence of identifying numbers and measures: since, for instance, \( \sqrt{8} \) is part of 8, it must have the same nature, and therefore be a number. For such numbers, moreover, we know how to calculate as many approximations as needed. Stevin faces the problem of the meaning of a radical such as \( \sqrt{8} \). To the rhetorical question, What is \( \sqrt{8} \)?, Stevin answers by pointing out all the similarities between radical and fractional numbers. In particular, the intelligibility of, say, 3/4, which comes from visualizing something divided into equal parts, is compared to the visualization of \( \sqrt{8} \) as the side of the square that measures 8. The commensurability of 3/4 with 1, says Stevin, does not make it different from \( \sqrt{8} \), since the latter in turn is commensurable with \( \sqrt{8} \), with \( \sqrt{2} \), and \( \sqrt{32} \) and infinitely many other numbers.\(^{16}\)

Notice that while Stevin’s theses are much clearer and more internally consistent than the ones found in the works of Tartaglia, Clavius, and Billingsley, yet it is impossible to deny the strong continuities between all the works analyzed above. Stevin’s definition of number presupposes the existence of numerical measures in the material world, something that was already taken for granted by Tartaglia, Clavius, and Billingsley. This presupposition Stevin never questions, as nobody did in the world of Renaissance mathematical practitioners. Notice also that Stevin’s numbers are not mere “abstractions” gained directly from material objects. On the contrary, they “abstract” the notion of measuring, which is an operation performed on objects, rather than a “raw” material object. Notice finally that in boldly departing from Euclid’s axiomatization of arithmetic and geometry, Stevin does not substitute his own axiomatization for the old one, but he rather appeals to the “common way” of conceptualizing—as Descartes would do 40 years later. This seems to be one of the first occasions in modern times in which a first-rank mathematician openly denied the principles and definitions of Euclid’s Elements the status of canonical founding grounds for the deductive structure of mathematics.

On what grounds does Stevin dare to oppose the centuries-old authority of Euclid? Recently H. Bos has rightly pointed out the likely influence of Pierre de la Ramée, the influential Huguenot reformer, on Stevin [Bos, 2001, 138]. Another clue (fully consistent with Ramist influences) may be found in Stevin’s views on the history of civilization as set forth in a text that is now known as “De Wysentyt” (“The Age of the Sages”). In fact Stevin did not write it as a separated tract but as part of the first, introductory book

\(^{16}\) Stevin [1585, 10]. Stevin includes a longer discussion with other supplementary arguments; see pp. 8–10.
to his *Geography*. It was originally published in Dutch within the 2-volume *Wisconstighe Ghedachtenissen* of 1605. Stevin was a practical man whose interests and substantial contributions to mechanics, hydrostatics, and mathematics were shaped by his engineering training and jobs. Yet his “The Age of the Sages” shows him partaking in the learned humanist’s belief in the Hermetic tradition of a Golden Age lost down in the deep past. In the very origins of human civilization Hermes and other philosophers of great learning taught humankind a *prisca philosophia* and a *prisca mathematica* that were afterwards corrupted and lost as part of the general moral and religious decay.

AGE OF THE SAGES we call that time in which exceptional learning was to be found among men, a fact which we perceive with certainty from certain signs, but without knowing among whom, where or when. [Stevin, 1961, 593, stress in the original]

Stevin’s views were in no way idiosyncratic in the Renaissance. In “De Wysentyt” he includes a long list of references by old authorities—all attesting to the great knowledge achieved a long time ago—that he received from the learned jurist and humanist Hugo Grotius [Stevin, 1634, 109–110]. We also know that Pierre de la Ramée believed that Euclid was not the author of the *Elements* but only an unfaithful reporter of the mathematical knowledge available in his times. Euclid’s version, according to La Ramée, corrupted the genuine *prisca mathematica* known before Greek times and contained in the original, lost *Elements*. It is also known that many Renaissance authors (not necessarily sharing La Ramée’s radical views) believed that the demonstrations in the *Elements* were not Euclid’s own [Cifoletti, 1996; Rose, 1975, 162–165]. Such views are obviously the appropriate context for Tartaglia’s and Clavius’s references to the corruption of the genuine *Elements* and the need for editing and improving with commentaries the corrupted versions available to them.

According to Stevin, several certain signs show that there existed great knowledge of astronomy that at the time of Hipparchus and Ptolemy “had almost disappeared” [Stevin, 1634, 106–107]. This idea not only is found in “De Wysentyt” but reappears more than once in Stevin’s astronomical treatise, *De Hemelloop*.18 There is also evidence, says Stevin, that “man formerly possessed” a wonderful knowledge of arithmetic, and in particular of algebra:

one of the curious peculiarities of [arithmetic] may be considered to be Algebra, which came to light again a few years ago (*sic*) from Arabic books, which subject, from the writings they left, is seen not to have been known to the Chaldeans, the Hebrews, the Greeks . . . or the Romans, all of whom were no arithmeticians worth the name. [Stevin, 1961, 599]

---

17 The *Wisconstighe Ghedachtenissen* was reprinted in 1608. In the same year a Latin translation by W. Snell appeared under the title *Hypomnemata mathematica*. The same year saw also in print a partial French translation by J. Tuning titled *Mathematical memoirs*. In 1634 parts of *Wisconstighe Ghedachtenissen* (including “De Wysentyt”) were translated into French by Stevin’s student Albert Girard and published in *Les Oeuvres Mathematiques* de Simon Stevin; here “de wysentyt” was translated as “le siecle sage.” In a move that vividly reflects the intellectual changes separating the 1630s from the first years of the 17th century in the Netherlands, Girard in 1634 manipulated Stevin’s “De Wysentyt” in at least two places in order to soften or disimulate Stevin’s sympathy for the Hermetic and magical traditions. I have also used Girard’s French translation, quoted as Stevin [1634], but when available all English quotations in this article come from the partial English translation in Stevin [1961].

18 *De Hemelloop* (*The Heavenly Motions*) is partially reproduced and translated into English in volume III of *The Principal Works* (see Stevin [1961]); for references to the lost knowledge of the Sages, see, pp. 55, 120, 209.
Arithmetic was therefore in need of reform to restore it to its pristine perfection. Other sources besides the Greek ones, particularly Arabic sources, must be taken into account, says Stevin, because they contain hints to and remains of what arithmetic was like before it fell victim to the general decline, and Euclid passed it down to us full of imperfections. The “authoritative authority” that must be followed in arithmetic are the arithmeticians of the Age of the Sages but not the Greeks:

[follow] the skilled arithmeticians of the Age of the Sages, and do not follow the Greeks, who were no arithmeticians, nor could perfectly be so, through lack of the proper numerals, . . . For though Euclid writes elegant arithmetical theorems, which had come down to him from the Age of the Sages, they include neither problems nor practice of arithmetic, . . . so that Euclid’s theorems bear witness to the Age of the Sages which had previously existed and did not then exist. [Stevin, 1961, 601]

In Euclid’s time, for instance, the “irrational” notion that radical quantities are “irrational, irregular, inexplicable, surd, and absurd” had already become fully accepted and such quantities were therefore excluded from the Elements. In particular, its Book X is now so difficult to understand, says Stevin, because it was originally worked out with radical numbers. Afterwards, however, the geometrical lines only were left, and these are now “imperfect” and difficult to handle because they are deprived of the radical numbers that served originally to “invent and describe” them [Stevin, 1585, 213–214]. Stevin also believes that the geometrical theorems were inherited from others as well, although the Elements is more faithful here than in the arithmetical books. In particular it has preserved for us the good deductive order needed for the organization of mathematics: “in [the Elements] besides the subject of geometry something very peculiar, extraordinary and useful is also to be noted and learned, viz. the systematic order observed by the Age of the Sages in the description of mathematics” [Stevin, 1961, 605]. Stevin does therefore confidently set forth a new understanding of the notion of number on the explicit assumption that he is restoring the *prisca mathematica* of the age of the sages.

8. Concluding remarks

The 16th-century editions of the *Elements* studied here show a remarkable synthesis of what used to be called the practical and the learned traditions of mathematical knowledge. These editions contain explicit and highly favorable references to Plato’s philosophy and ideal forms, yet they also acclaim the utility of mathematics for all kinds of technical endeavors. These texts show no opposition between a Platonic philosophy of mathematics centered on abstracting and contemplating pure forms out of the material world and a utilitarian understanding of mathematics. They also contain much erudition about the origins and philosophy of mathematics as well as a typically humanist interest for improving the corrupted versions of classical texts.

To judge by these editions, all of this fueled interest for the ever-growing number of practical applications of mathematics, and contributed to incorporate a new understanding of number and magnitude—one that the algebra treatises of the Middle Ages and the early Renaissance had taken for granted but failed to articulate explicitly. In particular in the editions of the *Elements* studied here it is assumed that Book II contains a “geometrical algebra” that actually provides foundations for algebraic rules. These editions were therefore instrumental in clearing the way for Stevin’s new understanding of numbers as measuring quantities, the first systematic attempt to enlarge the conceptual numerical framework of the *Elements*. 
Let us stress the decisive role that the intellectual as well as the social contexts play in the Renaissance transformation of the notions of number and magnitude. The many discourses, Hermetic and otherwise, that extolled the practical usefulness of mathematics legitimized the commercial and engineering perspective on arithmetic and geometry. Such perspective facilitated the introduction of as many kinds of numbers as different kinds of measures were needed. Moreover, the multifaceted, omnipresent 16th-century myth of the corruption and rebirth of knowledge is also decisively present in the Renaissance rewriting of Euclid’s *Elements*, as well as in Stevin’s radical departure from Euclid’s numbers and magnitudes. Notice, finally, that while the Renaissance editors of the *Elements* here considered did correct some medieval misunderstandings, yet they also added their own. That is to say “mathematical humanism,” as instantiated in the editions of the *Elements* here considered, was not so much the faithful philological reconstruction of the letter and meaning of Euclid’s *Elements* as the construction of a new Euclides that catered to the needs of Renaissance mathematicians. The conceptual development of mathematics in the 16th century would have hardly benefited from the faithful (from our point of view) reconstruction of the *Elements*—a reconstruction leading to a version in which the gulf between discrete numbers and continuous magnitudes is reinforced rather than closed. The results of such an enterprise would have been useless and unintelligible to Renaissance mathematicians.

References

Malet, A. Forthcoming. Numbers, polynomials, and algorithms in Stevin’s *Arithmétique* (1585).


